Math 161-03/06
10–10/11–2007

Review Problems for Test 2

These problems are provided to help you study. The fact that a problem occurs here does not mean that there will be a similar problem on the test. And the absence of a problem from this review sheet does not mean that there won't be a problem of that kind on the test.

1. Graph $y = 2x^{3/2} - 6x^{1/2}$.

2. Graph $y = \dfrac{x}{\sqrt{x^2+7}}$.

3. Graph $y = 5x^{2/5} + \dfrac{5}{7}x^{7/5}$.

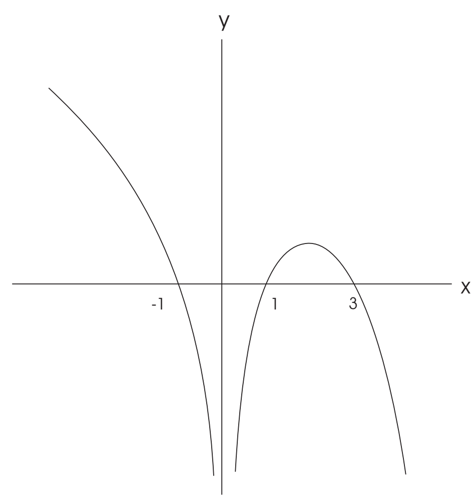
4. Graph $y = \dfrac{2x}{x^2-1}$.

5. Graph $f(x) = (x^2 - 4x + 5)e^x$.

6. Graph $f(x) = (x-2)(x-3) + 2\ln x$.

7. Sketch the graph of $y = |x^2 - 6x + 5|$ by first sketching the graph of $y = x^2 - 6x + 5$.

8. The function $y = f(x)$ is defined for all x. The graph of its derivative $y' = f'(x)$ is shown below:



Sketch the graph of $y = f(x)$.

9. A function $y = f(x)$ is defined for all x. In addition:

$$f(-1) = 0 \quad \text{and} \quad f'(3) \quad \text{is undefined,}$$

$$f'(x) \geq 0 \quad \text{for} \quad x \leq 2 \quad \text{and} \quad x > 3,$$
$$f'(x) \leq 0 \quad \text{for} \quad 2 \leq x < 3,$$
$$f''(x) < 0 \quad \text{for} \quad x < 3,$$
$$f''(x) > 0 \quad \text{for} \quad x > 3.$$

Sketch the graph of f.

10. Find the critical points of $y = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 4x + 5$ and classify them as local maxima or local minima using the Second Derivative Test.

11. Suppose $y = f(x)$ is a differentiable function, $f(3) = 4$, and $f'(3) = -6$. Use differentials to approximate $f(3.01)$.

12. The derivative of a function $y = f(x)$ is $y' = \frac{1}{x^4 + x^2 + 2}$. Approximate the change in the function as x goes from 1 to 0.99.

13. Use a linear approximation to approximate $\sqrt{1.99^3 + 1}$ to five decimal places.

14. The area of a sphere of radius r is $A = 4\pi r^2$. Suppose that the radius of a sphere is measured to be 5 meters with an error of ± 0.2 meters. Use a linear approximation to approximation the error in the area and the percentage error.

15. x and y are related by the equation

$$\frac{x^3}{y} - 4y^2 = 6xy - 8y.$$

Find the rate at which x is changing when $x = 2$ and $y = 1$, if y decreases at 21 units per second.

16. Let x and y be the two legs of a right triangle. Suppose the area is decreasing at 3 square units per second, and x is increasing at 5 units per second. Find the rate at which y is changing when $x = 6$ and $y = 20$.

17. A bagel (with lox and cream cheese) moves along the curve $y = x^2 + 1$ in such a way that its x-coordinate increases at 3 units per second. At what rate is its y-coordinate changing when it's at the point $(2, 5)$?

18. Bonzo ties Calvin to a large helium balloon, which floats away at a constant altitude of 600 feet. Bonzo pays out the rope attached to the balloon at 3 feet per second. How rapidly is the balloon moving horizontally at the instant when 1000 feet of rope have been let out?

19. A poster 6 feet high is mounted on a wall, with the bottom edge 5 feet above the ground. Calvin walks toward the picture at a constant rate of 2 feet per week. His eyes are level with the bottom edge of the picture. Let θ be the vertical angle subtended by the picture at Calvin's eyes. At what rate is θ changing when Calvin is 8 feet from the picture?

20. Use the Mean Value Theorem to show that if $0 < x < 1$, then

$$x + 1 < e^x < ex + 1.$$

- Hint: Apply the Mean Value Theorem to $f(x) = e^x$ with $a = 0$ and $b = x$. Use the fact that if $0 < c < x$, then $e^0 < e^c < e^1$.

21. (a) Prove that if $x > 1$, then $1 - \frac{1}{x} > 0$.

(b) Use the Mean Value Theorem to prove that if $x > 1$, then $x - 1 > \ln x$.

- Hint: Apply the Mean Value Theorem to $f(x) = x - \ln x$ with $a = 1$ and $b = x$, and use part (a).

22. Prove that the equation $x^5 + 7x^3 + 13x - 5 = 0$ has exactly one root.

23. Suppose that f is a differentiable function, $f(4) = 7$ and $|f'(x)| \leq 10$ for all x. Prove that $-13 \leq f(6) \leq 27$.

24. A differentiable function satisfies $f(3) = 0.2$ and $f'(3) = 10$. If Newton's method is applied to f starting at $x = 3$, what is the next value of x?

25. Use Newton's method to approximate a solution to $x^2 + 2x = \dfrac{5}{x}$. Do 5 iterations starting at $x = 1$, and do your computations to at least 5-place accuracy.

26. Find the absolute max and absolute min of $y = x^3 - 12x + 5$ on the interval $-1 \le x \le 4$.

27. Find the absolute max and absolute min of $y = 3x^{2/3}\left(\dfrac{1}{8}x^2 - \dfrac{1}{5}x - 1\right)$ on the interval $-2 \le x \le 8$.

