Math 211
9–21–2017

# Review Problems for Test 1

These problems are provided to help you study. The presence of a problem on this sheet does not imply that a similar problem will appear on the test. And the absence of a problem from this sheet does not imply that the test will not have a similar problem.

1. Compute $\displaystyle\int_0^{\pi/2} (\cos x)^3 \left(1 + (\sin x)^{1/2}\right)\, dx$.

2. Compute $\displaystyle\int (\cos 7x)^4\, dx$.

3. Compute $\displaystyle\int [\sin(x+3)]^2[\cos(x+3)]^2\, dx$.

4. Compute $\displaystyle\int (\sec 3x)^5 \tan 3x\, dx$.

5. Compute $\displaystyle\int (\tan x)^4\, dx$.

6. Compute $\displaystyle\int (\csc 2x)^3(\cot 2x)^3\, dx$.

7. Compute $\displaystyle\int \sqrt{x^2-1}\, dx$.

8. Compute $\displaystyle\int \ln(x^2+5)\, dx$.

9. Compute $\displaystyle\int x^2\sqrt{25-x^2}\, dx$.

10. Compute $\displaystyle\int x\sqrt{25-x^2}\, dx$.

11. Compute $\displaystyle\int \frac{x^2}{(x^2-9)^{3/2}}\, dx$.

12. Compute $\displaystyle\int \frac{x^2}{\sqrt{x^2+1}}\, dx$.

13. Compute $\displaystyle\int \frac{3+4x+5x^2+3x^3}{x^2(x+3)}\, dx$.

14. Compute $\displaystyle\int x^3 e^{4x}\, dx$.

15. Compute $\displaystyle\int e^{4x}\cos 2x\, dx$.

16. Compute $\displaystyle\int \cos 3x \sin 2x\, dx$.

17. Compute $\displaystyle\int \frac{1}{x^{1/2}(x^{1/3}+x^{1/4})}\, dx$.

18. Compute $\displaystyle\int \frac{1}{x^{7/8}+x^{5/8}}\,dx$.

19. Compute $\displaystyle\int \frac{x-2}{x^2-8x+25}\,dx$.

20. Compute $\displaystyle\int \frac{x+3}{\sqrt{-x^2-6x-8}}\,dx$.

21. Compute $\displaystyle\int \frac{1}{2x^2+8x+10}\,dx$.

22. Compute $\displaystyle\int \frac{6x^3-24x^2+16x+4}{x^4-4x^3+4x^2}\,dx$.

23. Compute $\displaystyle\int \frac{4x^3+2x^2+16x+11}{(x^2+1)(x^2+4)}\,dx$.

24. How would you try to decompose $\dfrac{2(x-2)^2}{x^4(x^2+4)^3}$ using partial fractions? (Just write out the fractions — you don't need to solve for the parameters.)

25. What is wrong with the following "partial fractions decomposition"?

$$\frac{5x}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x+1}?$$

26. What is wrong with the following "partial fractions decomposition"?

$$\frac{7}{x(x-1)} = \frac{A}{x} + \frac{B}{x} + \frac{C}{x-1}.$$

27. Find the partial fractions decomposition of

$$\frac{-3x^4+x^3-6x^2-3}{x(x^2+1)^2}.$$

28. (a) Compute $\displaystyle\int \frac{-x^2+8x-4}{x(x^2+x-2)}\,dx$.

(b) Calvin Butterball tries to use the antiderivative from (a) to compute

$$\int_{-1}^{1/2} \frac{-x^2+8x-4}{x(x^2+x-2)}\,dx.$$

He gets

$$\left[2\ln|x| + \ln|x-1| - 4\ln|x+2|\right]_{-1}^{1/2} = \left(2\ln\frac{1}{2} + \ln\frac{1}{2} - 4\ln 32\right) - (2\ln 1 + \ln 2 - 4\ln 1) \approx -4.39445.$$

Does this computation make sense? Why or why not?

29. Find the area of the region under $y = \dfrac{x}{(x^2+1)^2}$ from $x=0$ to $\infty$.

30. Compute $\displaystyle\int_4^6 \frac{1}{\sqrt{x-4}}\,dx$.

31. Compute $\displaystyle\int_{-\infty}^{0} xe^{x^2}\,dx$.

32. Compute $\displaystyle\int_{0}^{\infty} xe^{-3x}\,dx$.

33. Compute $\displaystyle\int_{0}^{\infty} \cos 3x\,dx$.

34. Compute $\displaystyle\int_{2}^{11} \frac{1}{\sqrt[3]{x-3}}\,dx$.

35. Prove that $\displaystyle\int_{0}^{\infty} e^{-x^4}\,dx$ converges. [Hint: Compare the integral to $\displaystyle\int_{0}^{\infty} e^{-x}\,dx$.]

36. Prove that $\displaystyle\int_{0}^{\infty} \frac{(\sin x)^2}{x^2+1}\,dx$ converges. [Hint: Use comparison, starting with the fact that $(\sin x)^2 \le 1$.]

37. (a) Show that the following integrals both diverge:

$$\int_{0}^{\infty} x\,dx \quad \text{and} \quad \int_{-\infty}^{0} x\,dx.$$

(It follows that $\displaystyle\int_{-\infty}^{\infty} x\,dx$ diverges as well.)

(b) Show that $\displaystyle\lim_{b\to\infty}\int_{-b}^{b} x\,dx$ converges. (This is called the **Cauchy principal value** of the integral; this problem shows that $\displaystyle\lim_{b\to\infty}\int_{-b}^{b} x\,dx$ is not the same as $\displaystyle\int_{-\infty}^{\infty} x\,dx$.)

