3-14-2019

Divisor Functions

Definition. The **sum of divisors** function is given by

$$\sigma(n) = \sum_{d \mid n} d.$$

As usual, the notation "$d \mid n$" as the range for a sum or product means that d ranges over the **positive** divisors of n.

The **number of divisors** function is given by

$$\tau(n) = \sum_{d \mid n} 1.$$

For example, the positive divisors of 15 are 1, 3, 5, and 15. So

$$\sigma(15) = 1 + 3 + 5 + 15 = 24 \quad \text{and} \quad \tau(15) = 4.$$

I want to find formulas for $\sigma(n)$ and $\tau(n)$ in terms of the prime factorization of n. This will be easy if I can show that σ and τ are multiplicative. I can do most of the work in the following theorem.

Theorem. The divisor sum of a multiplicative function is multiplicative.

Proof. Suppose f is multiplicative, and let $D(f)$ be the divisor sum of f. Suppose $(m, n) = 1$. Then

$$[D(f)](m) = \sum_{a \mid m} f(a) \quad \text{and} \quad [D(f)](n) = \sum_{b \mid n} f(b).$$

Then

$$[D(f)](m) \cdot [D(f)](n) = \left(\sum_{a \mid m} f(a)\right) \left(\sum_{b \mid n} f(b)\right) = \sum_{a \mid m} \sum_{b \mid n} f(a) f(b).$$

Now $(m, n) = 1$, so if $a \mid m$ and $b \mid n$, then $(a, b) = 1$. Therefore, multiplicativity of f implies

$$[D(f)](m) \cdot [D(f)](n) = \sum_{a \mid m} \sum_{b \mid n} f(ab).$$

Now every divisor d of mn can be written as $d = ab$, where $a \mid m$ and $b \mid n$. Going the other way, if $a \mid m$ and $b \mid n$ then $ab \mid mn$. So I may set $d = ab$, where $d \mid mn$, and replace the double sum with a single sum:

$$[D(f)](m) \cdot [D(f)](n) = \sum_{d \mid mn} f(d) = [D(f)](mn).$$

This proves that $D(f)$ is multiplicative. $\square$

Theorem. (a) The sum of divisors function σ is multiplicative.

(b) The number of divisors function τ is multiplicative.

Proof. (a) The identity function $\text{id}(x) = x$ is multiplicative: $\text{id}(mn) = mn = \text{id}(m) \cdot \text{id}(n)$ for *all* m, n, so obviously it's true for $(m, n) = 1$. Therefore, the divisor sum of id is multiplicative. But

$$[D(\text{id})](n) = \sum_{d \mid n} \text{id}(d) = \sum_{d \mid n} d = \sigma(n).$$

Hence, the sum of divisors function σ is multiplicative.

(b) The constant function $I(n) = 1$ is multiplicative: $I(mn) = 1 = 1 \cdot 1 = I(m) \cdot I(n)$ for *all* m, n, so obviously it's true for $(m, n) = 1$. Therefore, the divisor sum of I is multiplicative. But

$$[D(I)](n) = \sum_{d|n} I(d) = \sum_{d|n} 1 = \tau(n).$$

Hence, the number of divisors function τ is multiplicative. □

I'll use multiplicativity to obtain formulas for $\sigma(n)$ and $\tau(n)$ in terms of their prime factorizations (as I did with ϕ). First, I'll get the formulas in the case where n is a power of a prime.

Lemma. Let p be prime.

(a) $\sigma(p^k) = \dfrac{p^{k+1} - 1}{p - 1}$.

(b) $\tau(p^k) = k + 1$.

Proof. The divisors of p^k are 1, p, p^2, ..., p^k. So the sum of the divisors is

$$\sigma(p^k) = 1 + p + p^2 + \cdots + p^k = \frac{p^{k+1} - 1}{p - 1}.$$

And since the divisors of p^k are 1, p, p^2, ..., p^k, there are $k + 1$ of them, and

$$\tau(p^k) = k + 1. \quad \square$$

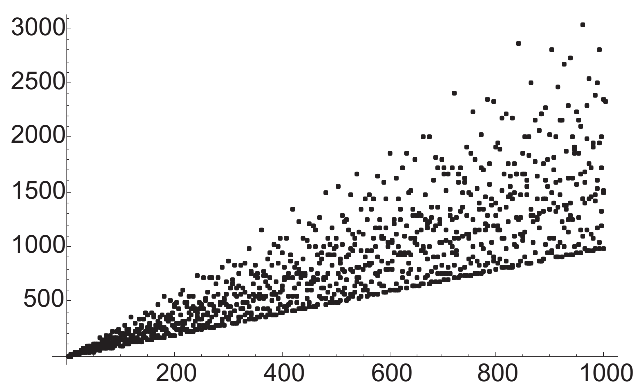
Theorem. Let $n = p_1^{r_1} \cdots p_k^{r_k}$, where the p's are distinct primes and $r_i \geq 1$ for all i. Then:

$$\sigma(n) = \left(\frac{p_1^{r_1+1} - 1}{p_1 - 1}\right) \cdots \left(\frac{p_k^{r_k+1} - 1}{p_k - 1}\right)$$

$$\tau(n) = (r_1 + 1) \cdots (r_k + 1)$$

Proof. These results follow from the preceding lemma, the fact that σ and τ are multiplicative, and the fact that the prime power factors $p_i^{r_i}$ are pairwise relatively prime. □

Here is a graph of $\sigma(n)$ for $1 \leq n \leq 1000$.