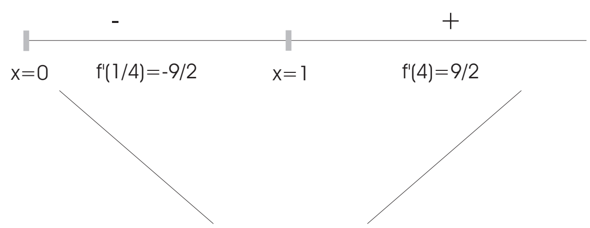
Solutions to the Review Problems for Test 2

1. Graph $y = 2x^{3/2} - 6x^{1/2}$.

The domain is $x \ge 0$.
$0 = y = 2x^{3/2} - 6x^{1/2} = 2x^{1/2}(x-3)$ gives the x-intercepts $x = 0$ and $x = 3$.
Set $x = 0$; the y-intercept is $y = 0$.
The derivatives are

$$y' = 3x^{1/2} - 3x^{-1/2} = \frac{3(x-1)}{x^{1/2}}, \quad y'' = \frac{3}{2}x^{-1/2} + \frac{3}{2}x^{-3/2} = \frac{3}{2} \cdot \frac{x+1}{x^{3/2}}.$$

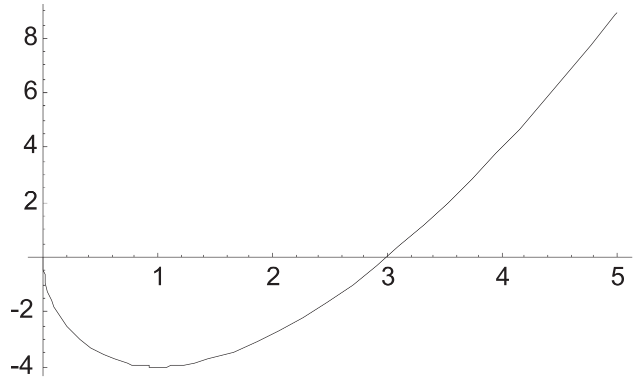
$y' = \dfrac{3(x-1)}{x^{1/2}} = 0$ for $x = 1$; y' is undefined for $x \le 0$. ($y = 0$ is an endpoint of the domain.)



The graph decreases for $0 \le x \le 1$ and increases for $x \ge 1$.
There is a local (endpoint) max at $(0,0)$ and a local (absolute) min at $(1,-4)$.
$y'' = \dfrac{3}{2} \cdot \dfrac{x+1}{x^{3/2}}$ is always positive, since the factors $\dfrac{3}{2}$, $x+1$, and $x^{3/2}$ are always positive. (Remember that the domain is $x \ge 0$!) Hence, the graph is always concave up, and there are no inflection points.
There are no vertical asymptotes. The domain ends at $x = 0$, but $f(0) = 0$.
Note that

$$\lim_{x\to+\infty}(2x^{3/2} - 6x^{1/2}) = +\infty.$$

Therefore, the graph rises on the far right. (It does not make sense to compute $\lim_{x\to-\infty}(2x^{3/2} - 6x^{1/2})$. Why?) Hence, there are no horizontal asymptotes.



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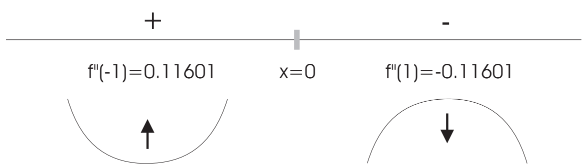
2. Graph $y = \dfrac{x}{\sqrt{x^2+7}}$.

The function is defined for all x.
The x-intercept is $x = 0$; the y-intercept is $y = 0$.
The derivatives are

$$y' = \frac{\sqrt{x^2+7} - \dfrac{x^2}{\sqrt{x^2+7}}}{x^2+7} = \frac{7}{(x^2+7)^{3/2}}, \quad y'' = \frac{-21x}{(x^2+7)^{5/2}}.$$

Since $(x^2+7)^{3/2} > 0$ for all x, y' is always positive. Hence, the graph is always increasing. There are no maxima or minima.

$y'' = \dfrac{-21x}{(x^2+7)^{5/2}} = 0$ for $x = 0$; y'' is defined for all x.



The graph is concave up for $x < 0$ and concave down for $x > 0$. $x = 0$ is a point of inflection.
There are no vertical asymptotes, since the function is defined for all x.
Observe that

$$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+7}} = 1.$$

Hence, the graph is asymptotic to $y = 1$ as $x \to +\infty$.
You may find it surprising that

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+7}} = -1.$$

Actually, it is no surprise if you think about it. Since $x \to -\infty$, x is taking on *negative* values. The numerator x is negative, but the denominator $\sqrt{x^2+7}$ is always positive, by definition. Hence, the limit must be negative, or at least not positive.

Algebraically, this is a result of the fact that

$$\sqrt{u^2} = |u|.$$

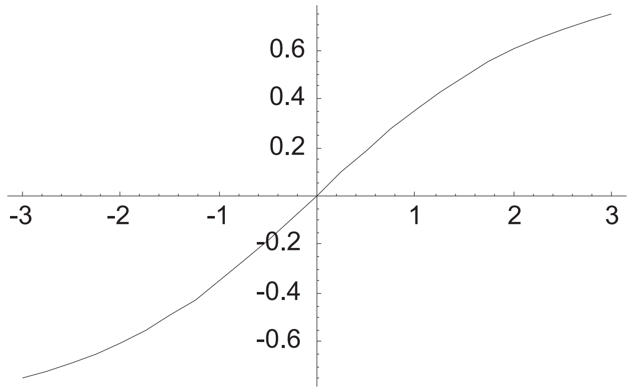
Moreover, $|u| = -u$ if u is negative. (The absolute value of a negative number is the negative of the number.)

