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Infinite Continued Fractions

- The value of an infinite continued fraction $[a_0; a_1, a_2, \ldots]$ is

$$\lim_{k\to\infty} c_k, \quad \text{where} \quad c_k \quad \text{is the } k\text{-th convergent.}$$

- If $[a_0; a_1, a_2, \ldots]$ is an infinite continued fraction with positive terms, then $\lim_{k\to\infty} c_k$ exists.
- The convergents of an infinite continued fraction with positive terms converge by oscillation to its value.
- If $[a_0; a_1, a_2, \ldots]$ is an infinite continued fraction with positive terms, then its value is an irrational number.
- Every irrational number x has a unique infinite continued fraction expansion $[a_0; a_1, a_2, \ldots]$ whose terms are given recursively by

$$x_0 = x, \quad \text{and} \quad a_k = [x_k], \quad x_{k+1} = \frac{1}{x_k - a_k} \text{ for } k \ge 1.$$

If $[a_0; a_1, a_2, \ldots]$ is an infinite continued fraction, I want to define its *value* to be the limit of the convergents:

$$\lim_{k\to\infty} c_k, \quad \text{where} \quad c_k \quad \text{is the } k\text{-th convergent.}$$

For this to make sense, I need to show that this limit exists.

In what follows, take as given an infinite continued fraction $[a_0; a_1, a_2, \ldots]$. Let

$$p_0 = a_0, \quad q_0 = 1$$

$$p_1 = a_1 a_0 + 1, \quad q_1 = a_1$$

$$p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}, \qquad k \ge 2,$$

$$c_k = \frac{p_k}{q_k}.$$

Lemma.

(a) $c_k - c_{k-1} = \dfrac{(-1)^{k-1}}{q_{k-1} q_k}$.

(b) $c_k - c_{k-2} = \dfrac{a_k (-1)^k}{q_{k-2} q_k}$.

Proof. For (a),

$$c_k - c_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_k q_{k-1} - p_{k-1} q_k}{q_{k-1} q_k} = \frac{(-1)^{k-1}}{q_{k-1} q_k}.$$

For (b),

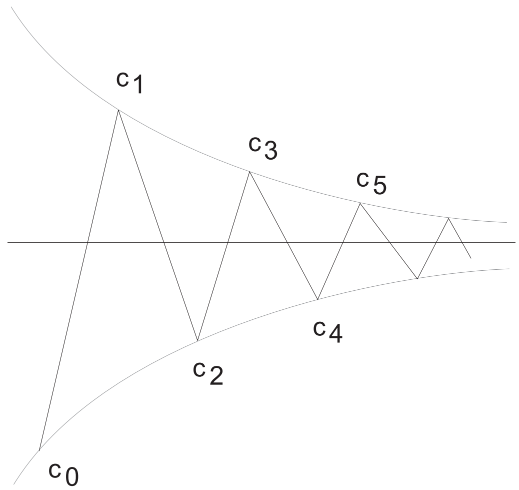
$$c_k - c_{k-2} = \frac{p_k}{q_k} - \frac{p_{k-2}}{q_{k-2}} = \frac{p_k q_{k-2} - p_{k-2} q_k}{q_{k-2} q_k} = \frac{(a_k p_{k-1} + p_{k-2}) q_{k-2} - p_{k-2}(a_k q_{k-1} + q_{k-2})}{q_{k-2} q_k} =$$

$$\frac{a_k (p_{k-1} q_{k-2} - p_{k-2} q_{k-1})}{q_{k-2} q_k} = \frac{a_k \cdot (-1)^{k-2}}{q_{k-2} q_k} = \frac{a_k (-1)^k}{q_{k-2} q_k}. \quad \square$$

Lemma.

$$c_1 > c_3 > c_5 > \cdots > c_4 > c_2 > c_0.$$

That is, the odd convergents get smaller, the even convergents get bigger, and any odd convergent is bigger than any even convergent.



Proof. If k is even, then

$$c_k - c_{k-2} = \frac{a_k(-1)^k}{q_{k-2}q_k} > 0, \quad \text{so} \quad c_k > c_{k-2}.$$

This shows that the even terms get bigger.

