Math 161
10-4-2020

Review Problems for Test 1

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. Compute $\lim_{x\to 9} \frac{x^2 - 81}{x - 9}$.

2. Compute $\lim_{x\to 2} \frac{x^2 - 4}{x^2 - x - 2}$.

3. Compute $\lim_{x\to 3} \frac{x^2 + 2x - 3}{x^2 - 9}$.

4. Compute $\lim_{x\to 3-} \frac{x^2 + 2x - 3}{x^2 - 9}$.

5. Compute $\lim_{x\to +\infty} \frac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^7}$.

6. Compute $\lim_{x\to +\infty} \frac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^6}$.

7. Compute $\lim_{x\to +\infty} (2x^3 + 5x + 1)$.

8. Compute $\lim_{x\to -\infty} (4 - 2x^2 - x^3)$.

9. Compute $\lim_{x\to +\infty} \frac{4x^{-11} + 7x^{-8}}{13x^{-8} + 5x^{-9}}$.

10. Compute $\lim_{x\to 1} \frac{x - 1}{\sqrt{x} - 1}$.

11. Compute $\lim_{x\to 2} \frac{\frac{1}{3} - \frac{1}{x+1}}{x^2 + 9x - 22}$.

12. Compute $\lim_{x\to 2} \sqrt[5]{\frac{2x + 5}{x - 7}}$.

13. Compute $\lim_{x\to 4} \frac{\frac{1}{7} - \frac{1}{x+3}}{\frac{1}{5} - \frac{1}{x+1}}$.

14. Compute $\lim_{x\to 3} \frac{\sqrt{x + 13} - 4}{x^2 + 2x - 15}$.

15. Compute $\lim_{x\to 0} \frac{7x \cos 5x}{8x + 3 \sin 3x}$.

16. Compute $\lim_{x\to 0} \frac{3x + 4 \sin 3x}{\tan 5x - x \cos 2x}$.

17. Suppose

$$f(x) = \begin{cases} 6 & \text{if } x \le 3 \\ \dfrac{x^2 - 9}{x - 3} & \text{if } x > 3 \end{cases}$$

Determine whether $\lim_{x \to 3} f(x)$ is defined. If it is, compute it.

18. Compute $\lim_{x \to 0} x^4 \cos \dfrac{1}{x^2}$.

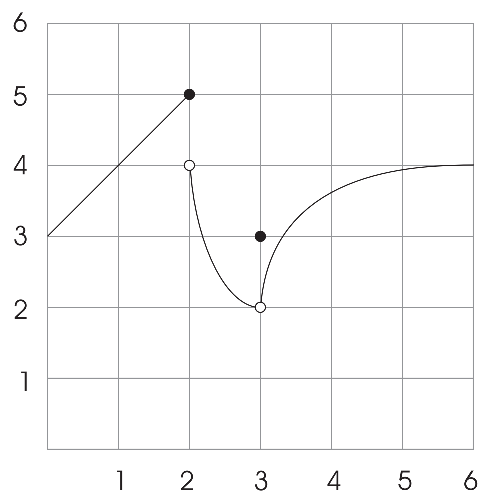
19. Suppose that

$$\lim_{x \to 2} g(x) = 3, \quad \lim_{x \to 2} h(x) = 9,$$

$$(x^2 + 8)g(x) \le f(x) \le (x + 2)h(x) \quad \text{for all} \quad x.$$

Compute $\lim_{x \to 2} f(x)$.

20. The picture below shows that graph of a function $y = f(x)$.



Compute:

(a) $\lim_{x \to 2-} f(x)$.

(b) $\lim_{x \to 2+} f(x)$.

(c) $\lim_{x \to 2} f(x)$.

(d) $\lim_{x \to 3-} f(x)$.

(e) $\lim_{x \to 3+} f(x)$.

(f) $\lim_{x \to 3} f(x)$.

(g) $f(3)$.

21. Locate the horizontal asymptotes and the vertical asymptotes of

$$y = \frac{x^2}{x^3 - 5x^2 - 6x}.$$

Justify your answer by computing the relevant limits.

22. Differentiate $f(x) = 8x^5 + 4x^2 - 9x + 1$.

23. Differentiate $f(x) = \dfrac{4}{x^3} - \dfrac{2}{\sqrt{x}}$.

24. Differentiate $y = 3\sqrt[7]{x^2} - 6\sqrt[5]{42}$.

25. Differentiate $g(u) = (u^5 - u^4 + u^3 + u^2 + u + 1)^{42}$.

26. Differentiate $h(x) = \left(\sqrt{x} + x - x^2\right)(x^6 + x^3 + x + 1)$.

27. Differentiate $y = \dfrac{x^6 + x^3 + 1}{x^4 + x^2 + 1}$.

28. Differentiate $f(x) = \left(3 + \left(3 + x^{-2}\right)^{-2}\right)^{-2}$.

29. Differentiate $f(x) = (\sin 3x + \tan 5x)^{50}$.

30. Differentiate $g(t) = \sin(\cos(\tan t)) + \sec\left(\dfrac{t}{\tan t}\right)$.

31. Differentiate $y = \sin(\cos x) + (\sin x)(\cos x)$.

32. Find $\dfrac{d}{dx}\left(\dfrac{x^7}{13} + 65x + \dfrac{4}{x^2} + 3\right)$.

33. Find $\dfrac{d}{dx}\left(\sqrt{3x} + \dfrac{7}{\sqrt[3]{x}} + \sqrt[7]{x^2}\right)$.

34. Compute $\dfrac{d}{dx}\left[\cos(3x+1)^2 + (\cos(3x+1))^2\right]$.

35. Compute $\dfrac{d}{dx}\dfrac{6x^3 + 3\sin x}{17}$.

36. Find $\dfrac{d}{dx}(3\sin x - 7x)(4 + x^2 - \tan x)$.

37. Find $f'(t)$ if $f(t) = \dfrac{3}{(t^2 + 3t + 1)^5}$.

38. Compute $\dfrac{d}{dx}(e^{3x} + 1)(\sin 2x)(x^4 + 2x + 1)$.

39. Find $\dfrac{d}{dx}\left(\dfrac{\sqrt{x} + \sec x}{3 + 5x - \cos x}\right)$.

40. Find y' if $y = \dfrac{(2 - 3x - x^5)e^{4x}}{\tan 5x + 7}$.

41. Compute $\dfrac{d}{dx}\dfrac{1 + \csc(x^2)}{(e^{-7x} + 3x)(7x^6 + x + 1)}$.

42. Compute $\dfrac{d}{dx}\left(\dfrac{x^2 + x + 1}{x^3 - 3x + 5}\right)^{11}$.

43. Compute $\dfrac{d}{dx}\left(e^{x^2} + 7^{x^2} + e^{7x} + 7^{e^x}\right)$.

44. Compute $\frac{d}{dx}\sqrt[3]{(x^2+e^x)(\ln x+42x)+x^3}$.

45. Compute $\frac{d}{dw}\ln\sqrt{e^{w^2}+e^{2w}+1}$.

46. Compute $\frac{d}{dx}\left(x^3+\left(x^3+(x^3+1)^3\right)^3\right)^3$.

47. Compute $\frac{d}{dx}\left(\frac{2x+5}{(x+2)^3}\right)$.

48. Compute the derivative with respect to t of $s(t)=\sec\left(\frac{5}{t^3+5t+8}\right)$.

49. Compute $\frac{d}{dx}[\log_5(x^2+x+1)]^7$.

50. Compute y''' for $y=(x^2+1)^4$.

51. Compute $y^{(50)}$ if $y=\frac{1}{4-x}$.

52. Compute $\frac{d}{dx}\sqrt{e^{f(x)^2}+1}$.

53. Compute $\frac{d}{dx}\ln(\sin(g(x))+1)$.

54. f and g are differentiable functions. The values of f, g, f', and g' at $x=2$ and $x=5$ are shown below:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	-6	5	4	10
5	0.5	-1	7	-3

Compute $(fg)'(5)$, $\left(\frac{f}{g}\right)'(2)$, and $[f(g)]'(2)$.

55. Find the equation of the tangent line to the graph of $f(x)=x^3+5x-10$ at $x=2$.

56. Find the equation of the tangent line to the graph of $y=\frac{1}{(x^2+4x+5)^2}$ at $x=1$.

57. For what value or values of c does the tangent line to the graph of $y=x^2+5x+1$ at $x=c$ pass through the point $(1,6)$?

58. The position of a cheeseburger at time t is given by

$$s(t)=t^3-12t^2+36t+2.$$

Find the value(s) of t for which the velocity is 0. Find the value(s) of t for which the acceleration is 0.

59. For what values of x is the function $f(x)=\frac{x-5}{x^2-3x-10}$ continuous?

60. For what values of x is the function $f(x)=\frac{1}{\sqrt{x^2+5x+6}}$ continuous?

61. Let

$$f(x)=\begin{cases}\dfrac{x^2-7x-8}{x+1} & \text{if } x\neq -1\\ 9 & \text{if } x=-1\end{cases}$$

Prove or disprove: f is continuous at $x = -1$.