Here is the computation in more detail:

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+7}} = \lim_{x\to-\infty} \frac{\dfrac{1}{x}\cdot x}{\dfrac{1}{x}\cdot\sqrt{x^2+7}} = \lim_{x\to-\infty} \frac{1}{-\sqrt{\dfrac{1}{x^2}\cdot(x^2+7)}} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\dfrac{7}{x^2}}} = -1.$$

Notice the negative sign that appeared when I pushed the $\dfrac{1}{x}$ into the square root. This is a result of the algebraic fact I mentioned above.

Anyway, the graph is asymptotic to $y = -1$ as $x \to -\infty$.



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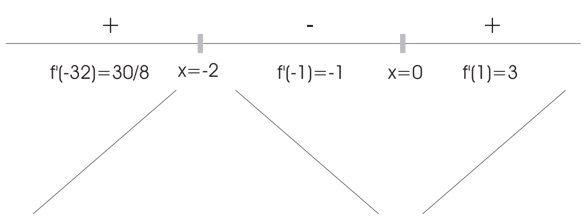
3. Graph $y = 5x^{2/5} + \frac{5}{7}x^{7/5}$.

The function is defined for all x.
$0 = y = 5x^{2/5} + \frac{5}{7}x^{7/5} = 5x^{2/5}(1 + \frac{1}{7}x)$ for $x = 0$ and $x = -7$. These are the x-intercepts.
Set $x = 0$; the y-intercept is $y = 0$.
The derivatives are

$$y' = 2x^{-3/5} + x^{2/5} = x^{-3/5}(2 + x) = \frac{x+2}{x^{3/5}}, \quad y'' = -\frac{6}{5}x^{-8/5} + \frac{2}{5}x^{-3/5} = \frac{1}{5}x^{-8/5}(-6 + 2x) = \frac{1}{5} \cdot \frac{2(x-3)}{x^{8/5}}.$$

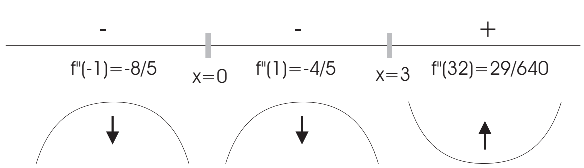
$y' = \frac{x+2}{x^{3/5}} = 0$ for $x = -2$. y' is undefined at $x = 0$.



The graph increases for $x \le -2$ and for $x \ge 0$. The graph decreases for $-2 \le x \le 0$.
There is a local max at $x = -2$; the approximate y-value is 4.71253. There is a local min at $(0, 0)$.
Note that

$$\lim_{x \to 0-} \frac{x+2}{x^{3/5}} = -\infty \quad \text{and} \quad \lim_{x \to 0+} \frac{x+2}{x^{3/5}} = +\infty.$$

There is a vertical tangent at the origin.
$y'' = \frac{1}{5} \cdot \frac{2(x-3)}{x^{8/5}} = 0$ at $x = 3$; y'' is undefined at $x = 0$.

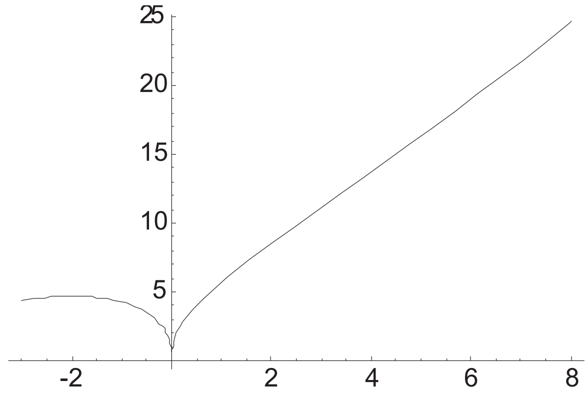


The graph is concave down for $x < 0$ and for $0 < x < 3$. The graph is concave up for $x > 3$. There is a point of inflection at $x = 3$.

There are no vertical asymptotes.

$$\lim_{x\to+\infty}(5x^{2/5}+\frac{5}{7}x^{7/5})=+\infty \quad \text{and} \quad \lim_{x\to-\infty}(5x^{2/5}+\frac{5}{7}x^{7/5})=-\infty.$$

The graph falls to $-\infty$ on the far left and rises to $+\infty$ on the far right.



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4. Graph $y=\dfrac{2x}{x^2-1}$.

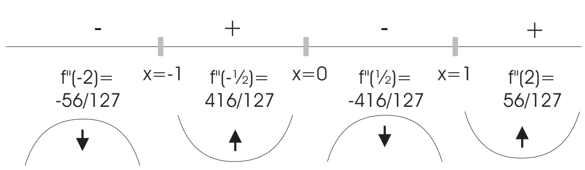
The domain is all x except $x=\pm 1$.
The x-intercept is $x=0$; the y-intercept is $y=0$.
The derivatives are

$$y'=\frac{(x^2-1)(2)-(2x)(2x)}{(x^2-1)^2}=\frac{-2(x^2+1)}{(x^2-1)^2},$$

$$y''=-2\cdot\frac{(x^2-1)^2(2x)-(x^2+1)(2)(x^2-1)(2x)}{(x^2-1)^4}=\frac{4x(x^2+3)}{(x^2-1)^3}.$$

$y'=\dfrac{-2(x^2+1)}{(x^2-1)^2}$ is always negative: -2 is negative, while x^2+1 and $(x^2-1)^2$ are always positive. The graph is decreasing everywhere; there are no local maxima or minima.

$y''=\dfrac{4x(x^2+3)}{(x^2-1)^3}=0$ for $x=0$. y'' is undefined at $x=\pm 1$.