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# Solutions to the Review Problems for Test 1

1. Compute $\displaystyle\int_{0}^{\pi/2} (\cos x)^3 \left(1 + (\sin x)^{1/2}\right)\,dx$.

I'll do the antiderivative first:

$$\int (\cos x)^3 \left(1 + (\sin x)^{1/2}\right)\,dx = \int (\cos x)^2 \left(1 + (\sin x)^{1/2}\right) \cos x\,dx =$$

$$\int \left(1 - (\sin x)^2\right) \left(1 + (\sin x)^{1/2}\right) \cos x\,dx =$$

$$\left[u = \sin x, \quad du = \cos x\,dx, \quad dx = \frac{du}{\cos x}\right]$$

$$\int (1-u^2)(1+u^{1/2})\,du = \int (1 + u^{1/2} - u^2 - u^{5/2})\,du = u + \frac{2}{3}u^{3/2} - \frac{1}{3}u^3 - \frac{2}{7}u^{7/2} + C =$$

$$\sin x + \frac{2}{3}(\sin x)^{3/2} - \frac{1}{3}(\sin x)^3 - \frac{2}{7}(\sin x)^{7/2} + C.$$

Therefore,

$$\int_{0}^{\pi/2} (\cos x)^3 \left(1 + (\sin x)^{1/2}\right)\,dx = \left[\sin x + \frac{2}{3}(\sin x)^{3/2} - \frac{1}{3}(\sin x)^3 - \frac{2}{7}(\sin x)^{7/2}\right]_0^{\pi/2} = \frac{22}{21}. \quad \square$$

2. Compute $\int (\cos 7x)^4\, dx$.

$$\int (\cos 7x)^4\, dx = \int (\cos 7x)^2 (\cos 7x)^2\, dx = \int \frac{1}{2}(1+\cos 14x)\cdot\frac{1}{2}(1+\cos 14x)\, dx =$$

$$\frac{1}{4}\int \left(1+2\cos 14x + (\cos 14x)^2\right)\, dx = \frac{1}{4}\int \left(1+2\cos 14x+\frac{1}{2}(1+\cos 28x)\right)\, dx =$$

$$\frac{1}{4}\left(x+\frac{1}{7}\sin 14x+\frac{1}{2}(x+\frac{1}{28}\sin 28x)\right)+C. \quad \square$$

3. Compute $\int [\sin(x+3)]^2[\cos(x+3)]^2\, dx$.

$$\int [\sin(x+3)]^2[\cos(x+3)]^2\, dx = \int \left(\frac{1}{2}[1-\cos 2(x+3)]\right)\left(\frac{1}{2}[1+\cos 2(x+3)]\right)\, dx =$$

$$\frac{1}{4}\int \left(1-[\cos 2(x+3)]^2\right)\, dx = \frac{1}{4}\int [\sin 2(x+3)]^2\, dx = \frac{1}{4}\int \frac{1}{2}[1-\cos 4(x+3)]\, dx =$$

$$\frac{1}{8}\left(x-\frac{1}{4}\sin 4(x+3)\right)+C. \quad \square$$

4. Compute $\int (\sec 3x)^5 \tan 3x\, dx$.

$$\int (\sec 3x)^5\tan 3x\, dx = \int (\sec 3x)^4(\sec 3x\tan 3x\, dx) = \frac{1}{3}\int u^4\, du = \frac{1}{15}u^5+C = \frac{1}{15}(\sec 3x)^5+C.$$

$$\left[u=\sec 3x,\quad du = 3\sec 3x\tan 3x\, dx,\quad dx = \frac{du}{3\sec 3x\tan 3x}\right] \quad \square$$

5. Compute $\int (\tan x)^4\, dx$.

$$\int (\tan x)^4\, dx = \int (\tan x)^2(\tan x)^2\, dx = \int (\tan x)^2\left((\sec x)^2-1\right)\, dx =$$

$$\int (\tan x)^2(\sec x)^2\, dx - \int (\tan x)^2\, dx = \int (\tan x)^2(\sec x)^2\, dx - \int \left((\sec x)^2-1\right)\, dx =$$

$$\int (\tan x)^2(\sec x)^2\, dx - \int (\sec x)^2\, dx + \int\, dx =$$

$$\left[u=\tan x,\quad du = (\sec x)^2\, dx,\quad dx = \frac{du}{(\sec x)^2}\right]$$

$$\int u^2\, du - \tan x + x + C = \frac{1}{3}u^3 - \tan x + x + C = \frac{1}{3}(\tan x)^3 - \tan x + x + C. \quad \square$$

6. Compute $\int (\csc 2x)^3 (\cot 2x)^3 \, dx$.

$$\int (\csc 2x)^3 (\cot 2x)^3 \, dx = \int (\csc 2x)^2 (\cot 2x)^2 (\csc 2x \cot 2x) \, dx =$$

$$\int (\csc 2x)^2 \left[ (\csc 2x)^2 - 1 \right] (\csc 2x \cot 2x) \, dx = \int u^2 (u^2 - 1)(\csc 2x \cot 2x) \cdot \frac{du}{-2 \csc 2x \cot 2x} =$$

$$\left[ u = \csc 2x, \quad du = -2 \csc 2x \cot 2x \, dx, \quad dx = \frac{du}{-2 \csc 2x \cot 2x} \right]$$

$$-\frac{1}{2} \int u^2 (u^2 - 1) \, du = -\frac{1}{2} \int (u^4 - u^2) \, du = -\frac{1}{2} \left( \frac{1}{5} u^5 - \frac{1}{3} u^3 \right) + C = -\frac{1}{2} \left( \frac{1}{5} (\csc 2x)^5 - \frac{1}{3} (\csc 2x)^3 \right) + C. \quad \square$$

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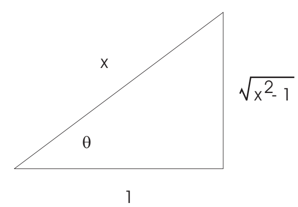
7. Compute $\int \sqrt{x^2 - 1} \, dx$.

$$\int \sqrt{x^2 - 1} \, dx = \int \sqrt{(\sec \theta)^2 - 1} \sec \theta \tan \theta \, d\theta = \int \sqrt{(\tan \theta)^2} \sec \theta \tan \theta \, d\theta = \int \sec \theta (\tan \theta)^2 \, d\theta =$$

$$[x = \sec \theta, \quad dx = \sec \theta \tan \theta \, d\theta]$$




$$\int \sec \theta \left( (\sec \theta)^2 - 1 \right) \, d\theta = \int \left( (\sec \theta)^3 - \sec \theta \right) \, d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta| + C =$$

$$\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = \frac{1}{2} x \sqrt{x^2 - 1} - \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| + C. \quad \square$$