62. Find the value of c which makes the following function continuous at $x = 2$:

$$f(x) = \begin{cases} 1 + 3x - cx^2 & \text{if } x < 2 \\ 7x + 3 & \text{if } x \geq 2 \end{cases}.$$

63. Prove that $f(x) = x^5 + 4x + 1$ has a root between $x = -1$ and $x = 0$.

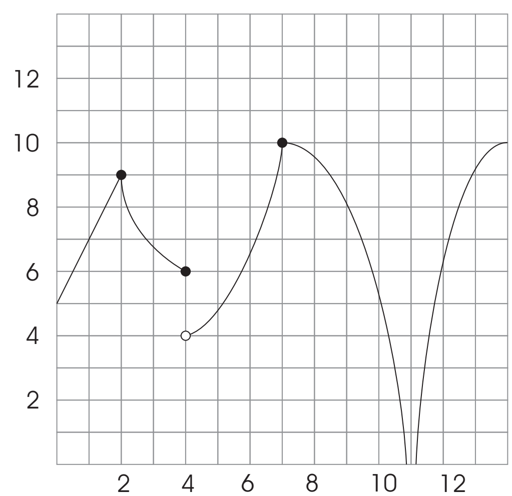
64. Suppose f is continuous, $f(2) = 13$, and $f(5) = 1$. Prove that there is a number x between 2 and 5 such that $x^3 \cdot f(x) = 110$.

65. Suppose $f(x) = \sqrt{x - 6}$. Use the limit definition of the derivative to compute $f'(15)$.

66. Suppose $f(x) = \dfrac{1}{x + 3}$. Use the limit definition of the derivative to compute $f'(x)$.

67. Suppose $f(x) = \dfrac{1}{\sqrt{x} + 1}$. Use the limit definition of the derivative to compute $f'(x)$.

68. The graph of a function $f(x)$ is pictured below. For what values of x is f continuous but not differentiable? For what values of f is f not continuous?



69. (a) Find the average rate of change of $f(x) = x^2 + 3x - 4$ on the interval $1 \leq x \leq 3$.

(b) Find the instantaneous rate of change of $f(x) = x^2 + 3x - 4$ at $x = 3$.

70. Give an example of a function $f(x)$ which is defined for all x and a number c such that

$$\lim_{x \to c} f(x) \neq f(c).$$

71. Suppose $f(x) = \dfrac{x + 1}{x + 2}$. Find $f^{-1}(x)$.

72. Suppose $f(2) = 5$ and $f'(2) = 7$. Find $(f^{-1})'(5)$.

73. Suppose $f(2) = 5$ and $f'(x) = \sqrt{x^4 + x^2 + 16}$. Find $(f^{-1})'(5)$.

74. Suppose $f(x) = (x^5 + 2x^3 + 5)^{1/3}$. Find $(f^{-1})'(2)$.

75. Compute $\dfrac{d}{dx}(x^2+3x+1)^{(x^2+6)}$.

76. Compute $\dfrac{d}{dx}(e^x+1)^{\sin x}$.

77. Find the equation of the tangent line to $y=(x^2+2)^x$ at $x=1$.

78. **(An alternate approach to logarithmic differentiation)**

(a) Use the identity $A=e^{\ln A}$ to write $(x^2+1)^{\tan x}$ using $e^{(\cdot)}$ and $\ln(\cdot)$.

(b) Use the Chain Rule to differentiate the expression you obtained in (a). (This gives another way of doing logarithmic differentiation involving powers.)

79. Find y' if y is defined implicitly by

$$\sin x+\cos y=3x+5y.$$

80. Find the equation of the tangent line to the curve

$$x^3+2xy^3=x^2-y^2+3 \quad \text{at the point} \quad (1,1).$$

81. Find the equation of the tangent line to the curve

$$\frac{x^2}{y}+y^2=7x-2y-9 \quad \text{at the point} \quad (3,1).$$

82. Find y'' at the point $(1,2)$ on the curve

$$x^3y+2x^2=y^3-2y.$$

Solutions to the Review Problems for Test 1

1. Compute $\displaystyle\lim_{x\to 9}\frac{x^2-81}{x-9}$.

Plugging in $x=9$ gives $\dfrac{0}{0}$. I factor the top, then cancel common factors of $x-9$:

$$\lim_{x\to 9}\frac{x^2-81}{x-9}=\lim_{x\to 9}\frac{(x-9)(x+9)}{x-9}=\lim_{x\to 9}(x+9)=18. \quad \square$$

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2. Compute $\displaystyle\lim_{x\to 2}\frac{x^2-4}{x^2-x-2}$.

Since plugging in $x=2$ gives $\dfrac{0}{0}$, it is reasonable to suppose that the zeros are being caused by a *common factor* on the top and the bottom. So factor the top and the bottom and cancel the $x-2$'s:

$$\lim_{x\to 2}\frac{x^2-4}{x^2-x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x+1)}=\lim_{x\to 2}\frac{x+2}{x+1}=\frac{4}{3}. \quad \square$$

3. Compute $\lim_{x \to 3} \dfrac{x^2 + 2x - 3}{x^2 - 9}$.

$$\lim_{x \to 3} \frac{x^2 + 2x - 3}{x^2 - 9} \quad \text{is undefined.}$$

If I plug in $x = 3$, I get $\dfrac{12}{0}$, a *nonzero* number over 0. In this case, the limit does not exist. (By the way, I hope you didn't try to use L'Hôpital's Rule here. It doesn't apply.) □

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4. Compute $\lim_{x \to 3-} \dfrac{x^2 + 2x - 3}{x^2 - 9}$.

$$\lim_{x \to 3-} \frac{x^2 + 2x - 3}{x^2 - 9} = -\infty.$$

This problem is different from the previous one because x is approaching 3 *from the left*. It would not be incorrect to say that the limit is undefined, but it is better to say that the limit is $-\infty$. There is a **vertical asymptote** at $x = 3$, and the graph goes *downward* (to $-\infty$) as it approaches the asymptote from the left.

How do you see algebraically that this is what it does? One way is to plug numbers close to 3, but less than 3, into $\dfrac{x^2 + 2x - 3}{x^2 - 9}$. For example, if $x = 2.99999$, $\dfrac{x^2 + 2x - 3}{x^2 - 9} \approx -199999$ — a big negative number. This doesn't *prove* that it's going to $-\infty$, but it's pretty good evidence.

Here is how I can analyze it. I know that plugging in $x = 3$ gives $\dfrac{12}{0}$. Therefore, I should be getting either $+\infty$ or $-\infty$ — the reciprocal of something small (≈ 0) should be something big.

Now let x approach 3 from the left. $x^2 + 2x - 3$ surely goes to 12, a positive number. $x^2 - 9$ goes to 0, but since $x < 3$, $x^2 - 9$ will be negative. Now $\dfrac{\text{positive}}{\text{negative}} = \text{negative}$, so the answer is $-\infty$. □

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5. Compute $\lim_{x \to +\infty} \dfrac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^7}$.

The idea is to divide the top and bottom by the highest power of x that occurs in either — in this case, x^7. Then use the fact that

$$\lim_{x \to \pm\infty} \frac{\text{number}}{\text{x}^{\text{positivepower}}} = 0.$$

I have

$$\lim_{x \to +\infty} \frac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^7} = \lim_{x \to +\infty} \frac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^7} \cdot \frac{\dfrac{1}{x^7}}{\dfrac{1}{x^7}} = \lim_{x \to +\infty} \frac{\dfrac{3x^7}{x^7} - \dfrac{5x^4}{x^7} + \dfrac{x}{x^7} - \dfrac{2}{x^7}}{\dfrac{10}{x^7} - \dfrac{2x^4}{x^7} - \dfrac{3x^7}{x^7}} =$$

$$\lim_{x \to +\infty} \frac{3 - \dfrac{5}{x^3} + \dfrac{1}{x^6} - \dfrac{2}{x^7}}{\dfrac{10}{x^7} - \dfrac{2}{x^3} - 3} = \frac{3 - 0 + 0 - 0}{0 - 0 - 3} = -1. \quad □$$

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6. Compute $\lim_{x \to +\infty} \dfrac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^6}$.

$$\lim_{x\to+\infty} \frac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^6} = \lim_{x\to+\infty} \frac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^6} \cdot \frac{\frac{1}{x^7}}{\frac{1}{x^7}} = \lim_{x\to+\infty} \frac{\frac{3x^7}{x^7} - \frac{5x^4}{x^7} + \frac{x}{x^7} - \frac{2}{x^7}}{\frac{10}{x^7} - \frac{2x^4}{x^7} - \frac{3x^6}{x^7}} =$$

$$\lim_{x\to+\infty} \frac{3 - \frac{5}{x^3} + \frac{1}{x^6} - \frac{2}{x^7}}{\frac{10}{x^7} - \frac{2}{x^3} - \frac{3}{x}} = \frac{3 - 0 + 0 - 0}{0 - 0 - 0} = -\infty.$$

There's little question that the limit is undefined; why is it $-\infty$? One approach is to note that the top and bottom are dominated by the highest powers of x. So this limits looks like

$$\lim_{x\to+\infty} \frac{3x^7}{-3x^6} = \lim_{x\to+\infty} -x = -\infty.$$

As a check, if you plug $x = 10^6$ into $\dfrac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^6}$, you get (approximately) -10^6. □

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7. Compute $\lim_{x\to+\infty} (2x^3 + 5x + 1)$.

As $x \to +\infty$, I have $2x^3 \to +\infty$ and $5x \to +\infty$. So

$$\lim_{x\to+\infty} (2x^3 + 5x + 1) = +\infty. \quad □$$

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8. Compute $\lim_{x\to-\infty} (4 - 2x^2 - x^3)$.