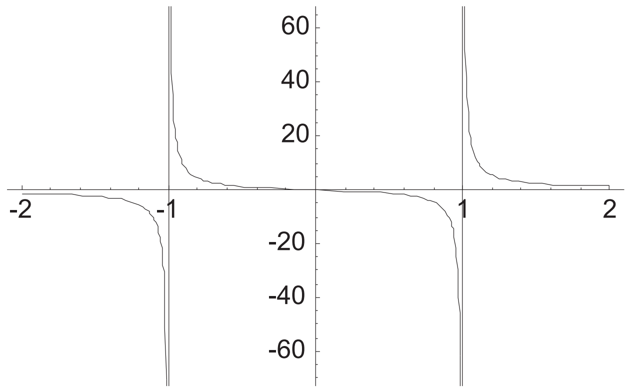
The graph is concave up for $-1<x<0$ and for $x>1$. The graph is concave down for $x<-1$ and for $0<x<1$. $x=0$ is the only inflection point.

There are vertical asymptotes at $x=\pm 1$. In fact,

$$\lim_{x\to 1+}\frac{2x}{x^2-1}=+\infty, \quad \lim_{x\to 1-}\frac{2x}{x^2-1}=-\infty,$$
$$\lim_{x\to -1+}\frac{2x}{x^2-1}=+\infty, \quad \lim_{x\to -1-}\frac{2x}{x^2-1}=-\infty.$$

The graph is asymptotic to $y=0$ as $x\to+\infty$ and as $x\to-\infty$:

$$\lim_{x\to+\infty}\frac{2x}{x^2-1}=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{2x}{x^2-1}=0.$$



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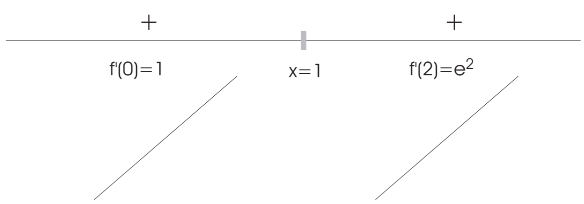
5. Graph $f(x) = (x^2 - 4x + 5)e^x$.

The domain is all real numbers.
Since $0 = (x^2 - 4x + 5)e^x$ has no solutions, there are no x-intercepts.
Setting $x = 0$ gives $y = 5$; the y-intercept is $y = 5$.
The derivatives are

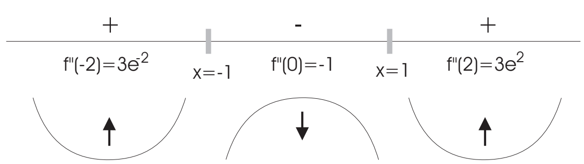
$$f'(x) = (x^2 - 4x + 5)e^x + (2x - 4)e^x = (x^2 - 2x + 1)e^x = (x - 1)^2 e^x,$$

$$f''(x) = (x^2 - 2x + 1)e^x + (2x - 2)e^x = (x^2 - 1)e^x = (x - 1)(x + 1)e^x.$$

$f'(x)$ is defined for all x. $f'(x) = 0$ for $x = 1$.



f increases for all x.
There are no local maxima or minima.
$f''(x)$ is defined for all x. $f''(x) = 0$ for $x = 1$ and $x = -1$.



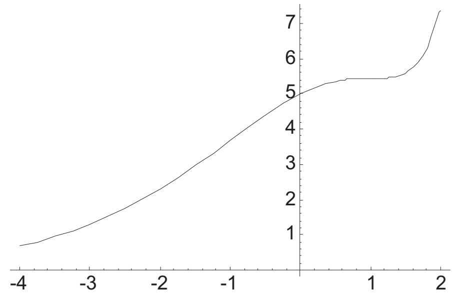
f is concave up for $x < -1$ and for $x > 1$. f is concave down for $-1 < x < 1$. $x = -1$ and $x = 1$ are inflection points.

There are no vertical asymptotes, since f is defined for all x.

$$\lim_{x \to +\infty} (x^2 - 4x + 5)e^x = +\infty \quad \text{and} \quad \lim_{x \to -\infty} (x^2 - 4x + 5)e^x = 0.$$

(You can verify the second limit empirically by plugging in a large negative number for x. For example, when $x = -100$, $(x^2 - 4x + 5)e^x \approx 3.87074 \times 10^{-40}$, which is pretty close to 0.)

$y = 0$ is a horizontal asymptote at $-\infty$.



6. Graph $f(x) = (x-2)(x-3) + 2\ln x$.

The domain is $x > 0$.
The x-intercept is $x \approx 0.0573709$. There are no y-intercepts.
Write

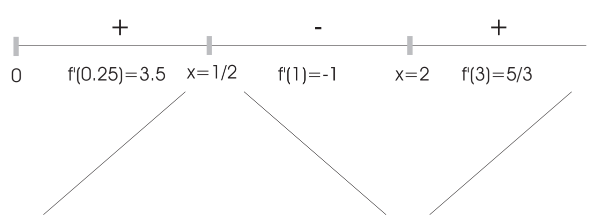
$$f(x) = x^2 - 5x + 6 + 2\ln x.$$

The derivatives are

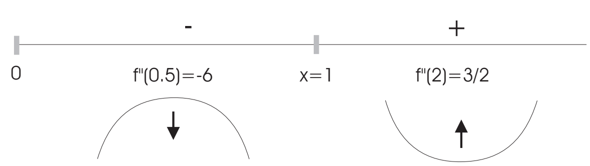
$$f'(x) = 2x - 5 + \frac{2}{x} = \frac{2x^2 - 5x + 2}{x} = \frac{(2x-1)(x-2)}{x},$$

$$f''(x) = 2 - \frac{2}{x^2} = \frac{2x^2 - 2}{x^2} = \frac{2(x-1)(x+1)}{x^2}.$$

$f'(x)$ is undefined for $x = 0$. $f'(x) = 0$ for $x = \frac{1}{2}$ and $x = 2$.



f increases for $0 < x \le \frac{1}{2}$ and for $x \ge 2$. f decreases for $\frac{1}{2} \le x \le 2$.
$x = \frac{1}{2}$ is a local max; $x = 2$ is a local min.
$f''(x)$ is undefined for $x = 0$. $f''(x) = 0$ for $x = 1$ and $x = -1$; however, $x = -1$ is not in the domain of f.