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8. Compute $\int \ln(x^2 + 5) \, dx$.

| | $\frac{d}{dx}$ | | $\int dx$ |
|---|---|---|---|
| $+$ | $\ln(x^2 + 5)$ | $\searrow$ | $1$ |
| $-$ | $\frac{2x}{x^2 + 5}$ | | $x$ |

$$\int \ln(x^2 + 5) \, dx = x \ln(x^2 + 5) - \int \frac{2x^2}{x^2 + 5} \, dx = x \ln(x^2 + 5) - \int \left( 2 - \frac{10}{x^2 + 5} \right) \, dx =$$

$$x \ln(x^2 + 5) - 2x + \frac{10}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} + C.$$

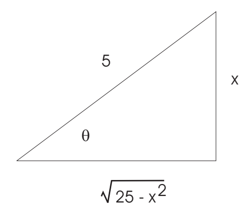
The second equality comes from dividing $2x^2$ by $x^2 + 5$ (long division). Alternatively, you can do this:

$$\frac{2x^2}{x^2 + 5} = \frac{2x^2 + 10 - 10}{x^2 + 5} = \frac{2x^2 + 10}{x^2 + 5} - \frac{10}{x^2 + 5} = \frac{2(x^2 + 5)}{x^2 + 5} - \frac{10}{x^2 + 5} = 2 - \frac{10}{x^2 + 5}. \quad \square$$

9. Compute $\int x^2\sqrt{25-x^2}\,dx$.

$$\int x^2\sqrt{25-x^2}\,dx = \int 25(\sin\theta)^2\sqrt{25-25(\sin\theta)^2}(5\cos\theta)\,d\theta = \int 25(\sin\theta)^2\sqrt{25(\cos\theta)^2}(5\cos\theta)\,d\theta =$$

$$[x = 5\sin\theta, \quad dx = 5\cos\theta\,d\theta]$$




$$625\int(\sin\theta)^2(\cos\theta)^2\,d\theta = 625\int\frac{1}{2}(1-\cos 2\theta)\cdot\frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{625}{4}\int\left(1-(\cos 2\theta)^2\right)\,d\theta =$$

$$\frac{625}{4}\int(\sin 2\theta)^2\,d\theta = \frac{625}{4}\int\frac{1}{2}(1-\cos 4\theta)\,d\theta = \frac{625}{8}(\theta - \frac{1}{4}\sin 4\theta) + C =$$

$$\frac{625}{8}\left(\theta - \sin\theta\cos\theta\left(2(\cos\theta)^2-1\right)\right) + C = \frac{625}{8}\arcsin\frac{x}{5} - \frac{1}{8}x\left(\sqrt{25-x^2}\right)(25-2x^2) + C. \quad \Box$$

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10. Compute $\int x\sqrt{25-x^2}\,dx$.

$$\int x\sqrt{25-x^2}\,dx = \int x\sqrt{u}\cdot\left(\frac{du}{-2x}\right) = -\frac{1}{2}\int u^{1/2}\,du = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(25-x^2)^{3/2} + C.$$

$$\left[u = 25-x^2, \quad du = -2x\,dx, \quad dx = \frac{du}{-2x}\right] \quad \Box$$

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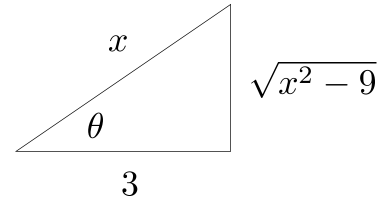
11. Compute $\int \frac{x^2}{(x^2-9)^{3/2}}\,dx$.

$$\int\frac{x^2}{(x^2-9)^{3/2}}\,dx = \int\frac{9(\sec\theta)^2}{(9(\sec\theta)^2-9)^{3/2}}\cdot 3\sec\theta\tan\theta\,d\theta = \int\frac{27(\sec\theta)^3\tan\theta}{27((\sec\theta)^2-1)^{3/2}}\,d\theta = \int\frac{(\sec\theta)^3\tan\theta}{((\tan\theta)^2)^{3/2}}\,d\theta =$$

$$[x = 3\sec\theta, \quad dx = 3\sec\theta\tan\theta\,d\theta]$$




$$\int\frac{(\sec\theta)^3\tan\theta}{(\tan\theta)^3}\,d\theta = \int\frac{(\sec\theta)^3}{(\tan\theta)^2}\,d\theta = \int\frac{1}{(\cos\theta)^3}\cdot\frac{(\cos\theta)^2}{(\sin\theta)^2}\,d\theta = \int\frac{1}{(\cos\theta)(\sin\theta)^2}\,d\theta =$$

$$\int \frac{(\sin\theta)^2 + (\cos\theta)^2}{(\cos\theta)(\sin\theta)^2}\,d\theta = \int \left(\frac{(\sin\theta)^2}{(\cos\theta)(\sin\theta)^2} + \frac{(\cos\theta)^2}{(\cos\theta)(\sin\theta)^2}\right)d\theta = \int \left(\frac{1}{\cos\theta} + \frac{\cos\theta}{(\sin\theta)^2}\right)d\theta =$$

$$\int (\sec\theta + \csc\theta\cot\theta)\,d\theta = \ln|\sec\theta + \tan\theta| - \csc\theta + c = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| - \frac{3}{\sqrt{x^2-9}} + c. \quad \square$$

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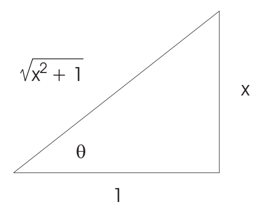
12. Compute $\int \frac{x^2}{\sqrt{x^2+1}}\,dx$.

$$\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \int \frac{(\tan\theta)^2}{\sqrt{(\tan\theta)^2+1}}(\sec\theta)^2\,d\theta = \int \frac{(\tan\theta)^2}{\sqrt{(\sec\theta)^2}}(\sec\theta)^2\,d\theta = \int \frac{(\tan\theta)^2}{\sec\theta}(\sec\theta)^2\,d\theta =$$

$$\left[x = \tan\theta, \quad dx = (\sec\theta)^2\,d\theta\right]$$




$$\int \sec\theta(\tan\theta)^2\,d\theta = \int \sec\theta\left((\sec\theta)^2 - 1\right)d\theta = \int \left((\sec\theta)^3 - \sec\theta\right)d\theta =$$

$$\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\ln|\sqrt{x^2+1} + x| + C. \quad \square$$

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13. Compute $\int \frac{3 + 4x + 5x^2 + 3x^3}{x^2(x+3)}\,dx$.