If k is odd, then

$$c_k - c_{k-2} = \frac{a_k(-1)^k}{q_{k-2}q_k} < 0, \quad \text{so} \quad c_k < c_{k-2}.$$

This shows that the odd terms get smaller.

Finally, I have to show any odd term is bigger than any even term. Note that it's true if the terms are adjacent:

$$c_{2n+1} - c_{2n} = \frac{(-1)^{(2n+1)-1}}{q_{k-1}q_k} > 0, \quad \text{so} \quad c_{2n+1} > c_{2n}.$$

Next, do the general case. Let c_{2n+1} be an odd term and let c_{2m} be an even term. Then

$$c_{2n+1} > c_{2n+2m+1} > c_{2n+2m} > c_{2m}.$$

The first inequality is true because odd terms decrease. The second inequality is true because I just observed that an odd term is bigger than an *adjacent* even. Finally, the last inequality is true because even terms increase.

I've shown that any odd term is bigger than any even term, and that completes the proof. □

Example. The first ten convergents for the golden ratio $\dfrac{1+\sqrt{5}}{2}$ are:

$$\underset{c_1}{2} > \underset{c_3}{1.66667} > \underset{c_5}{1.625} > \underset{c_7}{1.61905} > \underset{c_9}{1.61818}$$

$$\underset{c_8}{1.61765} > \underset{c_6}{1.61538} > \underset{c_4}{1.6} > \underset{c_2}{1.5} > \underset{c_0}{1}$$

The odd convergents get smaller, the even convergents get bigger, and any odd convergent is bigger than any even convergent. □

Lemma. $q_k \geq k$ for all $k \geq 1$.

Proof. I'll induct on k. $q_1 = a_1 \geq 1$, so the result holds for $k = 1$. Take $k > 1$, and assume it holds for numbers $\leq k$. I'll prove that it holds for $k + 1$.

$$q_{k+1} = a_{k+1}q_k + q_{k-1} \geq a_{k+1} \cdot k + (k-1) \geq 1 \cdot k + (k-1) = 2k - 1 = k + (k-1) \geq k + 1.$$

(The last inequality used $k \geq 2$.) This completes the induction step. □

Theorem. Let $[a_0; a_1, a_2, \ldots]$ be an infinite continued fraction with $a_k > 0$ for $k \geq 1$, and let c_k be the k-th convergent. Then

$$\lim_{k\to\infty} c_k \quad \text{exists.}$$

Proof. Consider the sequence of odd convergents

$$c_1 > c_3 > c_5 > \cdots.$$

This is a decreasing sequence of numbers, and it's bounded below — by any even convergent, for example. A standard result from analysis (see, for example, Theorem 3.14 of [1]) asserts that such a sequence must have a limit, so

$$\lim_{k\to\infty} c_{2k+1} \quad \text{exists.}$$

Likewise, consider the sequence of even convergents:

$$\cdots > c_4 > c_2 > c_0.$$

This is an increasing sequence of numbers that's bounded above — by any odd convergent, for example. The result from analysis mentioned above says that the sequence has a limit:

$$\lim_{k\to\infty} c_{2k} \quad \text{exists.}$$

I have to show that the two limits agree.

The previous lemma implies that $q_{2k} \geq 2k$ and $q_{2k+1} \geq 2k + 1$, so

$$0 \leq c_{2k+1} - c_{2k} = \frac{(-1)^{2k+1-1}}{q_{2k}q_{2k+1}} \leq \frac{1}{(2k)(2k+1)}.$$

Now let $k \to \infty$. $\dfrac{1}{(2k)(2k+1)} \to 0$, so by the Squeezing Theorem of calculus,

$$\lim_{k\to\infty} (c_{2k+1} - c_{2k}) = 0, \text{ i.e. } \lim_{k\to\infty} c_{2k+1} = \lim_{k\to\infty} c_{2k}.$$

Since the odd and even terms approach the same limit, $\lim\limits_{k\to\infty} c_k$ exists. □

Knowing this, I'm justified in *defining*

$$[a_0; a_1, a_2, \ldots] = \lim_{k\to\infty} c_k.$$

What can I say about its value?

Theorem. Let $[a_0; a_1, a_2, \ldots]$ be an infinite continued fraction with $a_k > 0$ for $k \geq 1$. Then $[a_0; a_1, a_2, \ldots]$ is irrational.