f is concave up for $x > 1$ and concave down for $0 < x < 1$. $x = 1$ is an inflection point.
Note that

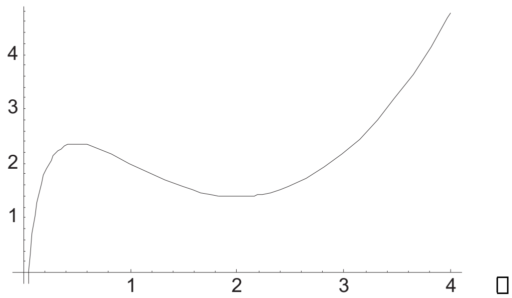
$$\lim_{x \to 0^+} ((x-2)(x-3) + 2\ln x) = -\infty.$$

Thus, there is a vertical asymptote at $x = 0$. (You can only approach 0 from the right, since f is only defined for $x > 0$.)

Also,

$$\lim_{x\to+\infty} ((x-2)(x-3) + 2\ln x) = +\infty.$$

Therefore, f does not have any horizontal asymptotes.



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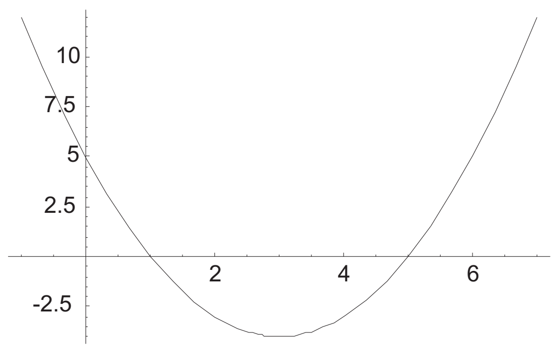
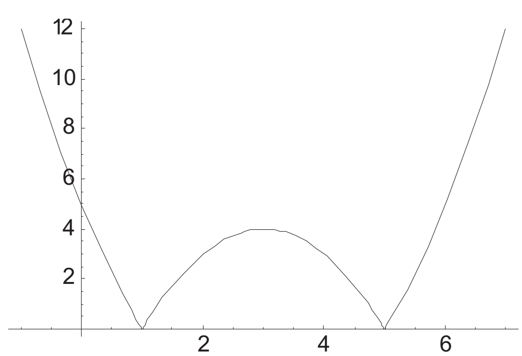
7. Sketch the graph of $y = |x^2 - 6x + 5|$ by first sketching the graph of $y = x^2 - 6x + 5$.

$y = x^2 - 6x + 5 = (x-1)(x-5)$ is a parabola with roots at $x = 1$ and at $x = 5$, opening upward:



The absolute value function leaves the positive parts alone and reflects the negative parts about the x-axis. Hence, the graph of $y = |x^2 - 6x + 5|$ is



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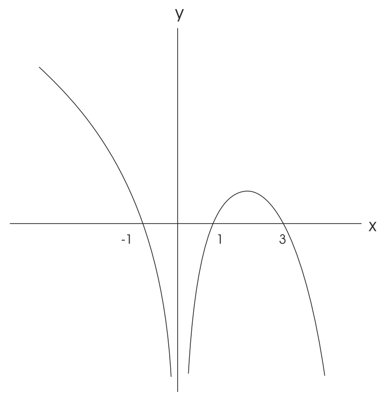
8. The function $y = f(x)$ is defined for all x. The graph of its derivative $y' = f'(x)$ is shown below:



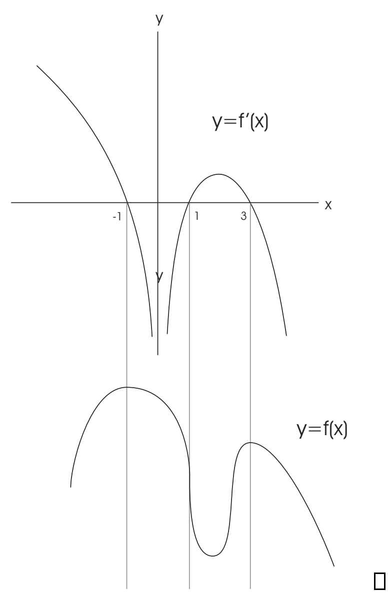
Sketch the graph of $y = f(x)$.

Since

$$\lim_{x\to 0+} f'(x) = \lim_{x\to 0-} f(x) = -\infty,$$

and since f is defined for all x (including $x = 0$), there is a vertical tangent at the origin.

Wherever $y' = 0$ — i.e. wherever y' crosses the x-axis, we have a critical point. In fact, there are local maxima at $x = -1$ and at $x = 3$, and there is a local minimum at $x = 1$.



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9. A function $y = f(x)$ is defined for all x. In addition:

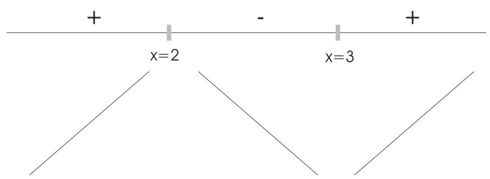
$$f(-1) = 0 \quad \text{and} \quad f'(3) \quad \text{is undefined,}$$

$$f'(x) \geq 0 \quad \text{for} \quad x \leq 2 \quad \text{and} \quad x > 3,$$
$$f'(x) \leq 0 \quad \text{for} \quad 2 \leq x < 3,$$
$$f''(x) < 0 \quad \text{for} \quad x < 3,$$

$$f''(x) > 0 \quad \text{for} \quad x > 3.$$

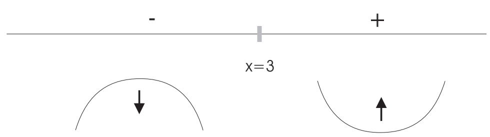
Sketch the graph of f.