As $x \to +\infty$, I have $2x^3 \to +\infty$ and $5x \to +\infty$. So

$$\lim_{x\to+\infty} (2x^3 + 5x + 1) = +\infty. \quad □$$

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9. Compute $\lim_{x\to+\infty} \dfrac{4x^{-11} + 7x^{-8}}{13x^{-8} + 5x^{-9}}$.

$$\lim_{x\to+\infty} \frac{4x^{-11} + 7x^{-8}}{13x^{-8} + 5x^{-9}} = \lim_{x\to+\infty} \frac{\frac{4}{x^{11}} + \frac{7}{x^8}}{\frac{13}{x^8} + \frac{5}{x^9}} = \lim_{x\to+\infty} \frac{\frac{4}{x^{11}} + \frac{7}{x^8}}{\frac{13}{x^8} + \frac{5}{x^9}} \cdot \frac{x^8}{x^8} =$$

$$\lim_{x\to+\infty} \frac{\frac{4x^8}{x^{11}} + \frac{7x^8}{x^8}}{\frac{13x^8}{x^8} + \frac{5x^8}{x^9}} = \lim_{x\to+\infty} \frac{\frac{4}{x^3} + 7}{13 + \frac{5}{x}} = \frac{0 + 7}{13 + 0} = \frac{7}{13}. \quad □$$

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10. Compute $\lim_{x\to 1} \dfrac{x - 1}{\sqrt{x} - 1}$.

Method 1. Multiply the top and bottom by the conjugate:

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}=\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}=\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}=$$

$$\lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-1}=\lim_{x\to 1}(\sqrt{x}+1)=1+1=2.$$

Method 2. Factor the top:

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}=\lim_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1}=\lim_{x\to 1}(\sqrt{x}+1)=1+1=2. \quad \square$$

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11. Compute $\lim_{x\to 2}\frac{\frac{1}{3}-\frac{1}{x+1}}{x^2+9x-22}$.

$$\lim_{x\to 2}\frac{\frac{1}{3}-\frac{1}{x+1}}{x^2+9x-22}=\lim_{x\to 2}\frac{\frac{1}{3}-\frac{1}{x+1}}{(x+11)(x-2)}=\lim_{x\to 2}\frac{\frac{1}{3}-\frac{1}{x+1}}{(x+11)(x-2)}\cdot\frac{3(x+1)}{3(x+1)}=$$

$$\lim_{x\to 2}\frac{\frac{3(x+1)}{3}-\frac{3(x+1)}{x+1}}{3(x+1)(x+11)(x-2)}=\lim_{x\to 2}\frac{x+1-3}{3(x+1)(x+11)(x-2)}=\lim_{x\to 2}\frac{x-2}{3(x+1)(x+11)(x-2)}=\lim_{x\to 2}\frac{1}{3(x+1)(x+11)}=\frac{1}{117}.$$

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12. Compute $\lim_{x\to 2}\sqrt[5]{\frac{2x+5}{x-7}}$.

$$\lim_{x\to 2}\sqrt[5]{\frac{2x+5}{x-7}}=\sqrt[5]{-\frac{9}{5}}.$$

Nothing complicated is going on here — I just plug in $x=2$. (By the way — Do you know how to compute $\sqrt[5]{-\frac{9}{5}}$ on your calculator? It's approximately -1.12475. You should always give exact answers unless told otherwise, however.) $\square$

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13. Compute $\lim_{x\to 4}\frac{\frac{1}{7}-\frac{1}{x+3}}{\frac{1}{5}-\frac{1}{x+1}}$.

$$\lim_{x\to 4}\frac{\frac{1}{7}-\frac{1}{x+3}}{\frac{1}{5}-\frac{1}{x+1}}=\lim_{x\to 4}\frac{\frac{1}{7}-\frac{1}{x+3}}{\frac{1}{5}-\frac{1}{x+1}}\cdot\frac{(5)(7)(x+3)(x+1)}{(5)(7)(x+3)(x+1)}=\lim_{x\to 4}\frac{5(x+1)\left(\frac{7(x+3)}{7}-\frac{7(x+3)}{x+3}\right)}{7(x+3)\left(\frac{5(x+1)}{5}-\frac{5(x+1)}{x+1}\right)}=$$

$$\lim_{x\to 4}\frac{5(x+1)[(x+3)-7]}{7(x+3)[(x+1)-5]}=\lim_{x\to 4}\frac{5(x+1)(x-4)}{7(x+3)(x-4)}=\lim_{x\to 4}\frac{5(x+1)}{7(x+3)}=\frac{25}{49}. \quad \square$$

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14. Compute $\lim_{x\to 3} \dfrac{\sqrt{x+13}-4}{x^2+2x-15}$.

$$\lim_{x\to 3} \frac{\sqrt{x+13}-4}{x^2+2x-15} = \lim_{x\to 3} \frac{\sqrt{x+13}-4}{(x+5)(x-3)} = \lim_{x\to 3} \frac{\sqrt{x+13}-4}{(x+5)(x-3)} \cdot \frac{\sqrt{x+13}+4}{\sqrt{x+13}+4} =$$

$$\lim_{x\to 3} \frac{(\sqrt{x+13})^2-4^2}{(x+5)(x-3)(\sqrt{x+13}+4)} = \lim_{x\to 3} \frac{x+13-16}{(x+5)(x-3)(\sqrt{x+13}+4)} = \lim_{x\to 3} \frac{x-3}{(x+5)(x-3)(\sqrt{x+13}+4)} =$$

$$\lim_{x\to 3} \frac{1}{(x+5)(\sqrt{x+13}+4)} = \frac{1}{64}. \quad \square$$

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15. Compute $\lim_{x\to 0} \dfrac{7x\cos 5x}{8x+3\sin 3x}$.

Remember that

$$\lim_{u\to 0} \frac{\sin u}{u} = 1.$$

In this problem, I'll let $u = 3x$. Now $x \to 0$ if and only if $u = 3x \to 0$, so

$$\lim_{x\to 0} \frac{\sin 3x}{3x} = 1.$$

Here's the solution:

$$\lim_{x\to 0} \frac{7x\cos 5x}{8x+3\sin 3x} = \lim_{x\to 0} \frac{7x\cos 5x}{8x+3\sin 3x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to 0} \frac{7\cos 5x}{8+3\frac{\sin 3x}{x}} = \lim_{x\to 0} \frac{7\cos 5x}{8+9\left(\frac{\sin 3x}{3x}\right)} =$$

$$\frac{7\cdot 1}{8+9\cdot 1} = \frac{7}{17}. \quad \square$$

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16. Compute $\lim_{x\to 0} \dfrac{3x+4\sin 3x}{\tan 5x - x\cos 2x}$.

$$\lim_{x\to 0} \frac{3x+4\sin 3x}{\tan 5x - x\cos 2x} = \lim_{x\to 0} \frac{3x+4\sin 3x}{\frac{\sin 5x}{\cos 5x} - x\cos 2x} = \lim_{x\to 0} \frac{3x+4\sin 3x}{\frac{\sin 5x}{\cos 5x} - x\cos 2x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} =$$

$$\lim_{x\to 0} \frac{3+4\frac{\sin 3x}{x}}{\frac{\sin 5x}{x}\frac{1}{\cos 5x} - \cos 2x} = \lim_{x\to 0} \frac{3+4\cdot 3\frac{\sin 3x}{3x}}{5\frac{\sin 5x}{5x}\frac{1}{\cos 5x} - \cos 2x} = \frac{3+12}{5-1} = \frac{15}{4}. \quad \square$$

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17. Suppose

$$f(x) = \begin{cases} 6 & \text{if } x \le 3 \\ \dfrac{x^2-9}{x-3} & \text{if } x > 3 \end{cases}$$

Determine whether $\lim_{x\to 3} f(x)$ is defined. If it is, compute it.

The function is made of "two pieces" glued together at $x = 3$, I'll compute $\lim_{x\to 3} f(x)$ by computing the limits from the left and right, and seeing if they're equal. (If they *aren't* equal, the limit is undefined.)

The left-hand limit is

$$\lim_{x\to 3-} f(x) = \lim_{x\to 3-} 6 = 6.$$

The right-hand limit is

$$\lim_{x\to 3+} f(x) = \lim_{x\to 3+} \frac{x^2-9}{x-3} = \lim_{x\to 3+} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3+} (x+3) = 6.$$

Since the left- and right-hand limits are equal, $\lim_{x\to 3} f(x)$ is defined, and

$$\lim_{x\to 3} f(x) = 6. \quad \square$$

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18. Compute $\displaystyle\lim_{x\to 0} x^4 \cos\frac{1}{x^2}$.

$$\begin{aligned} -1 &\le \cos\frac{1}{x^2} \le 1 \\ -x^4 &\le x^4\cos\frac{1}{x^2} \le x^4 \end{aligned}$$

Take the limit as $x \to 0$. I have

$$\lim_{x\to 0} x^4 = 0 \quad \text{and} \quad \lim_{x\to 0}(-x^4) = 0.$$

By the Squeezing Theorem,

$$\lim_{x\to 0} x^4\cos\frac{1}{x^2} = 0. \quad \square$$

---

19. Suppose that

$$\lim_{x\to 2} g(x) = 3, \quad \lim_{x\to 2} h(x) = 9,$$

$$(x^2+8)g(x) \le f(x) \le (x+2)h(x) \quad \text{for all} \quad x.$$

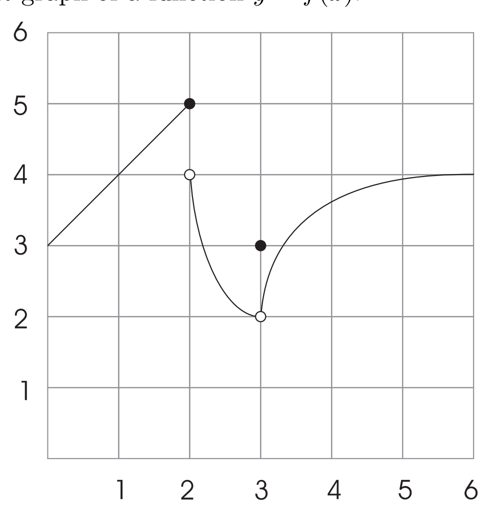
Compute $\displaystyle\lim_{x\to 2} f(x)$.

$$\lim_{x\to 2}(x^2+8)g(x) = 12\cdot 3 = 36 \quad \text{and} \quad \lim_{x\to 2}(x+2)h(x) = 4\cdot 9 = 36.$$

By the Squeezing Theorem, $\displaystyle\lim_{x\to 2} f(x) = 36.$ $\square$

---

20. The picture below shows that graph of a function $y = f(x)$.



Compute:

(a) $\lim_{x \to 2-} f(x)$.