The top and the bottom both have degree 3, so I must divide the top by the bottom:

$$\frac{3 + 4x + 5x^2 + 3x^3}{x^2(x+3)} = 3 + \frac{3 + 4x - 4x^2}{x^2(x+3)}.$$

I'll put the 3 aside for now, and work on the fraction:

$$\frac{3 + 4x - 4x^2}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}.$$

Clear denominators:

$$3 + 4x - 4x^2 = Ax(x+3) + B(x+3) + Cx^2.$$

Set $x = 0$: I get $3 = 3B$, so $B = 1$.
Set $x = -3$: I get $-45 = 9C$, so $C = -5$.
Plug $B$ and $C$ back in:

$$3 + 4x - 4x^2 = Ax(x+3) + (x+3) - 5x^2.$$

Differentiate:

$$4 - 8x = A(x+3) + Ax + 1 - 10x.$$

Set $x = 0$: I $4 = 3A + 1$, so $A = 1$.
Therefore,

$$\frac{3 + 4x - 4x^2}{x^2(x+3)} = \frac{1}{x} + \frac{1}{x^2} - \frac{5}{x+3}.$$

Hence,

$$\frac{3+4x+5x^2+3x^3}{x^2(x+3)} = 3 + \frac{1}{x} + \frac{1}{x^2} - \frac{5}{x+3}.$$

Finally,

$$\int \frac{3+4x+5x^2+3x^3}{x^2(x+3)}\,dx = \int \left(3 + \frac{1}{x} + \frac{1}{x^2} - \frac{5}{x+3}\right)\,dx = 3x + \ln|x| - \frac{1}{x} - 5\ln|x+3| + C. \quad \square$$

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14. Compute $\int x^3 e^{4x}\,dx$.

$$\begin{array}{ccccc} & \frac{d}{dx} & & \int dx \\ + & x^3 & & e^{4x} \\ & & \searrow & \\ - & 3x^2 & & \frac{1}{4}e^{4x} \\ & & \searrow & \\ + & 6x & & \frac{1}{16}e^{4x} \\ & & \searrow & \\ - & 6 & & \frac{1}{64}e^{4x} \\ & & \searrow & \\ + & 0 & & \frac{1}{256}e^{4x} \end{array}$$

$$\int x^3 e^{4x}\,dx = \frac{1}{4}x^3e^{4x} - \frac{3}{16}x^2e^{4x} + \frac{6}{64}xe^{4x} - \frac{6}{256}e^{4x} + C. \quad \square$$

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15. Compute $\int e^{4x}\cos 2x\,dx$.

$$\begin{array}{ccccc} & \frac{d}{dx} & & \int dx \\ + & \cos 2x & & e^{4x} \\ & & \searrow & \\ - & -2\sin 2x & & \frac{1}{4}e^{4x} \\ & & \searrow & \\ + & -4\cos 2x & \rightarrow & \frac{1}{16}e^{4x} \end{array}$$

$$\int e^{4x}\cos 2x\,dx = \frac{1}{4}e^{4x}\cos 2x + \frac{1}{8}e^{4x}\sin 2x - \frac{1}{4}\int e^{4x}\cos 2x\,dx,$$

$$\int e^{4x}\cos 2x\,dx + \frac{1}{4}\int e^{4x}\cos 2x\,dx = \frac{1}{4}e^{4x}\cos 2x + \frac{1}{8}e^{4x}\sin 2x - \frac{1}{4}\int e^{4x}\cos 2x\,dx + \frac{1}{4}\int e^{4x}\cos 2x\,dx,$$

$$\frac{5}{4}\int e^{4x}\cos 2x\,dx = \frac{1}{4}e^{4x}\cos 2x + \frac{1}{8}e^{4x}\sin 2x,$$

$$\frac{4}{5}\cdot\frac{5}{4}\int e^{4x}\cos 2x\,dx = \frac{4}{5}\cdot\frac{1}{4}e^{4x}\cos 2x + \frac{4}{5}\cdot\frac{1}{8}e^{4x}\sin 2x,$$

$$\int e^{4x}\cos 2x\,dx = \frac{1}{5}e^{4x}\cos 2x + \frac{1}{10}e^{4x}\sin 2x + C. \quad \square$$

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16. Compute $\displaystyle\int \cos 3x\sin 2x\,dx$.

| | $\dfrac{d}{dx}$ | | $\displaystyle\int dx$ |
|---|---|---|---|
| $+$ | $\cos 3x$ | | $\sin 2x$ |
| | | $\searrow$ | |
| $-$ | $-3\sin 3x$ | | $-\dfrac{1}{2}\cos 2x$ |
| | | $\searrow$ | |
| $+$ | $-9\cos 3x$ | $\rightarrow$ | $-\dfrac{1}{4}\sin 2x$ |

$$\int \cos 3x\sin 2x\,dx = -\frac{1}{2}\cos 3x\cos 2x - \frac{3}{4}\sin 3x\sin 2x + \frac{9}{4}\int \cos 3x\sin 2x\,dx,$$

$$-\frac{5}{4}\int \cos 3x\sin 2x\,dx = -\frac{1}{2}\cos 3x\cos 2x - \frac{3}{4}\sin 3x\sin 2x,$$

$$\int \cos 3x\sin 2x\,dx = \frac{2}{5}\cos 3x\cos 2x + \frac{3}{5}\sin 3x\sin 2x + C. \quad \square$$

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17. Compute $\displaystyle\int \frac{1}{x^{1/2}(x^{1/3}+x^{1/4})}\,dx$.