Here's the sign chart for f':



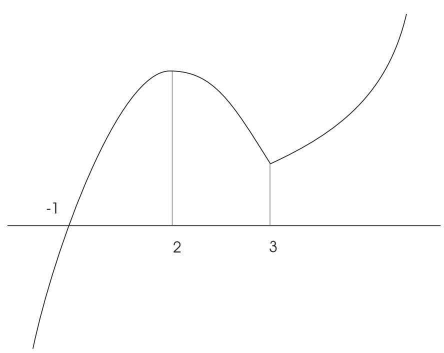
f increases for $x \le 2$ and for $x \ge 3$. f decreases for $2 \le x \le 3$. There's a local max at $x = 2$ and a local min at $x = 3$. Note that since f is defined for all x but $f'(3)$ is undefined, there is a corner in the graph at $x = 3$.

Here's the sign chart for f'':



f is concave up for $x > 3$ and concave down for $x < 3$. There is an inflection point at $x = 3$.

Here is a the graph of f:



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10. Find the critical points of $y = \frac{1}{3}x^3 + \frac{3}{2}x^2 - 4x + 5$ and classify them as local maxima or local minima using the Second Derivative Test.

$$y' = x^2 + 3x - 4 = (x + 4)(x - 1) \quad \text{and} \quad y'' = 2x + 3.$$

The critical points are $x = -4$ and $x = 1$.

x	$y'' = 2x + 3$	conclusion
1	5	local min
-4	-5	local max

□

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11. Suppose $y = f(x)$ is a differentiable function, $f(3) = 4$, and $f'(3) = -6$. Use differentials to approximate $f(3.01)$.

Use the formula

$$f(x + dx) \approx f(x) + f'(x)\,dx.$$

Here $x = 3$ and $x + dx = 3.01$, so

$$dx = (x + dx) - x = 3.01 - 3 = 0.01.$$

Therefore,

$$f(3.01) = f(x + dx) \approx f(x) + f'(x)\,dx = 4 + (-6)(0.01) = 4 - 0.06 = 3.94. \quad \square$$

---

12. The derivative of a function $y = f(x)$ is $y' = \dfrac{1}{x^4 + x^2 + 2}$. Approximate the change in the function as x goes from 1 to 0.99.

$y'(1) = \dfrac{1}{1^4 + 1^2 + 2} = 0.25$, and $dx = 0.99 - 1 = -0.01$. The change in the function is approximately

$$dy = f'(x)\,dx = (0.25)(-0.01) = -0.0025. \quad \square$$

---

13. Use a linear approximation to approximate $\sqrt{1.99^3 + 1}$ to five decimal places.

Let $f(x) = \sqrt{x^3 + 1}$, so $f(1.99) = \sqrt{1.99^3 + 1}$ and

$$f'(x) = \frac{3x^2}{2\sqrt{x^3 + 1}}.$$

Take $x = 2$ and $x + dx = 1.99$, so $dx = 1.99 - 2 = -0.01$. Then

$$f(1.99) \approx f(2) + f'(2)\,dx = \sqrt{9} + \left(\frac{12}{18}\right)(-0.01) \approx 2.99333. \quad \square$$

---

14. The area of a sphere of radius r is $A = 4\pi r^2$. Suppose that the radius of a sphere is measured to be 5 meters with an error of ± 0.2 meters. Use a linear approximation to approximation the error in the area and the percentage error.

$$dA = A'(r)\,dr, \quad \text{so} \quad dA = 8\pi r\,dr.$$

dA is the approximate error in the area, and dr is the approximate error in the radius. In this case, $r = 5$ and $dr = 0.2$. (I'm neglecting the sign, since I just care about the *size* of the error.) Then

$$dA = 8\pi \cdot 5 \cdot 0.2 = 8\pi \approx 25.13274.$$

The percentage area (or relative error) is approximately

$$\frac{dA}{A} = \frac{8\pi}{4\pi \cdot 5^2} = \frac{8\pi}{100\pi} = 0.08 = 8\%. \quad \square$$

---

15. x and y are related by the equation

$$\frac{x^3}{y} - 4y^2 = 6xy - 8y.$$

Find the rate at which x is changing when $x = 2$ and $y = 1$, if y decreases at 21 units per second.

Differentiate the equation with respect to t:

$$\frac{(y)\left(3x^2\frac{dx}{dt}\right) - (x^3)\left(\frac{dy}{dt}\right)}{y^2} - 8y\frac{dy}{dt} = 6x\frac{dy}{dt} + 6y\frac{dx}{dt} - 8\frac{dy}{dt}.$$

Plug in $x = 2$, $y = 1$, and $\frac{dy}{dt} = -21$:

$$12\frac{dx}{dt} + 168 + 168 = -252 + 6\frac{dx}{dt} + 168, \quad \frac{dx}{dt} = -70.$$

x decreases at 70 units per second. □

---

16. Let x and y be the two legs of a right triangle. Suppose the area is decreasing at 3 square units per second, and x is increasing at 5 units per second. Find the rate at which y is changing when $x = 6$ and $y = 20$.

The area of the triangle is

$$A = \frac{1}{2}xy.$$

Differentiate with respect to t:

$$\frac{dA}{dt} = \frac{1}{2}x\frac{dy}{dt} + \frac{1}{2}y\frac{dx}{dt}.$$

I have $\frac{dA}{dt} = -3$, $\frac{dx}{dt} = 5$, $x = 6$, and $y = 20$:

$$-3 = \frac{1}{2}\cdot(6)\left(\frac{dy}{dt}\right) + \frac{1}{2}\cdot(20)(5), \quad -3 = 3\frac{dy}{dt} + 50, \quad -53 = 3\frac{dy}{dt}, \quad \frac{dy}{dt} = -\frac{53}{3} \quad \text{units per second.}$$ □

---

17. A bagel (with lox and cream cheese) moves along the curve $y = x^2 + 1$ in such a way that its x-coordinate increases at 3 units per second. At what rate is its y-coordinate changing when it's at the point $(2, 5)$?