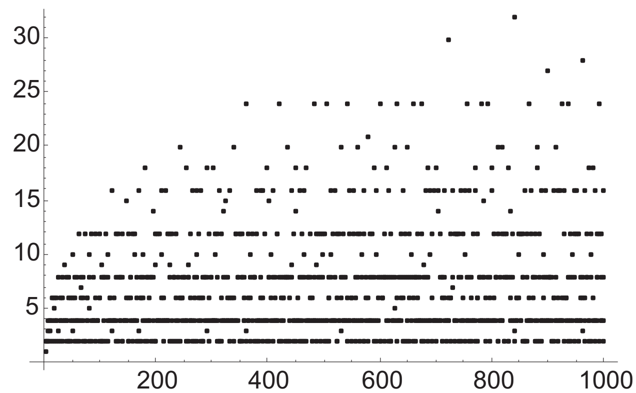
Note that if p is prime, $\sigma(p) = p + 1$. This gives the point $(p, p + 1)$, which lies on the line $y = x + 1$. This is the line that you see bounding the dots below.

For each n, there are only finitely many numbers k whose divisor sum is equal to n: that is, such that $\sigma(k) = n$. For k divides itself, so

$$n = \sigma(k) = (\text{other terms}) + k > k.$$

This says that k must be less than n. So if I'm looking for numbers whose divisors sum to n, I only need to look at numbers less than n. For example, if I want to find all numbers whose divisors sum to 42, I only need to look at $\{1, 2, \ldots, 41\}$.

Here is a graph of $\tau(n)$ for $1 \le n \le 1000$.



If p is prime, $\tau(p) = 2$. Thus, τ repeatedly returns to the horizontal line $y = 2$, which you can see bounding the dots below.

The formulas given in the theorem allow us to compute $\sigma(n)$ and $\tau(n)$ by hand for at least small values of n. For example, $720 = 2^4 \cdot 3^2 \cdot 5$, so

$$\sigma(720) = \left(\frac{2^5 - 1}{2 - 1}\right)\left(\frac{3^3 - 1}{3 - 1}\right)\left(\frac{5^2 - 1}{5 - 1}\right) = 2418,$$

$$\tau(720) = (4 + 1)(2 + 1)(1 + 1) = 30.$$

Example. Find all positive integers n such that $\sigma(n) = n + 8$.

Since $n = 1$ doesn't work, I can assume $n > 1$.
I have

$$\begin{aligned} n + 8 = \sigma(n) &= 1 + (\text{sum of divisors other than 1 and } n) + n \\ 7 &= (\text{sum of divisors other than 1 and } n) \end{aligned}$$

In other words, (sum of divisors other than 1 and n) is a sum of distinct positive integers other than 1 and n that is equal to 7. I have to consider all possible ways of doing this. I'll consider cases according to the largest element of this sum, which is the largest divisor d of n other than 1 and n.

Suppose $d = 7$.

$$(\text{sum of divisors other than 1 and } n) = 7.$$

Then the only divisor of n other than 1 and n is 7. Since $n \ne 7$, I know $n = 7^k$ for $k > 1$. But if $n > 49$, then 49 would be a divisor of n other than 1 and n. Hence, $n = 49$, and this is a solution.

Suppose $d = 6$. Then the expression (sum of divisors other than 1 and n) must have the form $6 + 1$, which contradicts the assumption that the sum does not include 1.

Suppose $d = 5$. Then the expression (sum of divisors other than 1 and n) must have the form $2 + 5$. In this case, $n = 2 \cdot 5 = 10$.

Suppose $d = 4$. Then

$$(\text{sum of divisors other than 1 and } n) = (\text{terms adding to 3}) + 4.$$

But if $4 \mid n$, then $2 \mid n$. So (terms adding to 3) must have the form $1 + 2$, contradicting the assumption that the sum does not include 1.

Suppose $d = 3$. Then

$$\text{(sum of divisors other than 1 and } n) = \text{(terms adding to 4)} + 3.$$

However, (terms adding to 4) can't include 1, and can't use 2 twice. Hence, this isn't possible.

Suppose $d = 2$. Then the remaining terms in (sum of divisors other than 1 and n) must sum to 5 and can only use 1, which is excluded by assumption. Hence, this isn't possible.

Therefore, $n = 10$ or $n = 49$. □