(b) $\lim_{x \to 2+} f(x)$.

(c) $\lim_{x \to 2} f(x)$.

(d) $\lim_{x \to 3-} f(x)$.

(e) $\lim_{x \to 3+} f(x)$.

(f) $\lim_{x \to 3} f(x)$.

(g) $f(3)$.

(a) $\displaystyle\lim_{x \to 2-} f(x) = 5$. □
(b) $\displaystyle\lim_{x \to 2+} f(x) = 4$. □
(c) $\displaystyle\lim_{x \to 2+} f(x)$ is undefined. □
(d) $\displaystyle\lim_{x \to 3-} f(x) = 2$. □
(e) $\displaystyle\lim_{x \to 3+} f(x) = 2$. □
(f) $\displaystyle\lim_{x \to 3} f(x) = 2$. □
(g) $f(3) = 3$. □

---

21. Locate the horizontal asymptotes and the vertical asymptotes of

$$y = \frac{x^2}{x^3 - 5x^2 - 6x}.$$

Justify your answer by computing the relevant limits.

$$\lim_{x \to \infty} \frac{x^2}{x^3 - 5x^2 - 6x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{x^2}{x^3 - 5x^2 - 6x} = 0.$$

Hence, $y = 0$ is a horizontal asymptote at $+\infty$ and at $-\infty$.
Factor the bottom:

$$y = \frac{x^2}{x(x-6)(x+1)}.$$

The function is undefined at $x = 0$, $x = 6$, and at $x = -1$.
Note that

$$\lim_{x \to 0} \frac{x^2}{x(x-6)(x+1)} = \lim_{x \to 0} \frac{x}{(x-6)(x+1)} = 0.$$

Hence, the graph does *not* have a vertical asymptote at $x = 0$.
Consider the limits at $x = 6$.

I know the one-sided limits will be either $+\infty$ or $-\infty$, since plugging in $x = 6$ gives $\dfrac{36}{0}$, a nonzero number divided by 0.

Take $x \to 6^+$ as an example. $x^2 \to 36$, $x \to 6$, $x + 1 \to 7$, and $x - 6 \to 0$. In the last case, since $x > 6$, $x - 6 > 0$, i.e. $x - 6$ goes to 0 through positive numbers. Since all the factors of $\dfrac{x^2}{x(x-6)(x+1)}$ are positive as $x \to 6^+$, the quotient must be positive, and the limit is $+\infty$.

Similar reasoning works for the left-hand limit.

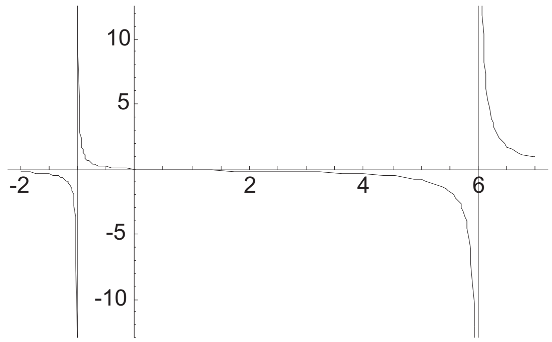
Thus,

$$\lim_{x\to 6^+} \frac{x^2}{x(x-6)(x+1)} = +\infty \quad \text{and} \quad \lim_{x\to 6^+} \frac{x^2}{x(x-6)(x+1)} = -\infty.$$

There is a vertical asymptote at $x = 6$.
Likewise,

$$\lim_{x\to -1^+} \frac{x^2}{x(x-6)(x+1)} = +\infty \quad \text{and} \quad \lim_{x\to -1^+} \frac{x^2}{x(x-6)(x+1)} = -\infty.$$

There is a vertical asymptote at $x = -1$.
The results are visible in the graph of the function:



□

---

22. Differentiate $f(x) = 8x^5 + 4x^2 - 9x + 1$.

$$f'(x) = 40x^4 + 8x - 9. \quad \square$$

---

23. Differentiate $f(x) = \dfrac{4}{x^3} - \dfrac{2}{\sqrt{x}}$.

$$f'(x) = -12x^{-4} + x^{-3/2}. \quad \square$$

---

24. Differentiate $y = 3\sqrt[7]{x^2} - 6\sqrt[5]{42}$.

$$y' = \frac{6}{7}x^{-5/7}. \quad \square$$

---

25. Differentiate $g(u) = (u^5 - u^4 + u^3 + u^2 + u + 1)^{42}$.

$$g'(u) = 42(u^5 - u^4 + u^3 + u^2 + u + 1)^{41}(5u^4 - 4u^3 + 3u^2 + 2u + 1). \quad \square$$

---

26. Differentiate $h(x) = \left(\sqrt{x} + x - x^2\right)(x^6 + x^3 + x + 1)$.

$$h'(x) = \left(\sqrt{x} + x - x^2\right)(6x^5 + 3x^2 + 1) + (x^6 + x^3 + x + 1)\left(\frac{1}{2}x^{-1/2} + 1 - 2x\right). \quad \square$$

---

27. Differentiate $y = \dfrac{x^6 + x^3 + 1}{x^4 + x^2 + 1}$.

$$y' = \frac{(x^4 + x^2 + 1)(6x^5 + 3x^2) - (x^6 + x^3 + 1)(4x^3 + 2x)}{(x^4 + x^2 + 1)^2}. \quad \square$$

---

28. Differentiate $f(x) = \left(3 + \left(3 + x^{-2}\right)^{-2}\right)^{-2}$.

$$f'(x) = (-2)\left(3 + \left(3 + x^{-2}\right)^{-2}\right)^{-3} \cdot (-2)\left(3 + x^{-2}\right)^{-3} \cdot (-2)x^{-3}. \quad \square$$

---

29. Differentiate $f(x) = (\sin 3x + \tan 5x)^{50}$.

$$f'(x) = 50(\sin 3x + \tan 5x)^{49}(3\cos 3x + 5(\sec 5x)^2). \quad \square$$

---

30. Differentiate $g(t) = \sin(\cos(\tan t)) + \sec\left(\dfrac{t}{\tan t}\right)$.

$$g'(t) = [\cos(\cos(\tan t))] \cdot (-\sin(\tan t)) \cdot (\sec t)^2 + \left(\sec\left(\frac{t}{\tan t}\right)\tan\left(\frac{t}{\tan t}\right)\right)\left(\frac{(\tan t)(1) - (t)(\sec t)^2}{(\tan t)^2}\right). \quad \square$$

---

31. Differentiate $y = \sin(\cos x) + (\sin x)(\cos x)$.

$$y' = (\cos(\cos x)) \cdot (-\sin x) + ((\sin x)(-\sin x) + (\cos x)(\cos x)). \quad \square$$

---

32. Find $\dfrac{d}{dx}\left(\dfrac{x^7}{13} + 65x + \dfrac{4}{x^2} + 3\right)$.

Note that
$$\frac{x^7}{13} + 65x + \frac{4}{x^2} + 3 = \frac{x^7}{13} + 65x + 4x^{-2} + 3.$$
Then
$$\frac{d}{dx}\left(\frac{x^7}{13} + 65x + 4x^{-2} + 3\right) = \frac{7}{13}x^6 + 65 - 8x^{-3} = \frac{7}{13}x^6 + 65 - \frac{8}{x^3}. \quad \square$$

---

33. Find $\dfrac{d}{dx}\left(\sqrt{3x} + \dfrac{7}{\sqrt[3]{x}} + \sqrt[7]{x^2}\right)$.

Note that

$$\sqrt{3x} + \frac{7}{\sqrt[3]{x}} + \sqrt[7]{x^2} = \sqrt{3}\sqrt{x} + 7x^{-1/3} + x^{2/7} = \sqrt{3} \cdot x^{1/2} + 7x^{-1/3} + x^{2/7}.$$

Then

$$\frac{d}{dx}\left(\sqrt{3} \cdot x^{1/2} + 7x^{-1/3} + x^{2/7}\right) = \frac{\sqrt{3}}{2}x^{-1/2} - \frac{7}{3}x^{-4/3} + \frac{2}{7}x^{-5/7}. \quad \square$$