Differentiating $y = x^2 + 1$ with respect to t, I get

$$\frac{dy}{dt} = 2x\frac{dx}{dt}.$$

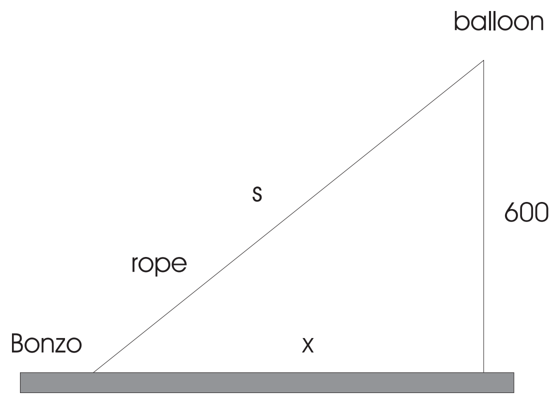
Plug in $x = 2$ and $\frac{dx}{dt} = 3$:

$$\frac{dy}{dt} = 2 \cdot 2 \cdot 3 = 12 \quad \text{units per second.}$$ □

---

18. Bonzo ties Calvin to a large helium balloon, which floats away at a constant altitude of 600 feet. Bonzo pays out the rope attached to the balloon at 3 feet per second. How rapidly is the balloon moving horizontally at the instant when 1000 feet of rope have been let out?

Let s be the length of the rope, and let x be the horizontal distance from the balloon to Bonzo.



By Pythagoras,

$$s^2 = x^2 + 600^2.$$

Differentiate with respect to t:

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt}, \quad s\frac{ds}{dt} = x\frac{dx}{dt}.$$

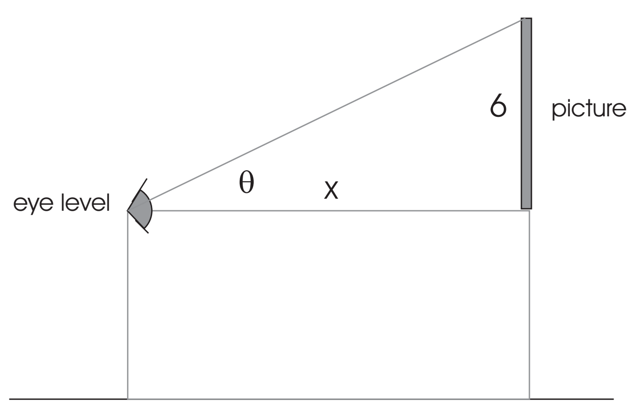
When $s = 1000$, $x = \sqrt{1000^2 - 600^2} = 800$. $\dfrac{ds}{dt} = 3$, so

$$3000 = 800\frac{dx}{dt}, \quad \frac{dx}{dt} = \frac{15}{4} \text{ feet/sec.} \quad \square$$

---

19. A poster 6 feet high is mounted on a wall, with the bottom edge 5 feet above the ground. Calvin walks toward the picture at a constant rate of 2 feet per week. His eyes are level with the bottom edge of the picture. Let θ be the vertical angle subtended by the picture at Calvin's eyes. At what rate is θ changing when Calvin is 8 feet from the picture?

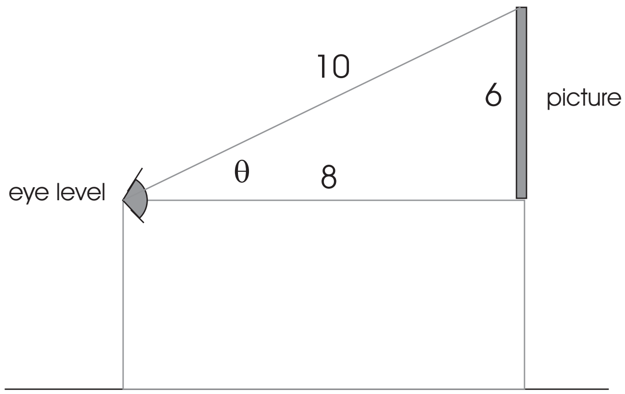
Let x be the distance from Calvin's eye to the base of the picture.



Now

$$\tan\theta = \frac{6}{x}, \quad \text{so} \quad (\sec\theta)^2 \frac{d\theta}{dt} = -\frac{6}{x^2} \cdot \frac{dx}{dt}.$$

Calvin walks toward the picture at 2 feet per week, so $\frac{dx}{dt} = -2$. (It's negative because the distance to the picture is *decreasing*.) When Calvin is 8 feet from the picture, $x = 8$. *At that instant*, the triangle looks like this:



(I got the 10 on the hypotenuse by Pythagoras.) Thus, $\sec\theta = \frac{10}{8} = \frac{5}{4}$, so

$$\left(\frac{5}{4}\right)^2 \frac{d\theta}{dt} = -\frac{6}{64} \cdot (-2), \quad \frac{d\theta}{dt} = \frac{3}{25} \text{ feet/week.} \quad \square$$

---

20. Use the Mean Value Theorem to show that if $0 < x < 1$, then

$$x + 1 < e^x < ex + 1.$$

Apply the Mean Value Theorem to $f(x) = e^x$ on the interval $[0, x]$, where $0 < x < 1$. The theorem says that there is a number c between 0 and x such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c).$$

Now $f(0) = e^0 = 1$, and $f'(c) = e^c$, so

$$\frac{e^x - 1}{x} = e^c.$$

Since $0 < c < x < 1$, and since e^x increases,

$$1 = e^0 < e^c < e^1 = e.$$

Therefore,

$$1 < \frac{e^x - 1}{x} < e, \quad \text{or} \quad x + 1 < e^x < ex + 1. \quad \square$$

---

21. (a) Prove that if $x > 1$, then $1 - \frac{1}{x} > 0$.

$$x > 1, \quad \frac{1}{x} < \frac{1}{1}, \quad \frac{1}{x} < 1, \quad 0 < 1 - \frac{1}{x}. \quad \square$$

(b) Use the Mean Value Theorem to prove that if $x > 1$, then $x - 1 > \ln x$.