Proof. Write $x = [a_0; a_1, a_2, \ldots]$ for short. I want to show that x is irrational. Suppose on the contrary that $x = \dfrac{p}{q}$, where p and q are integers. I will show this leads to a contradiction.

Since the odd convergents are bigger than x and the even convergents are smaller than x,

$$c_{2k+1} > x > c_{2k}.$$

Then

$$\begin{aligned}
c_{2k+1} - c_{2k} &> x - c_{2k} > 0, \\
\frac{(-1)^{2k}}{q_{2k}q_{2k+1}} &> x - c_{2k} > 0, \\
\frac{1}{q_{2k}q_{2k+1}} &> x - c_{2k} > 0, \\
\frac{1}{q_{2k}q_{2k+1}} &> x - \frac{p_{2k}}{q_{2k}} > 0, \\
\frac{1}{q_{2k+1}} &> xq_{2k} - p_{2k} > 0, \\
\frac{1}{q_{2k+1}} &> \frac{pq_{2k}}{q} - p_{2k} > 0, \\
\frac{q}{q_{2k+1}} &> pq_{2k} - p_{2k}q > 0.
\end{aligned}$$

Notice that this inequality is true for all k, and that the junk in the middle is an *integer*. But q is fixed, and $q_{2k+1} \geq 2k+1$, so if I make k sufficiently large eventually q_{2k+1} will become bigger than q. Then $\dfrac{q}{q_{2k+1}}$ will be a fraction less than 1, and I have an *integer* $pq_{2k} - p_{2k}q$ caught between 0 and a fraction less than 1. Since this is impossible, x can't be rational. □

Now I know that every infinite continued fraction made of positive integers represents an irrational number. The converse is also true true, and the next result gives an algorithm for computing the continued fraction expansion.

Theorem. Let $x \in \mathbb{R}$ be irrational. Let $x_0 = x$, and

$$a_k = [x_k], \quad x_{k+1} = \frac{1}{x_k - a_k} \text{ for } k \geq 0.$$

Then

$$x = [a_0; a_1, a_2, \ldots].$$

Proof.

Step 1. x_k is irrational for $k \geq 0$.

Since x is irrational and $x_0 = x$, the result is true for $k = 0$.

Assume that $k > 0$ and that the result is true for $k - 1$. I want to show that x_k is irrational.

Suppose on the contrary that $x_k = \dfrac{s}{t}$, where $s, t \in \mathbb{Z}$. Then

$$\frac{s}{t} = \frac{1}{x_{k-1} - a_{k-1}} \quad \text{so} \quad x_{k-1} = a_{k-1} + \frac{t}{s}.$$

Now all the a_k's are clearly integers (since $a_k = [x_k]$ means they're outputs of the greatest integer function), so $a_{k-1} + \dfrac{t}{s}$ is the sum of an integer and a rational number. Therefore, it's rational, so x_{k-1} is rational, contrary to the induction hypothesis.

It follows that x_k is irrational. By induction, x_k is irrational for all $k \geq 0$.

Step 2. The a_k's are positive integers for $k \geq 1$.

I already observed that the a_k's are integers.

Let $k \geq 0$. Since $a_k = [x_k]$, the definition of the greatest integer function gives

$$a_k \leq x_k < a_k + 1.$$

But x_k is irrational, so $a_k \neq x_k$. Hence,

$$a_k < x_k < a_k + 1,$$

$$0 < x_k - a_k < 1,$$

$$x_{k+1} = \frac{1}{x_k - a_k} > 1,$$

$$a_{k+1} = [x_{k+1}] \geq 1.$$

Since $k \geq 0$, this proves that the a_k's are positive integers for $k \geq 1$.

Step 3.

$$\lim_{k\to\infty} c_k = \lim_{k\to\infty} [a_0; a_1, \ldots, a_k] = x.$$

First, I'll get a formula for x in terms of the p's, q's, and a's.

Then I'll find $\left|x - \dfrac{p_k}{q_k}\right|$ and show that it's less than something which goes to 0.