---

34. Compute $\frac{d}{dx}\left[\cos(3x+1)^2 + (\cos(3x+1))^2\right]$.

$$\frac{d}{dx}\left[\cos(3x+1)^2 + (\cos(3x+1))^2\right] = (-\sin(3x+1)^2)(2)(3x+1)(3) + 2(\cos(3x+1))(-\sin(3x+1))(3) =$$

$$-6(3x+1)\sin(3x+1)^2 - 6\cos(3x+1)\sin(3x+1). \quad \square$$

---

35. Compute $\frac{d}{dx}\frac{6x^3 + 3\sin x}{17}$.

$$\frac{d}{dx}\frac{6x^3 + 3\sin x}{17} = \frac{d}{dx}\frac{1}{17}\left(6x^3 + 3\sin x\right) = \frac{1}{17}\left(18x^2 + 3\cos x\right).$$

It is *not* a good idea to use the Quotient Rule when either the top or the bottom of a fraction is a number. $\square$

---

36. Find $\frac{d}{dx}(3\sin x - 7x)(4 + x^2 - \tan x)$.

$$\frac{d}{dx}(3\sin x - 7x)(4 + x^2 - \tan x) = (3\sin x - 7x)(2x - (\sec x)^2) + (4 + x^2 - \tan x)(3\cos x - 7). \quad \square$$

---

37. Find $f'(t)$ if $f(t) = \frac{3}{(t^2 + 3t + 1)^5}$.

Rewrite the function as

$$f(t) = 3(t^2 + 3t + 1)^{-5}.$$

Then

$$f'(t) = (3)(-5)(t^2 + 3t + 1)^{-6}(2t + 3) = -\frac{15(2t+3)}{(t^2 + 3t + 1)^6}.$$

It is *not* a good idea to use the Quotient Rule when either the top or the bottom of a fraction is a number. $\square$

---

38. Compute $\frac{d}{dx}(e^{3x} + 1)(\sin 2x)(x^4 + 2x + 1)$.

For a product of three terms, the Product Rule says

$$\frac{d}{dx}(1^{\text{st}})(2^{\text{nd}})(3^{\text{rd}}) = (1^{\text{st}})(2^{\text{nd}})\left(\frac{d}{dx}(3^{\text{rd}})\right) + (1^{\text{st}})\left(\frac{d}{dx}(2^{\text{nd}})\right)(3^{\text{rd}}) + \left(\frac{d}{dx}(1^{\text{st}})\right)(2^{\text{nd}})(3^{\text{rd}}).$$

So

$$\frac{d}{dx}(e^{3x}+1)(\sin 2x)(x^4+2x+1) = (e^{3x}+1)(\sin 2x)(4x^3+2) + (e^{3x}+1)(2\cos 2x)(x^4+2x+1) +$$

$$(3e^{3x})(\sin 2x)(x^4+2x+1). \quad \square$$

---

39. Find $\dfrac{d}{dx}\left(\dfrac{\sqrt{x}+\sec x}{3+5x-\cos x}\right)$.

Note that

$$\frac{\sqrt{x}+\sec x}{3+5x-\cos x} = \frac{x^{1/2}+\sec x}{3+5x-\cos x}.$$

Then

$$\frac{d}{dx}\left(\frac{x^{1/2}+\sec x}{3+5x-\cos x}\right) = \frac{(3+5x-\cos x)\left(\frac{1}{2}x^{-1/2}+\sec x\tan x\right)) - (x^{1/2}+\sec x)(5+\sin x)}{(3+5x-\cos x)^2} =$$

$$\frac{(3+5x-\cos x)\left(\frac{1}{2\sqrt{x}}+\sec x\tan x\right) - (\sqrt{x}+\sec x)(5+\sin x)}{(3+5x-\cos x)^2}. \quad \square$$

---

40. Find y' if $y = \dfrac{(2-3x-x^5)e^{4x}}{\tan 5x+7}$.

$$y' = \frac{(\tan 5x+7)\left((2-3x-x^5)(4e^{4x}) + (e^{4x})(-3-5x^4)\right) - (2-3x-x^5)e^{4x}(5)(\sec 5x)^2}{(\tan 5x+7)^2}.$$

I applied the Quotient Rule to the original fraction. In taking the derivative of the top, I also needed to apply the Product Rule. $\square$

---

41. Compute $\dfrac{d}{dx}\dfrac{1+\csc(x^2)}{(e^{-7x}+3x)(7x^6+x+1)}$.

$$\frac{d}{dx}\frac{1+\csc(x^2)}{(e^{-7x}+3x)(7x^6+x+1)} =$$

$$\frac{(e^{-7x}+3x)(7x^6+x+1)(2x)(-\csc(x^2)\cot(x^2)) - (1+\csc(x^2))\left((e^{-7x}+3x)(42x^5+1) + (7x^6+x+1)(-7e^{-7x}+3)\right)}{(e^{-7x}+3x)^2(7x^6+x+1)^2}.$$

I applied the Quotient Rule to the original fraction. In taking the derivative of the bottom, I also needed to apply the Product Rule. $\square$

---

42. Compute $\dfrac{d}{dx}\left(\dfrac{x^2+x+1}{x^3-3x+5}\right)^{11}$.

$$\frac{d}{dx}\left(\frac{x^2+x+1}{x^3-3x+5}\right)^{11} = 11\left(\frac{x^2+x+1}{x^3-3x+5}\right)^{10}\left(\frac{(x^3-3x+5)(2x+1)-(x^2+x+1)(3x^2-3)}{(x^3-3x+5)^2}\right). \quad \square$$

---

43. Compute $\dfrac{d}{dx}\left(e^{x^2}+7^{x^2}+e^{7x}+7^{e^x}\right)$.

$$\frac{d}{dx}\left(e^{x^2}+7^{x^2}+e^{7x}+7^{e^x}\right) = 2xe^{x^2}+2x(\ln 7)7^{x^2}+7e^{7x}+(e^x)(\ln 7)7^{e^x}. \quad \square$$

---

44. Compute $\dfrac{d}{dx}\sqrt[3]{(x^2+e^x)(\ln x+42x)+x^3}$.

Note that

$$\sqrt[3]{(x^2+e^x)(\ln x+42x)+x^3} = \left((x^2+e^x)(\ln x+42x)+x^3\right)^{1/3}.$$

Then

$$\frac{d}{dx}\left((x^2+e^x)(\ln x+42x)+x^3\right)^{1/3} =$$

$$\frac{1}{3}\left((x^2+e^x)(\ln x+42x)+x^3\right)^{-2/3}\left((x^2+e^x)\left(\frac{1}{x}+42\right)+(\ln x+42x)(2x+e^x)+3x^2\right). \quad \square$$

---

45. Compute $\dfrac{d}{dw}\ln\sqrt{e^{w^2}+e^{2w}+1}$.

Note that

$$\ln\sqrt{e^{w^2}+e^{2w}+1} = \ln\left(e^{w^2}+e^{2w}+1\right)^{1/2} = \frac{1}{2}\ln\left(e^{w^2}+e^{2w}+1\right).$$

Then

$$\frac{d}{dw}\left(\frac{1}{2}\ln\left(e^{w^2}+e^{2w}+1\right)\right) = \frac{1}{2}\cdot\frac{2we^{w^2}+2e^{2w}}{e^{w^2}+e^{2w}+1} = \frac{we^{w^2}+e^{2w}}{e^{w^2}+e^{2w}+1}. \quad \square$$

---

46. Compute $\dfrac{d}{dx}\left(x^3+\left(x^3+(x^3+1)^3\right)^3\right)^3$.

$$\frac{d}{dx}\left(x^3+\left(x^3+(x^3+1)^3\right)^3\right)^3 =$$

$$3\left(x^3+\left(x^3+(x^3+1)^3\right)^3\right)^2\left(3x^2+3\left(x^3+(x^3+1)^3\right)^2\left(3x^2+3(x^3+1)^2(3x^2)\right)\right). \quad \square$$

---

47. Compute $\dfrac{d}{dx}\left(\dfrac{2x+5}{(x+2)^3}\right)$.

$$\frac{d}{dx}\left(\frac{2x+5}{(x+2)^3}\right) = \frac{(x+2)^3(2)-(2x+5)(3)(x+2)^2}{(x+2)^6} = \frac{(x+2)(2)-(2x+5)(3)}{(x+2)^4} =$$

$$\frac{2x + 4 - 6x - 15}{(x+2)^4} = \frac{-4x - 11}{(x+2)^4}.$$

In going from the second expression to the third, I cancelled a common factor of $(x+2)^2$. If you try to multiply before you cancel, you'll get a big mess, and it will be much harder to simplify. $\square$

---

48. Compute the derivative with respect to t of $s(t) = \sec\left(\dfrac{5}{t^3 + 5t + 8}\right)$.

Rewrite the function as

$$s(t) = \sec\left(5(t^3 + 5t + 8)^{-1}\right).$$

Then

$$s'(t) = \left(\sec\left(5(t^3 + 5t + 8)^{-1}\right) \tan\left(5(t^3 + 5t + 8)^{-1}\right)\right) \left((5)(-1)(t^3 + 5t + 8)^{-2}(3t^2 + 5)\right) =$$

$$\left(\sec\left(5(t^3 + 5t + 8)^{-1}\right) \tan\left(5(t^3 + 5t + 8)^{-1}\right)\right) \left(\frac{-5(3t^2 + 5)}{(t^3 + 5t + 8)^2}\right). \quad \square$$

---

49. Compute $\dfrac{d}{dx}[\log_5(x^2 + x + 1)]^7$.

$$\frac{d}{dx}[\log_5(x^2 + x + 1)]^7 = \left(7[\log_5(x^2 + x + 1)]^6\right) \left(\frac{2x + 1}{(x^2 + x + 1)\ln 5}\right). \quad \square$$

---

50. Compute y''' for $y = (x^2 + 1)^4$.

$$y' = 4(x^2 + 1)^3(2x) = 8x(x^2 + 1)^3,$$

$$y'' = 8\left((x)(3)(x^2 + 1)^2(2x) + (x^2 + 1)^3\right) = 48x^2(x^2 + 1)^2 + 8(x^2 + 1)^3,$$

$$y''' = 48\left((x^2)(2)(x^2 + 1)(2x) + (x^2 + 1)^2(2x)\right) + (8)(3)(x^2 + 1)^2(2x) =$$

$$192x^3(x^2 + 1) + 96x(x^2 + 1)^2 + 48x(x^2 + 1)^2 = 192x^3(x^2 + 1) + 144x(x^2 + 1)^2. \quad \square$$

---

51. Compute $y^{(50)}$ if $y = \dfrac{1}{4 - x}$.

I'll compute the first few derivatives until I see the pattern.

$$y' = \frac{1}{(4-x)^2},$$

$$y'' = \frac{2}{(4-x)^3},$$

$$y''' = \frac{2 \cdot 3}{(4-x)^4},$$

$$y^{(4)} = \frac{2 \cdot 3 \cdot 4}{(4-x)^5}.$$

Note that I don't get minus signs here; the powers are negative, but the Chain Rule requires the derivative of $4 - x$, which is -1. The two negatives cancel.

