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Periodic Continued Fractions

- A **quadratic irrational** is an irrational number which is a root of a quadratic equation with integer coefficients.
- Quadratic irrationals can be expressed in the form $\dfrac{p+\sqrt{q}}{r}$, where $p, q, r \in \mathbb{Z}$, $r \neq 0$, and q is positive and not a perfect square.
- Quadratic irrationals are exactly the real numbers which have infinite *periodic* continued fraction expansions.

Definition. A **quadratic irrational** is an irrational number which is a root of a quadratic equation

$$ax^2 + bx + c = 0, \quad a, b, c \in \mathbb{Z}, \quad a \neq 0.$$

Lemma. A number is a quadratic irrational if and only if it can be written in the form $\dfrac{p+\sqrt{q}}{r}$, where $p, q, r \in \mathbb{Z}$, $r \neq 0$, and q is positive and not a perfect square.

Proof. Suppose x is a quadratic irrational. Then x is a root of

$$ax^2 + bx + c = 0, \quad a, b, c \in \mathbb{Z}, a \neq 0.$$

By the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$-b$, $b^2 - 4ac$, and $2a$ are integers, and $2a \neq 0$, since $a \neq 0$.
If $b^2 - 4ac = 0$, then $x = -\dfrac{b}{2a}$, which is a rational number, contrary to assumption.
If $b^2 - 4ac < 0$, then x is complex, again contrary to assumption.
Hence, $b^2 - 4ac > 0$.

Finally, if $b^2 - 4ac$ is a perfect square, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is rational. Hence, $b^2 - 4ac$ is not a perfect square.

For the converse, suppose $x = \dfrac{p+\sqrt{q}}{r}$, where $p, q, r \in \mathbb{Z}$, $r \neq 0$, and q is positive and not a perfect square. Then

$$rx - p = \sqrt{q}, \quad (rx - p)^2 = q, \quad r^2x^2 - 2rpx + (p^2 - q) = 0.$$

This is a quadratic equation with integer coefficients, and $r^2 \neq 0$ since $r \neq 0$. Therefore, x is a quadratic irrational. □

Theorem. (Lagrange) The quadratic irrationals are exactly the real numbers which can be represented by infinite periodic continued fractions.

I'm going to prove one direction — that periodic continued fractions are quadratic irrationals. I need a series of lemmas; the lemmas are motivated by the informal procedure of the following example.

Example. Consider $x = [5; 2, 1, 2, 2, 1, 2, 2, \ldots] = [5; 2, \overline{1, 2, 2}]$.
I'll write x in closed form. Let $y = [\overline{1, 2, 2}]$. Then

$$x = 5 + \cfrac{1}{2 + \cfrac{1}{y}}.$$

On the other hand,

$$y = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{y}}}.$$

After some simplification, I get

$$5y^2 - 5y - 3 = 0, \quad y = \frac{5 \pm \sqrt{37}}{10}.$$

y must be positive, so $y = \dfrac{5 + \sqrt{37}}{10}$. Therefore,

$$x = 5 + \cfrac{1}{2 + \cfrac{1}{\cfrac{5 + \sqrt{37}}{10}}} = \frac{643 + 5\sqrt{37}}{126}. \quad \square$$

The idea of the lemmas is simply to emulate the algebra I just did.

Lemma 1. If x is a quadratic irrational and a_0 is an integer, then $a_0 + \dfrac{1}{x}$ is a quadratic irrational.

Proof. Write $x = \dfrac{a + \sqrt{b}}{c}$, where $a, b, c \in \mathbb{Z}$, $c \neq 0$, and b is positive and not a perfect square. Then

$$a_0 + \frac{1}{x} = a_0 + \cfrac{1}{\cfrac{a + \sqrt{b}}{c}} = a_0 + \frac{c}{a + \sqrt{b}} = \frac{(a_0 a^2 + ac - a_0 b) - c\sqrt{b}}{a^2 - b}.$$

(I've suppressed the ugly algebra involved in combining the fractions and rationalizing the denominator.) The last expression is a quadratic irrational; note that $a^2 - b \neq 0$, because b is not a perfect square. $\square$

Lemma 2. If x is a quadratic irrational and $a_0, a_1, \ldots, a_n$ are integers, then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}}$$

is a quadratic irrational.

