Continued Fractions

• A finite continued fraction is an expression of the form

\[ a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots + \frac{1}{a_n} }}} \]

It can also be written as \([a_0; a_1, a_2, \ldots, a_n]\).

• Finite continued fractions with integer terms represent rational numbers.

• Every rational number can be expressed as a finite continued fraction using the Euclidean Algorithm, but the expression is not unique.

• The \(k^{\text{th}}\) convergent of \([a_0; a_1, a_2, \ldots, a_n]\) is (the value of) the continued fraction \([a_0; a_1, a_2, \ldots, a_k]\).

• The convergents oscillate about the value of the continued fraction.

• The convergents of a continued fraction may be computed from the continued fraction using a recursive algorithm. The algorithm produces rational numbers in lowest terms.

Definition. Let \(a_0, \ldots a_n\) be real numbers, with \(a_1, \ldots, a_n\) positive. A finite continued fraction is an expression of the form

\[ a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots + \frac{1}{a_n} } } } \]

To make the writing easier, I’ll denote the continued fraction above by \([a_0; a_1, a_2, \ldots, a_n]\). In most cases, the \(a_i\)’s will be integers.

Example. \[ \frac{47}{17} = 2 + \frac{13}{17} = 2 + \frac{1}{\frac{17}{13}} = 2 + \frac{1}{1 + \frac{4}{13}} = 2 + \frac{1}{1 + \frac{1}{\frac{13}{4}}} = 2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{4}}} \].

In short form, \(\frac{47}{17} = [2; 1, 3, 4]\).

A little bit of thought should convince you that you can express any rational number as a finite continued fraction in this way. In fact, the expansion corresponds to the steps in the Euclidean algorithm. First,

\[ 47 = 2 \cdot 17 + 13 \]
17 = 1 \cdot 13 + 4
13 = 3 \cdot 4 + 1
4 = 4 \cdot 1 + 0

Rewrite these equations as

\[
\frac{47}{17} = 2 + \frac{13}{17}
\]
\[
\frac{17}{13} = 1 + \frac{4}{13}
\]
\[
\frac{13}{4} = 3 + \frac{1}{4}
\]

You can get the continued fraction I found above by substituting the third equation into the second, and then substituting the result into the first.

Since this is just the Euclidean algorithm, I can use the first two columns of the Extended Euclidean Algorithm table to get the numbers in the continued fraction expansion:

<table>
<thead>
<tr>
<th>a</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>47</td>
<td>-</td>
</tr>
<tr>
<td>17</td>
<td>2</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

Notice that the successive quotients 2, 1, 3, and 4 are the numbers in the continued fraction expansion.

Example. Find the finite continued fraction expansion for \( \frac{117}{17} \).

<table>
<thead>
<tr>
<th>a</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>117</td>
<td>-</td>
</tr>
<tr>
<td>17</td>
<td>6</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

\[
\frac{117}{17} = [6; 1, 7, 2] = 6 + \frac{1}{1 + \frac{1}{7 + \frac{1}{2}}}.
\]

Lemma. Every finite continued fraction with integer terms represents a rational number.

Proof. If \( a_0 \in \mathbb{Z} \), then \([a_0] \) is rational.
Inductively, suppose that a finite continued fraction with \( n - 1 \) “levels” is a rational number. I want to show that
\[
a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}
\]
is rational.

By induction,
\[
x = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}
\]
is rational.

So
\[
a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}} = a_0 + \frac{1}{x}
\]
is the sum of two rational numbers, which is rational as well. \( \square \)

**Example.** The continued fraction expansion of a rational number is not unique. For example,
\[
\frac{47}{17} = 2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{4}}} = 2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1}}}
\]
And in general,
\[
[a_0; a_1, a_2, \ldots, a_{n-1}, a_n] = [a_0; a_1, a_2, \ldots, a_{n-1}, a_n - 1, 1]. \quad \square
\]

I want to talk about **infinite continued fractions** — things that look like
\[
a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}
\]
In preparation for this, I’ll look at the effect of truncating a continued fraction.

**Definition.** The $k$\textsuperscript{th} **convergent** of the continued fraction \([a_0; a_1, a_2, \ldots, a_n]\) is

\[
c_k = [a_0; a_1, a_2, \ldots, a_k].
\]

Note that for \(k \geq n\), \(c_k = [a_0; a_1, a_2, \ldots, a_n]\).