Note also that, in order to see the pattern, I did *not* multiply out the numbers on the top.

Based on the pattern, I see that the power on the bottom is one more than the order of the derivative. On the top, I get the product of the numbers from 2 through the order of the derivative. So

$$y^{(50)} = \frac{2 \cdot 3 \cdot \cdots \cdot 50}{(4-x)^{51}}.$$

(You can also write the top as 50! (50-factorial).) □

---

52. Compute $\dfrac{d}{dx}\sqrt{e^{f(x)^2} + 1}$.

Since $f(x)$ is not given, the answer will come out in terms of $f(x)$ and $f'(x)$.

Note that

$$\sqrt{e^{f(x)^2} + 1} = \left(e^{f(x)^2} + 1\right)^{1/2}.$$

Then

$$\frac{d}{dx}\left(e^{f(x)^2} + 1\right)^{1/2} = \frac{1}{2}\left(e^{f(x)^2} + 1\right)^{-1/2}\left(e^{f(x)^2}\right)(2f(x))(f'(x)). \quad \square$$

---

53. Compute $\dfrac{d}{dx}\ln(\sin(g(x)) + 1)$.

Since $g(x)$ is not given, the answer will come out in terms of $g(x)$ and $g'(x)$.

$$\frac{d}{dx}\ln(\sin(g(x)) + 1) = \frac{(\cos(g(x))(g'(x))}{\sin(g(x)) + 1}. \quad \square$$

---

54. f and g are differentiable functions. The values of f, g, f', and g' at $x = 2$ and $x = 5$ are shown below:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	-6	5	4	10
5	0.5	-1	7	-3

Compute $(fg)'(5)$, $\left(\dfrac{f}{g}\right)'(2)$, and $[f(g)]'(2)$.

By the Product Rule,

$$(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (0.5)(-3) + (-1)(7) = -8.5.$$

By the Quotient Rule,

$$\left(\frac{f}{g}\right)'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{g(2)^2} = \frac{(5)(4) - (-6)(10)}{5^2} = \frac{16}{5}.$$

By the Chain Rule,

$$[f(g)]'(2) = f'(g(2)) \cdot g'(2) = f'(5) \cdot g'(2) = 7 \cdot 10 = 70. \quad \square$$

55. Find the equation of the tangent line to the graph of $f(x) = x^3 + 5x - 10$ at $x = 2$.

When $x = 2$,

$$f(2) = 2^3 + 5 \cdot 2 - 10 = 8.$$

The point of tangency is $(2, 8)$.
I have

$$f'(x) = 3x^2 + 5, \quad \text{so} \quad f'(2) = 17.$$

The tangent line is

$$17(x - 2) = y - 8, \quad \text{or} \quad y = 17x - 26. \quad \square$$

---

56. Find the equation of the tangent line to the graph of $y = \dfrac{1}{(x^2 + 4x + 5)^2}$ at $x = 1$.

When $x = 1$,

$$y\frac{1}{(1 + 4 + 5)^2} = \frac{1}{100}.$$

The point of tangency is $\left(1, \dfrac{1}{100}\right)$.
$y = (x^2 + 4x + 5)^{-2}$, so

$$y' = (-2)(x^2 + 4x + 5)^{-3}(2x + 4), \quad \text{and} \quad y'(1) = -\frac{3}{250}.$$

The tangent line is

$$-\frac{3}{250}(x - 1) = y - \frac{1}{100}. \quad \square$$

---

57. For what value or values of c does the tangent line to the graph of $y = x^2 + 5x + 1$ at $x = c$ pass through the point $(1, 6)$?

At $x = c$, I have $y = c^2 + 5c + 1$. Also,

$$y' = 2x + 5, \quad \text{so} \quad y'(c) = 2c + 5.$$

The tangent line to $y = x^2 + 5x + 1$ at $x = c$ is

$$y - (c^2 + 5c + 1) = (2c + 5)(x - c).$$

I want this line to pass through the point $(1, 6)$, so plug the point into the line equation and solve for c:

$$\begin{aligned} 6 - (c^2 + 5c + 1) &= (2c + 5)(1 - c) \\ -c^2 - 5c + 5 &= -2c^2 - 3c + 5 \\ c^2 - 2c &= 0 \\ c(c - 2) &= 0 \end{aligned}$$

The values of c are $c = 0$ and $c = 2$. $\square$

---

58. The position of a cheeseburger at time t is given by

$$s(t) = t^3 - 12t^2 + 36t + 2.$$

Find the value(s) of t for which the velocity is 0. Find the value(s) of t for which the acceleration is 0.

The velocity is the derivative of the position:

$$v(t) = s'(t) = 3t^2 - 24t + 36 = 3(t^2 - 8t + 12) = 3(t-2)(t-6).$$

I have $v(t) = 0$ for $t = 2$ and $t = 6$.
The acceleration is derivative of the velocity (or the second derivative of the position):

$$a(t) = v'(t) = 6t - 24 = 6(t-4).$$

(Note: I differentiated $v(t) = 3t^2 - 24t + 36$, *not* $v(t) = 3(t-2)(t-6)$. Differentiating the first expression is easier than differentiating the second.)
I have $a(t) = 0$ for $t = 4$. □

---

59. For what values of x is the function $f(x) = \dfrac{x-5}{x^2 - 3x - 10}$ continuous?

The function will be continuous where it's defined, so I have to find the domain.

$$f(x) = \frac{x-5}{(x-5)(x+2)}.$$

I can't divide by 0, and division by 0 occurs when $x = 5$ or $x = -2$.
(You can't cancel the $x - 5$'s, because that *assumes* $x - 5 \neq 0$.)
Thus, the domain is $x \neq -2, 5$, and that is where f is continuous. □

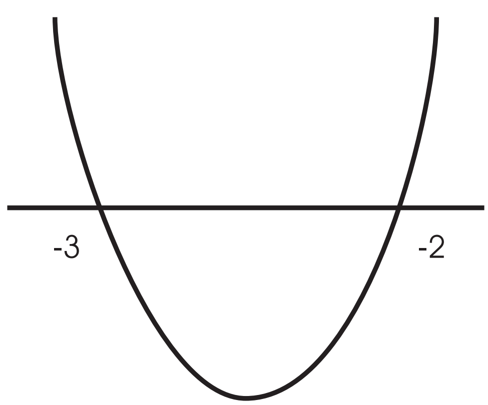
---

60. For what values of x is the function $f(x) = \dfrac{1}{\sqrt{x^2 + 5x + 6}}$ continuous?

The function will be continuous where it's defined, so I have to find the domain.

$$f(x) = \frac{1}{\sqrt{(x+2)(x+3)}}.$$

I can't divide by 0, and division by 0 occurs where $\sqrt{(x+2)(x+3)} = 0$. Square both sides: $(x+2)(x+3) = 0$. I get $x = -2$ or $x = -3$.
I can't take the square root of a negative number, and this occurs where $(x+2)(x+3) < 0$.
You can solve the inequality by constructing a sign chart.
Alternatively, graph the quadratic. $y = (x+2)(x+3)$ is a parabola opening upward (since it's $y = x^2 + 5x + 6$, which has a positive x^2-term). It has roots at $x = -2$ and at $x = -3$. Picture:



From the graph, you can see that $(x+2)(x+3) < 0$ for $-3 < x < -2$.

I throw out the bad points $x = -2$, $x = -3$, $-3 < x < -2$. The domain is $x < -3$ or $x > -2$, and that is where f is continuous. $\square$

---

61. Let

$$f(x) = \begin{cases} \dfrac{x^2 - 7x - 8}{x + 1} & \text{if } x \neq -1 \\ 9 & \text{if } x = -1 \end{cases}$$

Prove or disprove: f is continuous at $x = -1$.