Proof. I'll use induction. The case $n = 0$ was done in Lemma 1.

Suppose $n > 0$, and suppose the result is true for $n - 1$. Then in

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}}, \quad \text{the subfraction} \quad a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}$$

is a quadratic irrational by the induction hypothesis.

But the original fraction is just $a_0 + \dfrac{1}{(\text{the subfraction})}$, so it's a quadratic irrational by Lemma 1. This completes the induction step, so the result is true for all $n \geq 0$. □

Lemma 3. Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}}, \quad \text{can be written as} \quad \frac{ax+b}{cx+d},$$

where $a, b, c, d \in \mathbb{Z}$.

Proof. Your experience with algebra should tell you this is obvious, but I'll give the proof by induction anyway.

For $n = 0$, I have

$$a_0 + \frac{1}{x} = \frac{a_0 x + 1}{x}.$$

This has the right form.

Take $n > 0$, and assume the result is true for $n - 1$. Then in

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}}, \quad \text{the subfraction} \quad a_1 + \cfrac{1}{a_2 + \cfrac{}{\ddots + \cfrac{1}{a_n + \cfrac{1}{x}}}}$$

can be written as $\dfrac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{Z}$, by induction.

The original fraction is therefore

$$a_0 + \cfrac{1}{\cfrac{ax+b}{cx+d}} = \frac{(a_0 a + c)x + (a_0 b + d)}{ax+b}.$$

(I've suppressed some easy but ugly algebra again.) The last fraction is in the right form, so this completes the induction step. The result is therefore true for all $n \geq 0$. □

I'm ready to prove that periodic continued fractions are quadratic irrationals. First, I'll consider those that start repeating immediately.

Lemma 4. If $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, then

$$x = [\overline{a_0; a_1, \ldots, a_n}]$$

is a quadratic irrational.

Proof. First, x is irrational, because it is an *infinite* continued fraction.

By Lemma 3,

$$x = [\overline{a_0; a_1, \ldots, a_n}] = [a_0; a_1, \ldots, a_n, x] = \frac{ax+b}{cx+d},$$

where $a, b, c, d \in \mathbb{Z}$.

Hence,

$$cx^2 + dx = ax + b, \quad cx^2 + (d - a)x - b = 0.$$

Therefore, x is a quadratic irrational. □

In the general case, the fraction does not start repeating immediately.

Proposition. If $b_0, b_1, \ldots, b_m, a_0, a_1, \ldots, a_n \in \mathbb{Z}$, then

$$x = [b_0; b_1, \ldots, b_m, \overline{a_0, a_1, \ldots, a_n}]$$

is a quadratic irrational.

Proof. $[\overline{a_0, a_1, \ldots, a_n}]$ is a quadratic irrational by Lemma 4. Therefore,

$$x = [b_0; b_1, \ldots, b_m, x]$$

is a quadratic irrational by Lemma 2. □

The converse states the quadratic irrationals give rise to periodic continued fractions. I won't give the proof; however, here's an example which shows how you can go from a quadratic equation to a periodic continued fraction (at least in this case).

Example. Suppose x is a quadratic irrational satisfying $x^2 + x - 1 = 0$. Rewrite the equation as

$$x(x + 1) - 1 = 0, \quad \text{and then} \quad x = \frac{1}{1 + x}.$$

Now substitute $x = \dfrac{1}{1 + x}$ for x *in the right side*:

$$x = \cfrac{1}{1 + \cfrac{1}{1 + x}}.$$

Do it again:

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + x}}}.$$

It's clear that you can keep going, and so $x = [0; \overline{1}]$. □