Since the least common multiple of 2, 3, and 4 is 12, I'll let $x = u^{12}$:

$$\int \frac{1}{x^{1/2}(x^{1/3}+x^{1/4})}\,dx = \int \frac{12u^{11}\,du}{u^6(u^4+u^3)} = 12\int \frac{u^2}{u+1}\,du = 12\int \left(u - 1 + \frac{1}{u+1}\right)du =$$

$$\left[x = u^{12}, \quad dx = 12u^{11}\,du\right]$$

$$12\left(\frac{1}{2}u^2 - u + \ln|u+1|\right) + C = 12\left(\frac{1}{2}x^{1/6} - x^{1/12} + \ln|x^{1/12}+1|\right) + C. \quad \square$$

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18. Compute $\displaystyle\int \frac{1}{x^{7/8}+x^{5/8}}\,dx$.

$$\int \frac{1}{x^{7/8}+x^{5/8}}\,dx = \int \frac{1}{u^7+u^5}\cdot 8u^7\,du = 8\int \frac{u^2}{u^2+1}\,du = 8\int \frac{(u^2+1)-1}{u^2+1}\,du =$$

$$\left[x = u^8, \quad dx = 8u^7\,du; \quad u = x^{1/8}\right]$$

$$8\int \left(\frac{u^2+1}{u^2+1} - \frac{1}{u^2+1}\right)du = 8\int \left(1 - \frac{1}{u^2+1}\right)du = 8\left(u - \tan^{-1}u\right) + c = 8\left(x^{1/8} - \tan^{-1}x^{1/8}\right) + c. \quad \square$$

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19. Compute $\displaystyle\int \frac{x-2}{x^2-8x+25}\,dx$.

Since $\frac{1}{2} \cdot (-8) = -4$ and $(-4)^2 = 16$, I have

$$x^2 - 8x + 25 = x^2 - 8x + 16 + 9 = (x-4)^2 + 9.$$

Therefore,

$$\int \frac{x-2}{x^2 - 8x + 25}\,dx = \int \frac{x-2}{(x-4)^2+9}\,dx = \int \frac{(u+4)-2}{u^2+9}\,du = \int \frac{u+2}{u^2+9}\,du =$$

$$\left[u = x-4, \quad du = dx; \quad x = u+4\right]$$

$$\int \frac{u}{u^2+9}\,du + \int \frac{2}{u^2+9}\,du = \frac{1}{2}\ln|u^2+9| + \frac{2}{3}\tan^{-1}\frac{u}{3} + C = \frac{1}{2}\ln|(x-4)^2+9| + \frac{2}{3}\tan^{-1}\frac{x-4}{3} + C.$$

I did the first part of the $u$-integral using the substitution $w = u^2 + 9$. □

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20. Compute $\displaystyle\int \frac{x+3}{\sqrt{-x^2-6x-8}}\,dx$.

First,

$$-x^2 - 6x - 8 = -(x^2+6x+8) = -\left[(x^2+6x+9) - 1\right] = -\left[(x+3)^2 - 1\right] = 1 - (x+3)^2.$$

I note that $\frac{6}{2} = 3$ and $3^2 = 9$, so I needed 9 to complete the square.
Thus,

$$\int \frac{x+3}{\sqrt{-x^2-6x-8}}\,dx = \int \frac{x+3}{\sqrt{1-(x+3)^3}}\,dx = \int \frac{u}{\sqrt{1-u^2}}\,du = \int \frac{u}{\sqrt{w}} \cdot \frac{dw}{-2u} =$$

$$\left[u = x+3, \quad du = dx; w = 1-u^2, \quad dw = -2u\,du, \quad du = \frac{dw}{-2u}\right]$$

$$-\frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw = -\frac{1}{2} \cdot 2\sqrt{w} + C = -\sqrt{1-u^2} + C = -\sqrt{1-(x+3)^2} + C. \quad \square$$

---

21. Compute $\displaystyle\int \frac{1}{2x^2+8x+10}\,dx$.

$$\int \frac{1}{2x^2+8x+10}\,dx = \frac{1}{2}\int \frac{1}{x^2+4x+5}\,dx = \frac{1}{2}\int \frac{1}{(x^2+4x+4)+1}\,dx = \frac{1}{2}\int \frac{1}{(x+2)^2+1}\,dx =$$

$$\frac{1}{2}\tan^{-1}(x+2) + C.$$

I completed the square by noting that $\frac{4}{2} = 2$ and $2^2 = 4$. You can do the integral using $u = x+2$. □

---

22. Compute $\displaystyle\int \frac{6x^3 - 24x^2 + 16x + 4}{x^4 - 4x^3 + 4x^2}\,dx$.

First, $x^4 - 4x^3 + 4x^2 = x^2(x-2)^2$.

$$\frac{6x^3 - 24x^2 + 16x + 4}{x^2(x-2)^2} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-2} + \frac{d}{(x-2)^2}$$
$$6x^3 - 24x^2 + 16x + 4 = ax(x-2)^2 + b(x-2)^2 + cx^2(x-2) + dx^2$$

Set $x = 0$. I get $4 = 4b$, so $b = 1$.
Set $x = 2$. I get $-12 = 4d$, so $d = -3$.
Then
$$6x^3 - 24x^2 + 16x + 4 = ax(x-2)^2 + (x-2)^2 + cx^2(x-2) - 3x^2.$$

At this point, you can plug other numbers in for $x$, or differentiate the equation and then plug numbers in. The idea is to get equations for $a$ and $c$ which you can solve.

For example, set $x = 1$. I get
$$2 = a + 1 - c - 3, \quad \text{or} \quad 4 = a - c.$$

Set $x = -1$. I get
$$-42 = -9a + 9 - 3c - 3, \quad \text{or} \quad 16 = 3a + c.$$

I have to solve $4 = a - c$ and $16 = 3a + c$. You can do this in various ways.

For instance, if I add the equations $4 = a - c$ and $16 = 3a + c$, I get $20 = 4a$, so $a = 5$. Then plugging $a = 5$ into $4 = a - c$, I get $4 = 5 - c$, so $c = 1$.