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 - 7x - 8}{x + 1} = \lim_{x \to -1} \frac{(x-8)(x+1)}{x+1} = \lim_{x \to -1} (x - 8) = -9.$$

On the other hand, $f(-1) = 9$. Since $\lim_{x \to -1} f(x) \neq f(-1)$, f is *not* continuous at $x = -1$. $\square$

---

62. Find the value of c which makes the following function continuous at $x = 2$:

$$f(x) = \begin{cases} 1 + 3x - cx^2 & \text{if } x < 2 \\ 7x + 3 & \text{if } x \geq 2 \end{cases}.$$

For f to be continuous at $x = 2$, $\lim_{x \to 2} f(x)$ must be defined. This will happen if the left and right-hand limits at $x = 2$ are equal. Compute the limits:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (7x + 3) = 17,$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (1 + 3x - cx^2) = 7 - 4c.$$

Set the left and right-hand limits equal and solve for c:

$$7 - 4c = 17, \quad -4c = 10, \quad c = -\frac{5}{2}.$$

This value of c will make the left and right-hand limits equal to 17, so in this case, $\lim_{x \to 2} f(x) = 17$. Since $f(2) = 17$, it follows that $\lim_{x \to 2} f(x) = f(2)$, and f is continuous at $x = 2$. $\square$

---

63. Prove that $f(x) = x^5 + 4x + 1$ has a root between $x = -1$ and $x = 0$.

f is continuous, since it's a polynomial.

x	$f(x) = x^5 + 4x + 1$
-1	-4
0	1

Since f is negative when $x = -1$ and f is positive when $x = 0$, the Intermediate Value Theorem implies that $f(c) = 0$ for some number c between -1 and 0. $\square$

---

64. Suppose f is continuous, $f(2) = 13$, and $f(5) = 1$. Prove that there is a number x between 2 and 5 such that $x^3 \cdot f(x) = 110$.

x^3 and $f(x)$ are continuous, so $x^3 \cdot f(x)$ is continuous.

x	$x^3 \cdot f(x)$
2	$8 \cdot 13 = 104$
5	$125 \cdot 1 = 125$

Since $x^3 f(x)$ is continuous, and since 110 is between 104 and 125, the Intermediate Value Theorem says that there is a number x between 2 and 5 such that $x^3 \cdot f(x) = 110$. □

---

65. Suppose $f(x) = \sqrt{x-6}$. Use the limit definition of the derivative to compute $f'(15)$.

$$f'(15) = \lim_{h \to 0} \frac{f(15+h) - f(15)}{h}.$$

Now

$$f(15) = \sqrt{15-6} = \sqrt{9} = 3,$$

$$f(15+h) = \sqrt{(15+h)-6} = \sqrt{9+h}.$$

So

$$f'(15) = \lim_{h \to 0} \frac{\sqrt{9+h}-3}{h} = \lim_{h \to 0} \frac{\sqrt{9+h}-3}{h} \cdot \frac{\sqrt{9+h}+3}{\sqrt{9+h}+3} =$$

$$\lim_{h \to 0} \frac{(\sqrt{9+h})^2 - 3^2}{h(\sqrt{9+h}+3)} = \lim_{h \to 0} \frac{(9+h)-9}{h(\sqrt{9+h}+3)} = \lim_{h \to 0} \frac{h}{h(\sqrt{9+h}+3)} = \lim_{h \to 0} \frac{1}{\sqrt{9+h}+3} = \frac{1}{6}. \quad \square$$

---

66. Suppose $f(x) = \dfrac{1}{x+3}$. Use the limit definition of the derivative to compute $f'(x)$.

I'll simplify the quotient $\dfrac{f(x+h) - f(x)}{h}$, then plug it into the limit.

$$f(x+h) - f(x) = \frac{1}{x+h+3} - \frac{1}{x+3} = \frac{(x+3) - (x+h+3)}{(x+h+3)(x+3)} = \frac{-h}{(x+h+3)(x+3)}.$$

Therefore,

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{-h}{(x+h+3)(x+3)}}{h} = \frac{-h}{h(x+h+3)(x+3)} = \frac{-1}{(x+h+3)(x+3)}.$$

Hence,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-1}{(x+h+3)(x+3)} = -\frac{1}{(x+3)^2}. \quad \square$$

---

67. Suppose $f(x) = \dfrac{1}{\sqrt{x}+1}$. Use the limit definition of the derivative to compute $f'(x)$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{\sqrt{x+h}+1}-\dfrac{1}{\sqrt{x}+1}}{h} =$$

$$\lim_{h\to 0}\frac{\dfrac{1}{\sqrt{x+h}+1}-\dfrac{1}{\sqrt{x}+1}}{h}\cdot\frac{(\sqrt{x}+1)(\sqrt{x+h}+1)}{(\sqrt{x}+1)(\sqrt{x+h}+1)} = \lim_{h\to 0}\frac{(\sqrt{x}+1)-(\sqrt{x+h}+1)}{h(\sqrt{x}+1)(\sqrt{x+h}+1)} =$$

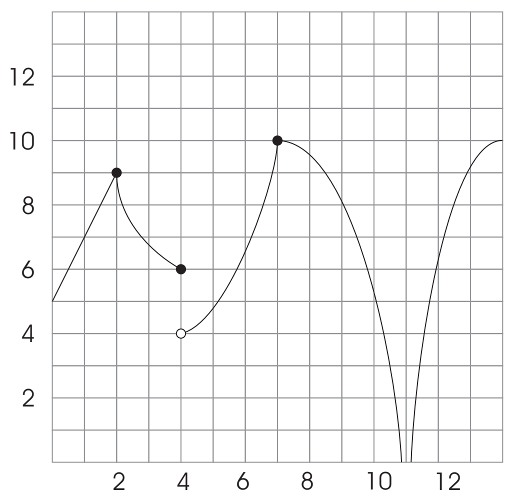
$$\lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h(\sqrt{x}+1)(\sqrt{x+h}+1)} = \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h(\sqrt{x}+1)(\sqrt{x+h}+1)}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} =$$

$$\lim_{h\to 0}\frac{x-(x+h)}{h(\sqrt{x}+1)(\sqrt{x+h}+1)(\sqrt{x}+\sqrt{x+h})} = \lim_{h\to 0}\frac{-h}{h(\sqrt{x}+1)(\sqrt{x+h}+1)(\sqrt{x}+\sqrt{x+h})} =$$

$$\lim_{h\to 0}\frac{-1}{(\sqrt{x}+1)(\sqrt{x+h}+1)(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{2(\sqrt{x}+1)^2\sqrt{x}}. \quad \square$$

---

68. The graph of a function $f(x)$ is pictured below. For what values of x is f continuous but not differentiable? For what values of f is f not continuous?



f is continuous but not differentiable at $x = 2$ and at $x = 7$, since at those points the graph is unbroken but has corners.

f is not continuous at $x = 4$ and at $x = 11$. $\square$

---

69. (a) Find the average rate of change of $f(x) = x^2 + 3x - 4$ on the interval $1 \le x \le 3$.

(b) Find the instantaneous rate of change of $f(x) = x^2 + 3x - 4$ at $x = 3$.

(a)

$$\frac{f(3)-f(1)}{3-1} = \frac{14-0}{3-1} = 7. \quad \square$$

(b)

$$f'(x) = 2x + 3, \quad \text{so} \quad f'(3) = 6 + 3 = 9. \quad \square$$

---

70. Give an example of a function $f(x)$ which is defined for all x and a number c such that

$$\lim_{x \to c} f(x) \neq f(c).$$

There are lots of possible answers to this question. For example, let

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0, \quad \text{but} \quad f(0) = 1.$$

Of course, the condition $\lim_{x \to c} f(x) \neq f(c)$ means that the function is **not continuous** at c. The value of $\lim_{x \to c} f(x)$ (or even its existence) does not depend on the value of $f(c)$ (or even its existence). □

---

71. Suppose $f(x) = \dfrac{x+1}{x+2}$. Find $f^{-1}(x)$.

Let $y = \dfrac{x+1}{x+2}$. Swap x's and y's and solve for y:

$$\begin{aligned}
x &= \frac{y+1}{y+2} \\
(y+2) \cdot x &= (y+2) \cdot \frac{y+1}{y+2} \\
x(y+2) &= y+1 \\
xy + 2x &= y+1 \\
xy + 2x - 2x - y &= y + 1 - 2x - y \\
xy - y &= 1 - 2x \\
y(x-1) &= 1 - 2x \\
\frac{1}{x-1} \cdot y(x-1) &= \frac{1}{x-1} \cdot (1-2x) \\
y &= \frac{1-2x}{x-1}
\end{aligned}$$

Therefore, $f^{-1}(x) = \dfrac{1-2x}{x-1}$. □

---

72. Suppose $f(2) = 5$ and $f'(2) = 7$. Find $(f^{-1})'(5)$.

First, notice that $f(2) = 5$ means $f^{-1}(5) = 2$. I'll use the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Setting $x = 5$, I get

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(2)} = \frac{1}{7}. \quad □$$

---

73. Suppose $f(2) = 5$ and $f'(x) = \sqrt{x^4 + x^2 + 16}$. Find $(f^{-1})'(5)$.

First, notice that $f(2)=5$ means $f^{-1}(5)=2$. I'll use the formula

$$(f^{-1})'(x)=\frac{1}{f'(f^{-1}(x))}.$$

Setting $x=5$, I get

$$(f^{-1})'(5)=\frac{1}{f'(f^{-1}(5))}=\frac{1}{f'(2)}=\frac{1}{\sqrt{2^4+2^2+16}}=\frac{1}{6}.\quad\square$$

---

74. Suppose $f(x)=(x^5+2x^3+5)^{1/3}$. Find $(f^{-1})'(2)$.

It looks like I need f' in this problem, so I'll do that first. By the Chain Rule,

$$f'(x)=\frac{1}{3}(x^5+2x^3+5)^{-2/3}(5x^4+6x^2).$$

I'll use the formula

$$(f^{-1})'(x)=\frac{1}{f'(f^{-1}(x))}.$$

Setting $x=2$, I get

$$(f^{-1})'(2)=\frac{1}{f'(f^{-1}(2))}.\qquad (*)$$

I need to find $f^{-1}(2)$. Remember that

$$f^{-1}(2)=A\quad\text{means}\quad f(A)=2.$$

So I want to find a number A such that $f(A)=2$, or

$$(A^5+2A^3+5)^{1/3}=2.$$

If you try to solve this algebraically for A, you might get as far as

$$\begin{aligned}(A^5+2A^3+5)^{1/3}&=2\\ [(A^5+2A^3+5)^{1/3}]^3&=2^3\\ A^5+2A^3+5&=8\\ A^5+2A^3-3&=0\end{aligned}$$

But you probably don't know any method or formula for solving this kind of equation.