To get the formula for x, start with

$$x_{k+1} = \frac{1}{x_k - a_k}.$$

Do some algebra to get

$$x_k = a_k + \frac{1}{x_{k+1}}.$$

Write out this equation for a few values of k:

$$\begin{aligned} x_0 &= a_0 + \frac{1}{x_1} \\ x_1 &= a_1 + \frac{1}{x_2} \\ x_2 &= a_2 + \frac{1}{x_3} \\ x_3 &= a_3 + \frac{1}{x_4} \\ &\vdots \end{aligned}$$

Substituting the second equation of the set into the first gives

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{x_2}}.$$

Substituting $x_2 = a_2 + \dfrac{1}{x_3}$ into this equation gives

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{x_3}}}.$$

Substituting $x_3 = a_3 + \dfrac{1}{x_4}$ into this equation gives

$$x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{x_4}}}}.$$

You get the idea. In general,

$$x = x_0 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_k + \cfrac{1}{x_{k+1}}}}} = [a_0; a_1, a_2, \ldots, a_k, x_{k+1}].$$

In other words, the x_k's are the "infinite tails" of the continued fraction.
Recall the recursion formulas for convergents:

$$p_k = a_k p_{k-1} + p_{k-2} \quad \text{and} \quad q_k = a_k q_{k-1} + q_{k-2}.$$

The right sides only involve terms up to a_k (and p's and q's of smaller indices). Therefore, the fractions

$$[a_0; a_1, a_2, \ldots, a_k, x_{k+1}] \quad \text{and} \quad [a_0; a_1, a_2, \ldots, a_k, a_{k+1}, \ldots]$$

have the same p's and q's through index k.

Using the recursion formula for convergents, I get

$$x = x_0 = [a_0; a_1, a_2, \ldots, a_k, x_{k+1}] = \frac{x_{k+1} p_k + p_{k-1}}{x_{k+1} q_k + q_{k-1}}.$$

Therefore,

$$x - \frac{p_k}{q_k} = \frac{x_{k+1} p_k + p_{k-1}}{x_{k+1} q_k + q_{k-1}} - \frac{p_k}{q_k} = \frac{x_{k+1} p_k q_k + p_{k-1} q_k - x_{k+1} p_k q_k - p_k q_{k-1}}{(x_{k+1} q_k + q_{k-1}) q_k} =$$

$$\frac{p_{k-1} q_k - p_k q_{k-1}}{(x_{k+1} q_k + q_{k-1}) q_k} = \frac{(-1)^k}{(x_{k+1} q_k + q_{k-1}) q_k}.$$

Take absolute values:

$$\left| x - \frac{p_k}{q_k} \right| = \frac{1}{(x_{k+1} q_k + q_{k-1}) q_k}.$$

Now

$$x_{k+1} > [x_{k+1}] = a_{k+1}, \quad \text{so} \quad x_{k+1} q_k + q_{k-1} > a_{k+1} q_k + q_{k-1} = q_{k+1}.$$

Therefore,

$$\frac{1}{x_{k+1}q_k + q_{k-1}} < \frac{1}{q_{k+1}},$$

$$\frac{1}{(x_{k+1}q_k + q_{k-1})q_k} < \frac{1}{q_{k+1}q_k},$$

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_{k+1}q_k}.$$

By an earlier lemma, $q_k \ge k$ and $q_{k+1} \ge k+1$, so

$$\left|x - \frac{p_k}{q_k}\right| < \frac{1}{q_{k+1}q_k} \le \frac{1}{k(k+1)}.$$

Now $\lim_{k\to\infty} \dfrac{1}{k(k+1)} = 0$, so by the Squeezing Theorem

$$\lim_{k\to\infty} \left|x - \frac{p_k}{q_k}\right| = 0.$$

This implies that

$$\lim_{k\to\infty} \frac{p_k}{q_k} = x. \quad \square$$

Example. I'll compute the continued fraction expansion of π. Here are the first two steps:

$$x_0 = \pi, \quad a_0 = [x_0] = [\pi] = 3$$

$$x_1 = \frac{1}{x_0 - a_0} \approx 7.06251, \quad a_1 = [x_1] = 7$$

Continuing in this way, I obtain:

x_k	a_k	p_k	q_k	c_k
π	3	3	1	3
$7.06251\ldots$	7	22	7	$\frac{22}{7}$
$15.99659\ldots$	15	333	106	$\frac{333}{106}$
$1.00341\ldots$	1	355	113	$\frac{355}{113}$
$292.63459\ldots$	292	103993	33102	$\frac{103993}{33102}$