Thus,
$$\int \frac{6x^3 - 24x^2 + 16x + 4}{x^4 - 4x^3 + 4x^2}\,dx = \int \left(\frac{5}{x} + \frac{1}{x^2} + \frac{1}{x-2} - \frac{3}{(x-2)^2}\right) dx = 5\ln|x| - \frac{1}{x} + \ln|x-2| + \frac{3}{x-2} + C. \quad \square$$

---

23. Compute $\displaystyle\int \frac{4x^3 + 2x^2 + 16x + 11}{(x^2+1)(x^2+4)}\,dx$.

$$\frac{4x^3 + 2x^2 + 16x + 11}{(x^2+1)(x^2+4)} = \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2+4}$$
$$4x^3 + 2x^2 + 16x + 11 = (ax+b)(x^2+4) + (cx+d)(x^2+1)$$

Set $x = 0$: This gives
$$11 = 4b + d. \tag{1}$$

Differentiate the last $x$-equation (using the Product Rule on the two terms on the right):
$$12x^2 + 4x + 16 = (ax+b)(2x) + a(x^2+4) + (cx+d)(2x) + c(x^2+1).$$

Set $x = 0$:
$$16 = 4a + c. \tag{2}$$

Differentiate the last $x$-equation:
$$24x + 4 = (ax+b)(2) + (a)(2x) + (a)(2x) + (cx+d)(2) + (c)(2x) + (c)(2x).$$

Set $x = 0$:
$$4 = 2b + 2d, \quad \text{so} \quad 2 = b + d. \tag{3}$$

Differentiate the last $x$-equation:
$$24 = 2a + 2a + 2a + 2c + 2c + 2c, \quad \text{so} \quad 4 = a + c. \tag{4}$$

Solving (1) ($11 = 4b + d$) together with (3) ($2 = b + d$) gives $b = 3$ and $d = -1$.
Solving (2) ($16 = 4a + c$) together with (4) ($4 = a + c$) gives $a = 4$ and $c = 0$.
Thus, I have

$$\int \frac{4x^3 + 2x^2 + 16x + 11}{(x^2+1)(x^2+4)}\,dx = \int \left(\frac{4x}{x^2+1} + \frac{3}{x^2+1} - \frac{1}{x^2+4}\right) dx = 2\ln(x^2+1) + 3\tan^{-1} x - \frac{1}{2}\tan^{-1}\frac{x}{2} + C.$$

The first integral is computed using $u = x^2 + 1$; the second and third use the inverse tangent formula:

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C. \quad \square$$

---

24. How would you try to decompose $\dfrac{2(x-2)^2}{x^4(x^2+4)^3}$ using partial fractions?

$$\frac{2(x-2)^2}{x^4(x^2+4)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{Ex+F}{x^2+4} + \frac{Gx+H}{(x^2+4)^2} + \frac{Ix+J}{(x^2+4)^3}. \quad \square$$

---

25. What is wrong with the following "partial fractions decomposition"?

$$\frac{5x}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x+1}?$$

Partial fractions is the opposite of combining fractions over a common denominator. In this case, the question is: "What fractions would add up to $\dfrac{5x}{(x-1)^2(x+1)}$?" The decompositions above *could* occur, since it has $(x-1)^2(x+1)$ as the common denominator.

However, since you don't know beforehand what the fractions are, you must assume the "worst case" — namely, that there might be an $\dfrac{A}{x-1}$ term. And in fact, there is — if you work out the decomposition, it comes out to

$$\frac{5x}{(x-1)^2(x+1)} = -\frac{5}{4}\frac{1}{x+1} + \frac{5}{4}\frac{1}{x-1} + \frac{5}{2}\frac{1}{(x-1)^2}.$$

Notice the term $\dfrac{5}{4}\dfrac{1}{x-1}$. $\square$

---

26. What is wrong with the following "partial fractions decomposition"?

$$\frac{7}{x(x-1)} = \frac{A}{x} + \frac{B}{x} + \frac{C}{x-1}.$$

The first two terms could be combined into a single term $\dfrac{D}{x}$, so they're redundant. There is no reason to list the same denominator twice. $\square$

---

27. Find the partial fractions decomposition of

$$\frac{-3x^4 + x^3 - 6x^2 - 3}{x(x^2+1)^2}.$$

Try the decomposition

$$\frac{-3x^4 + x^3 - 6x^2 - 3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{(x^2+1)^2}.$$

Clear denominators:

$$-3x^4 + x^3 - 6x^2 - 3 = A(x^2+1)^2 + (Bx + C)(x)(x^2+1) + (Dx + E)(x).$$

Set $x = 0$: I get $A = -3$. Plug it back in:

$$-3x^4 + x^3 - 6x^2 - 3 = -3(x^2+1)^2 + (Bx + C)(x)(x^2+1) + (Dx + E)(x).$$

Differentiate:

$$12x^3 + 3x^2 - 12x = -12x(x^2+1) + B(x^3 + x) + (Bx + C)(3x^2 + 1) + 2Dx + E.$$

Set $x = 0$: I get $C + E = 0$.
Differentiate again:

$$-36x^2 + 6x - 12 = -36x^2 - 12 + B(3x^2 + 1) + B(3x^2 + 1) + (Bx + C)(6x) + 2D.$$

Set $x = 0$: I get $B + D = 0$.
Cancel the $-36x^2$ and $-12$ terms in the previous equation, then differentiate:

$$6x = B(3x^2 + 1) + B(3x^2 + 1) + (Bx + C)(6x) + 2D,$$

$$6 = 6Bx + 6Bx + 6Bx + (Bx + C)(6).$$

Set $x = 0$: I get $C = 1$. Since $C + E = 0$, it follows that $E = -1$.
Plug $C = 1$ back in, then simplify the equation:

$$6 = 24Bx + 6, \qquad \text{or} \qquad 0 = 24Bx.$$

Set $x = 1$: I get $B = 0$. But $B + D = 0$, so $D = 0$.
Hence,

$$\frac{-3x^4 + x^3 - 6x^2 - 3}{x(x^2+1)^2} = -\frac{3}{x} + \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}. \quad \square$$

---

28. (a) Compute $\displaystyle\int \frac{-x^2 + 8x - 4}{x(x^2 + x - 2)}\,dx$.

(b) Calvin Butterball tries to use the antiderivative from (a) to compute

$$\int_{-1}^{1/2} \frac{-x^2 + 8x - 4}{x(x^2 + x - 2)}\,dx.$$

He gets

$$\left[2\ln|x| + \ln|x-1| - 4\ln|x+2|\right]_{-1}^{1/2} = \left(2\ln\frac{1}{2} + \ln\frac{1}{2} - 4\ln 32\right) - (2\ln 1 + \ln 2 - 4\ln 1) \approx -4.39445.$$

Does this computation make sense? Why or why not?

