Nonlinear Diophantine Equations

In general, solving a nonlinear Diophantine equation can be very difficult. In this section, we'll look at some examples of solving such an equation, and showing that such an equation can't be solved.

Example. Find all pairs of nonnegative integers (x, y) such that

$$(xy+4)^2 = x^2 + y^2.$$

$$\begin{aligned}
(xy+4)^2 &= x^2 + y^2 \\
x^2y^2 + 8xy + 16 &= x^2 + y^2 \\
x^2y^2 + 6xy + 16 &= x^2 - 2xy + y^2 \\
x^2y^2 + 6xy + 9 + 7 &= (x-y)^2 \\
(xy+3)^2 + 7 &= (x-y)^2 \\
(xy+3)^2 - (x-y)^2 &= -7 \\
[(xy+3)+(x-y)][(xy+3)-(x-y)] &= -7
\end{aligned}$$

Case 1. $(xy+3)+(x-y) = 7$ and $(xy+3)-(x-y) = -1$.

Adding the two equations gives

$$2(xy+3) = 6 \quad \text{so} \quad xy+3 = 3.$$

Thus, $xy = 0$.
Subtracting the two equations gives

$$2(x-y) = 8 \quad \text{so} \quad x-y = 4.$$

The second equation gives $x = y+4$. Plugging this into $xy = 0$ gives

$$(y+4)y = 0.$$

$y = -4$ gives $x = 0$ and $y = 0$ gives $x = 4$. The two solutions in this case are $(0, -4)$ and $(4, 0)$.

Case 2. $(xy+3)+(x-y) = -1$ and $(xy+3)-(x-y) = 7$.

Adding the two equations gives

$$2(xy+3) = 6 \quad \text{so} \quad xy+3 = 3.$$

Thus, $xy = 0$.
Subtracting the two equations gives

$$2(x-y) = -8 \quad \text{so} \quad x-y = -4.$$

The second equation gives $x = y-4$. Plugging this into $xy = 0$ gives

$$(y-4)y = 0.$$

$y = 4$ gives $x = 0$ and $y = 0$ gives $x = -4$. The two solutions in this case are $(0, 4)$ and $(-4, 0)$.

Case 3. $(xy+3)+(x-y) = -7$ and $(xy+3)-(x-y) = 1$.

Adding the two equations gives

$$2(xy+3) = -6 \quad \text{so} \quad xy+3 = -3.$$

Thus, $xy = -6$.
Subtracting the two equations gives

$$2(x - y) = -8 \quad \text{so} \quad x - y = -4.$$

The second equation gives $x = y - 4$. Plugging this into $xy = -6$ gives

$$(y - 4)y = -6, \quad \text{or} \quad y^2 - 4y + 6 = 0.$$

This equation has no real solutions.

Case 4. $(xy + 3) + (x - y) = 1$ and $(xy + 3) - (x - y) = -7$.

Adding the two equations gives

$$2(xy + 3) = -6 \quad \text{so} \quad xy + 3 = -3.$$

Thus, $xy = -6$.
Subtracting the two equations gives

$$2(x - y) = 8 \quad \text{so} \quad x - y = 4.$$

The second equation gives $x = y + 4$. Plugging this into $xy = -6$ gives

$$(y + 4)y = -6, \quad \text{or} \quad y^2 + 4y + 6 = 0.$$

This equation has no real solutions.

The solutions are $(0, -4)$, $(4, 0)$, $(0, 4)$, and $(-4, 0)$. □

Example. Prove that the following Diophantine equation has no solutions:

$$x^2 - 15y^2 = 22.$$

I reduce the equation mod 5 to obtain

$$x^2 = 2 \pmod{5}.$$

I construct a table of squares mod 5:

$x \pmod{5}$	0	1	2	3	4
$x^2 \pmod{5}$	0	1	4	4	1

This shows that 2 is not a square mod 5. Hence, the original Diophantine equation has no solutions. □