However, the problem was intended to be doable, so if you can't solve systematically, it must be intended that you should use trial and error. And if it's trial and error, the answer is probably not something like "$A=517$". So try a few small numbers for A — plug them into $A^5+2A^3-3=0$ and see if you can find one that works.

If you do this, you find that $A=1$ solves the equation. Thus, $f(1)=2$, so $f^{-1}(2)=1$. Now going back to (*), I get

$$(f^{-1})'(2)=\frac{1}{f'(f^{-1}(2))}=\frac{1}{f'(1)}=\frac{1}{\left(\frac{1}{3}\cdot(8^{-2/3})\cdot 11\right)}=\frac{1}{\left(\frac{11}{12}\right)}=\frac{12}{11}.\quad\square$$

---

75. Compute $\dfrac{d}{dx}(x^2+3x+1)^{(x^2+6)}$.

Let $y = (x^2 + 3x + 1)^{(x^2+6)}$. Take the log of both sides, then differentiate:

$$\begin{aligned}
\ln y &= \ln(x^2 + 3x + 1)^{(x^2+6)} \\
\ln y &= (x^2 + 6)\ln(x^2 + 3x + 1) \\
\frac{y'}{y} &= (x^2 + 6)\left(\frac{2x + 3}{x^2 + 3x + 1}\right) + [\ln(x^2 + 3x + 1)](2x) \\
y' &= y\left[(x^2 + 6)\left(\frac{2x + 3}{x^2 + 3x + 1}\right) + 2x\ln(x^2 + 3x + 1)\right] \\
y' &= (x^2 + 3x + 1)^{(x^2+6)}\left[(x^2 + 6)\left(\frac{2x + 3}{x^2 + 3x + 1}\right) + 2x\ln(x^2 + 3x + 1)\right]
\end{aligned}$$

Note that in the next-to-the-last step, I changed $[\ln(x^2 + 3x + 1)](2x)$ to $2x\ln(x^2 + 3x + 1)$. If you don't do this, you must have the $[\cdot]$'s around $\ln(x^2 + 3x + 1)$. Writing "$\ln(x^2 + 3x + 1)(2x)$" is not correct. □

---

76. Compute $\dfrac{d}{dx}(e^x + 1)^{\sin x}$.

Let $y = (e^x + 1)^{\sin x}$. Take the log of both sides, then differentiate:

$$\begin{aligned}
\ln y &= \ln(e^x + 1)^{\sin x} \\
\ln y &= (\sin x)\ln(e^x + 1) \\
\frac{y'}{y} &= (\sin x)\left(\frac{e^x}{e^x + 1}\right) + [\ln(e^x + 1)](\cos x) \\
y' &= y\left((\sin x)\left(\frac{e^x}{e^x + 1}\right) + [\ln(e^x + 1)](\cos x)\right) \\
y' &= (e^x + 1)^{\sin x}\left((\sin x)\left(\frac{e^x}{e^x + 1}\right) + [\ln(e^x + 1)](\cos x)\right)
\end{aligned}$$ □

---

77. Find the equation of the tangent line to $y = (x^2 + 2)^x$ at $x = 1$.

When $x = 1$, I have $y = (1 + 2)^1 = 3$, so the point of tangency is $(1, 3)$.
Use logarithmic differentiation to find y':

$$\begin{aligned}
y &= (x^2 + 2)^x \\
\ln y &= \ln(x^2 + 2)^x \\
\ln y &= x\ln(x^2 + 2) \\
\frac{y'}{y} &= x\left(\frac{2x}{x^2 + 2}\right) + \ln(x^2 + 2) \\
y' &= y\left[x\left(\frac{2x}{x^2 + 2}\right) + \ln(x^2 + 2)\right] \\
y' &= (x^2 + 2)^x \cdot \left[x\left(\frac{2x}{x^2 + 2}\right) + \ln(x^2 + 2)\right]
\end{aligned}$$

Plug in $x = 1$:

$$y' = 3 \cdot \left(1 \cdot \frac{2}{3} + \ln 3\right) = 2 + 3\ln 3.$$

The line is

$$y - 3 = (2 + 3\ln 3)(x - 1). \quad \square$$

---

78. **(An alternate approach to logarithmic differentiation)**

(a) Use the identity $A = e^{\ln A}$ to write $(x^2+1)^{\tan x}$ using $e^{(\cdot)}$ and $\ln(\cdot)$.

(b) Use the Chain Rule to differentiate the expression you obtained in (a). (This gives another way of doing logarithmic differentiation involving powers.)

(a)

$$(x^2+1)^{\tan x} = e^{\ln(x^2+1)^{\tan x}} = e^{(\tan x)\ln(x^2+1)}. \quad \square$$

(b) I'll differentiate $e^{(\tan x)\ln(x^2+1)}$ using the Chain Rule. The outer function is the exponential $e^{(\cdot)}$; to differentiate the inner function $(\tan x)\ln(x^2+1)$, I'll use the Product Rule.

$$\frac{d}{dx}e^{(\tan x)\ln(x^2+1)} = \left(e^{(\tan x)\ln(x^2+1)}\right)\left[(\tan x)\left(\frac{2x}{x^2+1}\right) + \left(\ln(x^2+1)\right)\left((\sec x)^2\right)\right] =$$

$$(x^2+1)^{\tan x}\left[(\tan x)\left(\frac{2x}{x^2+1}\right) + \left(\ln(x^2+1)\right)\left((\sec x)^2\right)\right].$$

In other words,

$$\frac{d}{dx}(x^2+1)^{\tan x} = (x^2+1)^{\tan x}\left[(\tan x)\left(\frac{2x}{x^2+1}\right) + \left(\ln(x^2+1)\right)\left((\sec x)^2\right)\right]. \quad \square$$

---

79. Find y' if y is defined implicitly by

$$\sin x + \cos y = 3x + 5y.$$

Differentiate implicitly and solve for y:

$$\begin{aligned} \cos x + (-\sin y)y' &= 3 + 5y' \\ (-\sin y)y' - 5y' &= 3 - \cos x \\ y'(-\sin y - 5) &= 3 - \cos x \\ y' &= \frac{3 - \cos x}{-\sin y - 5} = \frac{\cos x - 3}{\sin y + 5} \quad \square \end{aligned}$$

---

80. Find the equation of the tangent line to the curve

$$x^3 + 2xy^3 = x^2 - y^2 + 3 \quad \text{at the point} \quad (1,1).$$

Differentiate implicitly:

$$3x^2 + 2y^3 + 6xy^2y' = 2x - 2yy'.$$

Note: If you have a point to plug in, plug the point in **before** you solve for y'.

Plug in $x = 1$, $y = 1$:

$$3 + 2 + 6y' = 2 - 2y', \quad y' = -\frac{3}{8}.$$

The tangent line is

$$-\frac{3}{8}(x-1) = y - 1. \quad \square$$

---

81. Find the equation of the tangent line to the curve

$$\frac{x^2}{y} + y^2 = 7x - 2y - 9 \quad \text{at the point} \quad (3, 1).$$

Differentiate implicitly:

$$\frac{y \cdot 2x - x^2 y'}{y^2} + 2yy' = 7 - 2y'.$$

Note: If you have a point to plug in, plug the point in **before** you solve for y'.

Plug in $x = 3$, $y = 1$:

$$(6 - 9y') + 2y' = 7 - 2y', \quad -5y' = 1, \quad y' = -\frac{1}{5}.$$

The tangent line is

$$y - 1 = -\frac{1}{5}(x - 3). \quad \square$$

---

82. Find y'' at the point $(1, 2)$ on the curve

$$x^3 y + 2x^2 = y^3 - 2y.$$

Differentiate implicitly:

$$x^3 y' + 3x^2 y + 4x = 3y^2 y' - 2y'. \qquad (*)$$

Plug in $x = 1$ and $y = 2$ and solve for y':

$$y' + 6 + 4 = 12y' - 2y', \quad y' = \frac{10}{9}.$$

Differentiate (*) implicitly:

$$x^3 y'' + 3x^2 y' + 3x^2 y' + 6xy + 4 = 3y^2 y'' + 6y(y')^2 - 2y''.$$

Plug in $x = 1$, $y = 2$, and $y' = \frac{10}{9}$ and solve for y'':

$$y'' + 3\left(\frac{10}{9}\right) + 3\left(\frac{10}{9}\right) + 12 + 4 = 12y'' + 12\left(\frac{10}{9}\right)^2 - 2y'', \quad y'' = \frac{212}{243}. \quad \square$$

He who climbs onto the roof should not push away the ladder. - Karol Irzykowski