(a) $x(x^2+x-2) = x(x-1)(x+2)$, so I try

$$\frac{-x^2+8x-4}{x(x^2+x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}.$$

Clear denominators:

$$-x^2+8x-4 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1).$$

Set $x = 0$: I get $-4 = -2A$, or $A = 2$.
Set $x = 1$: I get $3 = 3B$, or $B = 1$.
Set $x = -2$: I get $-24 = 6C$, or $C = -4$.
Therefore,

$$\int \frac{-x^2+8x-4}{x(x^2+x-2)}\,dx = \int \left(\frac{2}{x} + \frac{1}{x-1} - \frac{4}{x+2}\right) dx = 2\ln|x| + \ln|x-1| - 4\ln|x+2| + C. \quad \square$$

(b) The computation is incorrect, because the antiderivative is valid only within intervals which don't contain the singularities at $x = -2$, $x = 0$, and $x = 1$. The interval $-1 \le x \le \dfrac{1}{2}$ includes the singularity at $x = 0$. It is not legal to simply "plug in the endpoints" — to do this definite integral correctly, you should set it up as two **improper integrals**. $\square$

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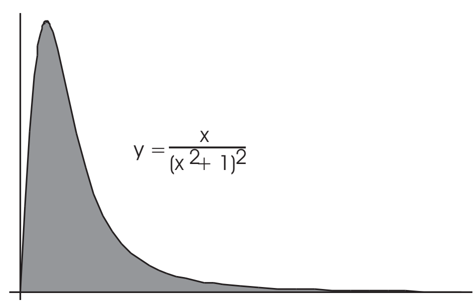
29. Find the area of the region under $y = \dfrac{x}{(x^2+1)^2}$ from $x = 0$ to $\infty$.




The area is

$$A = \int_0^\infty \frac{x}{(x^2+1)^2}\,dx = \lim_{c\to\infty} \int_0^c \frac{x}{(x^2+1)^2}\,dx = \lim_{c\to\infty} \left[-\frac{1}{2(x^2+1)}\right]_0^c = -\frac{1}{2}\lim_{c\to\infty}\left(\frac{1}{c^2+1} - 1\right) = \frac{1}{2}.$$

Here's the work for the antiderivative:

$$\int \frac{x}{(x^2+1)^2}\,dx = \int \frac{x}{u^2}\cdot\frac{du}{2x} = \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u} + C = -\frac{1}{2(x^2+1)} + C.$$

$$\left[u = x^2+1, \quad du = 2x\,dx, \quad dx = \frac{du}{2x}\right] \quad \square$$

---

30. Compute $\displaystyle\int_4^6 \frac{1}{\sqrt{x-4}}\,dx$.

Since $\dfrac{1}{\sqrt{x-4}}$ is undefined at $x = 4$ and $x = 4$ is in the interval of integration (it's one of the endpoints), the integral is improper. I replace the "4" with a parameter $a$, then take the limit as $a$ approaches 4 from the right.

$$\int_4^6 \frac{1}{\sqrt{x-4}}\,dx = \lim_{a\to 4^+}\int_a^6 \frac{1}{\sqrt{x-4}}\,dx = \lim_{a\to 4^+}\left[2\sqrt{x-4}\right]_a^6 = 2\lim_{a\to 4^+}\left(\sqrt{2}-\sqrt{a-4}\right) = 2\sqrt{2}.\quad \square$$

---

31. Compute $\displaystyle\int_{-\infty}^0 xe^{x^2}\,dx$.

$$\int_{-\infty}^0 xe^{x^2}\,dx = \lim_{b\to-\infty}\int_b^0 xe^{x^2}\,dx = \lim_{b\to-\infty}\int_{b^2}^0 xe^u\cdot\frac{du}{2x} = \frac{1}{2}\lim_{b\to-\infty}\int_{b^2}^0 e^u\,du =$$

$$\left[u = x^2,\quad du = 2x\,dx,\quad dx = \frac{du}{2x};\quad x = b,\quad u = b^2;\quad x = 0,\quad u = 0\right]$$

$$\frac{1}{2}\lim_{b\to-\infty}\left[e^u\right]_{b^2}^0 = \frac{1}{2}\lim_{b\to-\infty}\left(1-e^{b^2}\right).$$

As $b \to -\infty$, I have $b^2 \to \infty$, and $e^{b^2} \to \infty$. Therefore, the integral diverges (to $-\infty$, since the $e^{b^2}$ term was negated). $\square$

---

32. Compute $\displaystyle\int_0^\infty xe^{-3x}\,dx$.

$$\int_0^\infty xe^{-3x}\,dx = \lim_{b\to+\infty}\int_0^b xe^{-3x}\,dx = \lim_{b\to+\infty}\left[-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x}\right]_0^b =$$

$$\lim_{b\to+\infty}\left(-\frac{1}{3}be^{-3b} - \frac{1}{9}e^{-3b} + \frac{1}{9}\right) = -0 - 0 + \frac{1}{9} = \frac{1}{9}.$$

Here's the work for the two limits:

$$\lim_{b\to+\infty} e^{-3b} = \lim_{b\to+\infty}\frac{1}{e^{3b}} = 0,$$

$$\lim_{b\to+\infty} be^{-3b} = \lim_{b\to+\infty}\frac{b}{e^{3b}} = \lim_{b\to+\infty}\frac{1}{3e^{3b}} = 0.$$

I used L'Hôpital's Rule to compute the second limit.
Here's the work for the antiderivative:

| | $\dfrac{d}{dx}$ | | $\displaystyle\int dx$ |
|---|---|---|---|
| $+$ | $x$ | | $e^{-3x}$ |
| | | $\searrow$ | |
| $-$ | $1$ | | $-\dfrac{1}{3}e^{-3x}$ |
| | | $\searrow$ | |
| $+$ | $0$ | | $\dfrac{1}{9}e^{-3x}$ |

$$\int xe^{-3x}\,dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C.\quad \square$$