**Example.** [1; 2, 3, 2]

\[
c_0 = 1 \\
c_1 = 1 + \frac{1}{2} = \frac{3}{2} \\
c_2 = 1 + \frac{1}{\frac{2}{1} + \frac{1}{3}} = \frac{10}{7} \\
c_3 = 1 + \frac{1}{\frac{2}{1} + \frac{1}{\frac{3}{2}}} = \frac{23}{16}
\]

And \(c_4 = c_5 = \cdots = \frac{23}{16}\) as well. \(\Box\)

The next result gives an algorithm for computing the convergents of a continued fraction. It’s important for theoretical reasons, too — I’ll need it for several of the proofs that follow. For the theorem, I won’t assume that the \(a_i\’s\) are integers, since I will need the general result later on.

**Theorem.** Let \(a_0, a_1, \ldots, a_n\) be positive real numbers. Let

\[
p_0 = a_0, \quad q_0 = 1 \\
p_1 = a_1a_0 + 1, \quad q_1 = a_1 \\
p_k = a_kp_{k-1} + p_{k-2}, \quad q_k = a_kq_{k-1} + q_{k-2}, \quad k \geq 2.
\]

Then the \(k\)-th convergent of \([a_0; a_1, a_2, \ldots, a_n]\) is \(c_k = \frac{p_k}{q_k}\).

**Proof.** First, note that

\[
[b_0; b_1, \ldots, b_k, b_{k+1}] = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{\ddots + \frac{1}{b_{k-1} + \frac{1}{b_k + \frac{1}{b_{k+1}}}}}}}}
\]

by regarding the last two terms as a single term.

Note also that \(p_0, \ldots, p_{k-1}\) and \(q_0, \ldots, q_{k-1}\) are the same for these two fractions, since they only differ in the \(k\)-th term.
Now I’ll start the proof — it will go by induction on $k$. For $k = 0$,

$$c_0 = a_0 = \frac{a_0}{1} = \frac{p_0}{q_0}. $$

And for $k = 1$,

$$c_1 = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1} = \frac{p_1}{q_1}. $$

Suppose $k \geq 2$, and assume that result holds through the $k$-th convergent. Then

$$c_{k+1} = \left[ a_0; a_1, \ldots, a_k, a_{k+1} \right] = \left[ a_0; a_1, \ldots, a_k + \frac{1}{a_{k+1}} \right]. $$

Now this is the $k$-th convergent of a continued fraction, so by induction this is $\frac{p_k}{q_k}$, where $p_k$ and $q_k$ refer to $\left[ a_0; a_1, \ldots, a_k + \frac{1}{a_{k+1}} \right]$ (as opposed to $\left[ a_0; a_1, \ldots, a_k, a_{k+1} \right]$). But what are the $p_k$ and $q_k$ for this fraction? They’re given inductively by

$$p_k = (k\text{-th term})p_{k-1} + p_{k-2}, \quad q_k = (k\text{-th term})q_{k-1} + q_{k-2}. $$

Now $p_{k-2}, p_{k-1}, q_{k-2}, q_{k-1}$ are the same for $\left[ a_0; a_1, \ldots, a_k + \frac{1}{a_{k+1}} \right]$ and $\left[ a_0; a_1, \ldots, a_k, a_{k+1} \right]$, as I noted at the start. On the other hand, the $k$-th term of $\left[ a_0; a_1, \ldots, a_k + \frac{1}{a_{k+1}} \right]$ is $a_k + \frac{1}{a_{k+1}}$. So

$$c_{k+1} = \frac{(a_k + \frac{1}{a_{k+1}})p_{k-1} + p_{k-2}}{(a_k + \frac{1}{a_{k+1}})q_{k-1} + q_{k-2}} = \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} = \frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}} = \frac{p_{k+1}}{q_{k+1}}. $$

(The next to the last equality also follows by induction.) This shows that the result holds for $k + 1$, so the induction step is complete.

---

**Example.** $[1; 2, 1, 2, 1]$

<table>
<thead>
<tr>
<th>$a_k$</th>
<th>$p_k$</th>
<th>$q_k$</th>
<th>$c_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>$\frac{3}{2} = 1.5$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>3</td>
<td>$\frac{4}{3} \approx 1.33333$</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>8</td>
<td>$\frac{11}{8} = 1.375$</td>
</tr>
<tr>
<td>1</td>
<td>15</td>
<td>11</td>
<td>$\frac{15}{11} \approx 1.36364$</td>
</tr>
</tbody>
</table>

There is a pattern to the computation of the $p$’s and $q$’s which makes things pretty easy. To get the
next \( p \), for instance, multiply the current \( a \) by the last \( p \) and add the next-to-the-last \( p \).

\[
\begin{array}{ccc}
\hline
a & p & q \\
1 & 1 & 1 \\
2 & 3 & 2 \\
1 & 4 & 3 \\
2 & 11 & 8 \\
1 & & \\
\hline
\end{array}
\begin{array}{ccc}
\hline
a & p & q \\
1 & 1 & 1 \\
2 & 3 & 2 \\
1 & 4 & 3 \\
2 & 11 & 8 \\
1 & & \\
\hline
\end{array}
\]