$\square$

Example. I'll compute the continued fraction expansion of $\sqrt{5}$.

$$x_0 = \sqrt{5} \quad a_0 = [\sqrt{5}] \approx [2.23607] = 2$$

$$x_1 = \frac{1}{\sqrt{5} - 2}, \quad a_1 = \left[\frac{1}{\sqrt{5} - 2}\right] \approx [4.23607] = 4$$

Here are the first 5 terms:

x	a
$\sqrt{5}$	2
$4.23606\ldots$	4
$4.23606\ldots$	4
$4.23606\ldots$	4
$4.23606\ldots$	4

In fact, the continued fraction expansion for $\sqrt{5}$ is $[2; 4, 4, 4, \ldots]$. $\square$

Theorem. The continued fraction expansion of an irrational number is unique.

Proof. Suppose

$$[a_0; a_1, a_2, \ldots] = x = [b_0; b_1, b_2, \ldots]$$

are two continued fractions for the irrational number x, where $a_k, b_k \in \mathbb{Z}$ and $a_k, b_k \geq 1$ for $k \geq 1$. I want to show that $a_k = b_k$ for all k.

Recall that:

- The even convergents are smaller than x.
- The odd convergents are greater than x.

Therefore,

$$a_0 < x < a_0 + \frac{1}{a_1}.$$

Now

$$\begin{aligned} a_1 &\geq 1 \\ \frac{1}{a_1} &\leq 1 \\ a_0 + \frac{1}{a_1} &\leq a_0 + 1 \\ x &< a_0 + 1 \end{aligned}$$

Thus, a_0 *is an integer less than* x*, and the next larger integer* $a_0 + 1$ *is greater than* x. This means that $a_0 = [x]$.

The same reasoning applies to the b's. Therefore, $b_0 = [x]$, so $a_0 = b_0$.

Hence,

$$\begin{aligned} [a_0; a_1, a_2, \ldots] &= [b_0; b_1, b_2, \ldots] \\ a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} &= b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}} \\ \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} &= \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}} \\ [a_1; a_2, a_3, \ldots] &= [b_1; b_2, b_3, \ldots] \end{aligned}$$

I can continue in the same way to show that $a_k = b_k$ for all k. □

Here's a summary of some of the important results on infinite continued fractions:

1. An irrational number has a unique infinite continued fraction expansion.

2. The algorithm for computing the continued fraction expansion of an irrational number x is:

$$x_0 = x, \quad \text{and} \quad a_k = [x_k], \quad x_{k+1} = \frac{1}{x_k - a_k} \quad \text{for} \quad k \geq 1.$$

Then

$$x = [a_0; a_1, a_2, \ldots].$$

3. If $[a_0; a_1, a_2, \ldots]$ is the continued fraction expansion of an irrational number, then a_k is a positive integer for $k \geq 1$.

4. If $x = [a_0; a_1, a_2, \ldots]$ is the continued fraction expansion of an irrational number and $\{p_k\}$ and $\{q_k\}$ are defined by the recursion formulas for convergents, then

$$\left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_{k+1} q_k}.$$

Example. Here is the continued fraction expansion for $e + \pi$.

x_k	a_k	p_k	q_k	c_k
$5.85987\ldots$	5	5	1	5
$1.16296\ldots$	1	6	1	6
$6.13646\ldots$	6	41	7	$\frac{41}{7}$
$7.32821\ldots$	7	293	50	$\frac{293}{50}$
$3.0468\ldots$	3	920	157	$\frac{920}{157}$
$21.3697\ldots$	21	19613	3347	$\frac{19613}{3347}$

Now

$$e + \pi - \frac{920}{157} \approx 0.000001871 \quad \text{while} \quad \frac{1}{157 \cdot 3347} \approx 0.000001903.$$

Thus, in this case,

$$\left| e + \pi - \frac{920}{157} \right| < \frac{1}{157 \cdot 3347}. \quad \square$$

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[1] Walter Rudin, *Principles of Mathematical Analysis* (3rd edition). New York: McGraw-Hill Book Company, 1976.