33. Compute $\int_0^\infty \cos 3x\,dx$.

$$\int_0^\infty \cos 3x\,dx = \lim_{k\to\infty}\int_0^k \cos 3x\,dx = \lim_{k\to\infty}\left[\frac{1}{3}\sin 3x\right]_0^k = \lim_{k\to\infty}\frac{1}{3}\sin 3k.$$

$\lim_{k\to\infty}\sin 3k$ is undefined. Hence, the integral diverges. (It doesn't diverge to $\infty$ or $-\infty$; the limit is simply undefined.) $\square$

---

34. Compute $\int_2^{11} \frac{1}{\sqrt[3]{x-3}}\,dx$.

$$\int_2^{11} \frac{1}{\sqrt[3]{x-3}}\,dx = \int_2^{3} \frac{1}{\sqrt[3]{x-3}}\,dx + \int_3^{11} \frac{1}{\sqrt[3]{x-3}}\,dx.$$

The first integral is

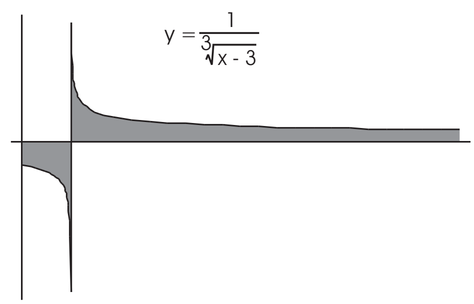
$$\int_2^{3} \frac{1}{\sqrt[3]{x-3}}\,dx = \lim_{a\to 3-}\int_2^{a} \frac{1}{\sqrt[3]{x-3}}\,dx = \lim_{a\to 3-}\left[\frac{3}{2}(x-3)^{2/3}\right]_2^a = \frac{3}{2}\lim_{a\to 3-}\left((a-3)^{2/3}-1\right) = -\frac{3}{2}.$$

The second integral is

$$\int_3^{11} \frac{1}{\sqrt[3]{x-3}}\,dx = \lim_{b\to 3+}\int_b^{11} \frac{1}{\sqrt[3]{x-3}}\,dx = \lim_{b\to 3+}\left[\frac{3}{2}(x-3)^{2/3}\right]_b^{11} = \frac{3}{2}\lim_{b\to 3+}\left(4-(b-3)^{2/3}\right) = 6.$$

Therefore,

$$\int_2^{11} \frac{1}{\sqrt[3]{x-3}}\,dx = -\frac{3}{2}+6 = \frac{9}{2}.$$




The graph of $\frac{1}{\sqrt[3]{x-3}}$ has a vertical asymptote at $x = 3$, but the (signed) area on each side is finite. The negative area to the left of $x = 3$ partially cancels the positive area to the right of $x = 3$. Thus, the integral in this problem does not represent the actual area bounded by the graph, the $x$-axis, and the lines $x = 2$, $x = 3$, and $x = 11$. $\square$

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In the next few problems, I'll use the following fact. If $f$ and $g$ are integrable functions on every finite interval $[a, b]$ and $f(x) \ge g(x) \ge 0$, then if $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ converges.

Intuitively, if the bigger function's integral converges to a number, then the smaller function's integral must converge, because it's caught between that number and 0.

35. Prove that $\displaystyle\int_0^\infty e^{-x^4}\,dx$ converges.

The interval of integration is $x \ge 0$. On this interval, $x \le x^4$, so $-x \ge -x^4$, and $e^{-x} \ge e^{-x^4}$. Therefore, if $\displaystyle\int_0^\infty e^{-x}\,dx$ converges, then $\displaystyle\int_0^\infty e^{-x^4}\,dx$ cconverges.

Now

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}\left(-e^{-b}+1\right) = -0+1 = 1.$$

Since $\displaystyle\int_0^\infty e^{-x}\,dx$ converges, it follows that $\displaystyle\int_0^\infty e^{-x^4}\,dx$ converges as well. □

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36. Prove that $\displaystyle\int_0^\infty \frac{(\sin x)^2}{x^2+1}\,dx$ converges.

I have

$$(\sin x)^2 \le 1$$

$$\frac{(\sin x)^2}{x^2+1} \le \frac{1}{x^2+1}$$

(I built up from a known fact about trig functions to get an inequality with the function I'm trying to integrate on the "small" side.)

Now

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\left[\tan^{-1} x\right]_0^b = \frac{\pi}{2}.$$

Since $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx$ converges and $\dfrac{1}{x^2+1} \ge \dfrac{(\sin x)^2}{x^2+1} \ge 0$, it follows that $\displaystyle\int_0^\infty \frac{(\sin x)^2}{x^2+1}\,dx$ converges as well. □

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37. (a) Show that the following integrals both diverge:

$$\int_0^\infty x\,dx \quad\text{and}\quad \int_{-\infty}^0 x\,dx.$$

(It follows that $\displaystyle\int_{-\infty}^\infty x\,dx$ diverges as well.)

(b) Show that $\displaystyle\lim_{b\to\infty}\int_{-b}^b x\,dx$ converges. (This is called the **Cauchy principal value** of the integral; this problem shows that $\displaystyle\lim_{b\to\infty}\int_{-b}^b x\,dx$ is not the same as $\displaystyle\int_{-\infty}^\infty x\,dx$.)

(a)

$$\int_0^\infty x\,dx = \lim_{b\to\infty}\int_0^b x\,dx = \lim_{b\to\infty}\left[\frac{1}{2}x^2\right]_0^b = \lim_{b\to\infty}\frac{1}{2}b^2 = \infty.$$

$$\int_{-\infty}^0 x\,dx = \lim_{b\to-\infty}\int_b^0 x\,dx = \lim_{b\to-\infty}\left[\frac{1}{2}x^2\right]_b^0 = \lim_{b\to-\infty}\left(-\frac{1}{2}b^2\right) = -\infty.\quad □$$

(b)

$$\lim_{b\to\infty}\int_{-b}^b x\,dx = \lim_{b\to\infty}\left[\frac{1}{2}x^2\right]_{-b}^b = \lim_{b\to\infty}\left(\frac{1}{2}b^2 - \frac{1}{2}b^2\right) = \lim_{b\to\infty} 0 = 0.\quad □$$

*Whatever is worth doing at all is worth doing well.* - PHILLIP STANHOPE