Fill in the \( a \)'s, \( p_0 \), \( q_0 \), \( p_1 \), and \( q_1 \).

\[
p_2 = (1)(3) + 1 = 4
q_2 = (1)(2) + 1 = 3
\]

\[
p_3 = (2)(4) + 3 = 11
q_3 = (2)(3) + 2 = 8
\]

Notice that the convergents oscillate, and that the fractions which give the convergents are always in lowest terms. □

**Example.** \([1; 1, 3, 1, 3]\)

\[
\begin{array}{cccc}
\hline
a_k & p_k & q_k & c_k \\
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 2 \\
3 & 7 & 4 & \frac{7}{4} = 1.75 \\
1 & 9 & 5 & \frac{9}{5} = 1.8 \\
3 & 34 & 19 & \frac{34}{19} \approx 1.78947 \\
\hline
\end{array}
\]

Again, notice that the convergents oscillate, and that the fractions for the convergents are always in lowest terms. □

I’ll prove that the convergent fractions are in lowest terms first.

**Theorem.** Let \( a_0, a_1, \ldots, a_n \) be positive real numbers. Let

\[
p_0 = a_0, \quad q_0 = 1
\]

\[
p_1 = a_1a_0 + 1, \quad q_1 = a_1
\]

\[
p_k = a_kp_{k-1} + p_{k-2}, \quad q_k = a_kq_{k-1} + q_{k-2}, \quad k \geq 2.
\]

Then

\[
p_kq_{k-1} - p_{k-1}q_k = (-1)^{k-1}.
\]
Corollary. Let \(a_0, a_1, \ldots, a_n\) be positive integers. Let
\[
\begin{align*}
p_0 &= a_0, \quad q_0 = 1 \\
p_1 &= a_1a_0 + 1, \quad q_1 = a_1 \\
p_k &= a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}, \quad k \geq 2.
\end{align*}
\]
For \(k \geq 1\), \(\frac{p_k}{q_k}\) is in lowest terms.

Proof. \(p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1} = \pm 1\) implies that \((p_k, q_k) = 1\).  

Proof of the Theorem. I’ll induct on \(k\). For \(k = 1\),
\[
p_1q_0 - p_0q_1 = (a_1a_0 + 1)(1) - (a_0)(a_1) = 1 = 1^1 - 1.
\]
Take \(k > 1\), and assume the result holds for \(k\). Then
\[
p_{k+1}q_k - p_kq_{k+1} = (a_{k+1}p_k + p_{k-1})q_k - p_k(a_{k+1}q_k + q_{k-1}) = p_{k-1}q_k - p_kq_{k-1} =
\]
\[
- (p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}) = -(-1)^{k-1} = (-1)^k.
\]
This proves the result for \(k + 1\), so the general result is true by induction.

Example. I’ll show later that \(\frac{1 + \sqrt{5}}{2}\) (the golden ratio) has the infinite continued fraction expansion \([1; 1, 1, \ldots]\). Here are the first ten convergents:

<table>
<thead>
<tr>
<th>(a_k)</th>
<th>(p_k)</th>
<th>(q_k)</th>
<th>(c_k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>1.5</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>3</td>
<td>1.6667</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>5</td>
<td>1.6</td>
</tr>
<tr>
<td>1</td>
<td>13</td>
<td>8</td>
<td>1.625</td>
</tr>
<tr>
<td>1</td>
<td>21</td>
<td>13</td>
<td>1.61538</td>
</tr>
<tr>
<td>1</td>
<td>34</td>
<td>21</td>
<td>1.61905</td>
</tr>
<tr>
<td>1</td>
<td>55</td>
<td>34</td>
<td>1.61765</td>
</tr>
<tr>
<td>1</td>
<td>89</td>
<td>55</td>
<td>1.61818</td>
</tr>
</tbody>
</table>

In fact, \(\frac{1 + \sqrt{5}}{2} \approx 1.61803\). In this case, you can see formally that \([1; 1, 1, \ldots]\) should be \(\frac{1 + \sqrt{5}}{2}\). Let
\[
x = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ddots}}}
\]
Notice that \(x\) contains a copy of itself as the bottom of the first fraction! So
\[
x = 1 + \frac{1}{x}, \quad x^2 = x + 1, \quad x^2 - x - 1 = 0.
\]
The roots are \(\frac{1 \pm \sqrt{5}}{2}\). Since the fraction is positive, take the positive root to obtain \(x = \frac{1 + \sqrt{5}}{2}\).  

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