Apply the Mean Value Theorem to $f(x) = x - \ln x$ with $a = 1$ and $b = x$. The theorem says that there is a point c such that $1 < c < x$ and

$$\frac{f(x) - f(1)}{x - 1} = f'(c).$$

Now $f(1) = 1$ and $f'(c) = 1 - \dfrac{1}{c}$, so

$$\frac{x - \ln x - 1}{x - 1} = 1 - \frac{1}{c}.$$

Since $1 - \dfrac{1}{c} > 0$ by part (a), I have

$$\begin{aligned} \frac{x - \ln x - 1}{x - 1} &> 0 \\ x - \ln x - 1 &> 0 \\ x - 1 &> \ln x \quad \square \end{aligned}$$

---

22. Prove that the equation $x^5 + 7x^3 + 13x - 5 = 0$ has exactly one root.

Let $f(x) = x^5 + 7x^3 + 13x - 5$. I have to show that f has exactly one root.

First, I'll show that f has *at least* one root. Then I'll show that it can't have *more than* one root.

Note that

$$f(0) = -5 \quad \text{and} \quad f(1) = 16.$$

By the Intermediate Value Theorem, f must have *at least* one root between 0 and 1.

Now suppose that f has *more than* one root. Then it has at least two roots, so let a and b be roots of f. Thus, $f(a) = 0$ and $f(b) = 0$, and by Rolle's theorem, f must have a horizontal tangent between a and b.

However, the derivative is

$$f'(x) = 5x^4 + 21x^2 + 13.$$

Since all the powers are even and the coefficients are positive,

$$f'(x) = 5x^4 + 21x^2 + 13 > 0 \quad \text{for all} \quad x.$$

In particular, $f'(x)$ is never 0, so f has no horizontal tangents.

Since I've reached a contradiction, my assumption that f has *more than* one root must be wrong. Therefore, f can't have more than one root.

Since I already know f has *at least* one root, it must have *exactly* one root. $\square$

---

23. Suppose that f is a differentiable function, $f(4) = 7$ and $|f'(x)| \le 10$ for all x. Prove that $-13 \le f(6) \le 27$.

Applying the Mean Value Theorem to f on the interval $4 \le x \le 6$, I find that there is a number c such that $4 < c < 6$ and

$$\frac{f(6) - f(4)}{6 - 4} = f'(c), \quad \text{or} \quad \frac{f(6) - 7}{2} = f'(c).$$

Thus,

$$\left| \frac{f(6) - 7}{2} \right| = |f'(c)| \le 10, \quad \text{so} \quad |f(6) - 7| \le 20.$$

The inequality says that the distance from $f(6)$ to 7 is less than or equal to 20. Since $7-20=-13$ and $7+20=27$, it follows that

$$-13 \le f(6) \le 27. \quad \square$$

---

24. A differentiable function satisfies $f(3)=0.2$ and $f'(3)=10$. If Newton's method is applied to f starting at $x=3$, what is the next value of x?

$$3-\frac{f(3)}{f'(3)}=3-\frac{0.2}{10}=3-0.02=2.98. \quad \square$$

---

25. Use Newton's method to approximate a solution to $x^2+2x=\dfrac{5}{x}$. Do 5 iterations starting at $x=1$, and do your computations to at least 5-place accuracy.

Rewrite the equation:

$$x^2+2x=\frac{5}{x}, \quad x^3+2x^2=5, \quad x^3+2x^2-5=0.$$

Let $f(x)=x^3+2x^2-5$. The Newton function is

$$[N(f)](x)=x-\frac{x^3+2x^2-5}{3x^2+4x}.$$

Iterating this function starting at $x=1$ produces the iterates

$$1, 1.28571, 1.24300, 1.24190, 1.24190.$$

The root is $x\approx 1.24190$. $\square$

---

26. Find the absolute max and absolute min of $y=x^3-12x+5$ on the interval $-1\le x\le 4$.

The derivative is

$$y'=3x^2-12=3(x-2)(x+2).$$

y' is defined for all x; $y'=0$ for $x=2$ or $x=-2$. I only consider $x=2$, since $x=-2$ is not in the interval $-1\le x\le 4$. Plug the critical point and the endpoints into the function:

x	-1	2	4
$f(x)$	16	-11	21

The absolute max is $y=21$ at $x=4$; the absolute min is $y=-11$ at $x=2$. $\square$

---

27. Find the absolute max and absolute min of $y=3x^{2/3}\left(\dfrac{1}{8}x^2-\dfrac{1}{5}x-1\right)$ on the interval $-2\le x\le 8$.

First, multiply out:

$$y=\frac{3}{8}x^{8/3}-\frac{3}{5}x^{5/3}-3x^{2/3}.$$

(This makes it easier to differentiate.) The derivative is

$$y' = x^{5/3} - x^{2/3} - 2x^{-1/3}.$$

Simplify by writing the negative power as a fraction, combining over a common denominator, then factoring:

$$y' = x^{5/3} - x^{2/3} - 2x^{-1/3} = x^{5/3} - x^{2/3} - \frac{2}{x^{1/3}} = x^{5/3} \cdot \frac{x^{1/3}}{x^{1/3}} - x^{2/3} \cdot \frac{x^{1/3}}{x^{1/3}} - \frac{2}{x^{1/3}} = \frac{x^2 - x - 2}{x^{1/3}} = \frac{(x-2)(x+1)}{x^{1/3}}.$$

$y' = 0$ for $x = 2$ and for $x = -1$. y' is undefined for $x = 0$. All of these points are in the interval $-2 \le x \le 8$, so all need to be tested.

x	-2	-1	0	2	8
y	-0.47622	-2.025	0	-4.28598	64.8

The absolute max is at $x = 8$; the absolute min is at $x = 2$. □

The flower in the vase still smiles, but no longer laughs. - Malcolm de Chazal