Review Problems for Test 2

These problems are provided to help you study. The presence of a problem on this sheet does not imply that a similar problem will appear on the test. And the absence of a problem from this sheet does not imply that the test will not have a similar problem.

1. Find the area of the region bounded by the graphs of \( y = x^2 - 3x \) and \( y = 15 - x \).

2. Find the area of the region between \( y = x^2 - x \) and \( y = x + 8 \) from \( x = 0 \) to \( x = 5 \).

3. The region bounded by \( y = 4x - x^2 \) and the \( x \)-axis is revolved about the \( x \)-axis. Find the volume of the solid that is generated.

4. Consider the region in the \( x-y \) plane bounded by \( y = e^x \), the line \( y = 1 \), and the line \( x = 1 \). Find the volume generated by revolving the region:
   (a) About the line \( y = 1 \).
   (b) About the line \( x = 2 \).
   (c) About the line \( y = e \).

5. The base of a solid is the region in the \( x-y \) plane bounded by the curves \( y = x^2 \) and \( y = x + 2 \). The cross-sections of the solid perpendicular to the \( x-y \) plane and the \( x \)-axis are isosceles right triangles with one leg in the \( x-y \) plane. Find the volume of the solid.

6. The region which lies above the \( x \)-axis and below the graph of \( y = \frac{1}{x^2 + 1} \), \(-\infty < x < \infty \), is revolved about the \( x \)-axis. Find the volume of the solid which is generated.
   \[ \int \frac{1}{(x^2 + 1)^2} \, dx = \frac{1}{2} \frac{x}{x^2 + 1} + \frac{1}{2} \tan^{-1} x + C. \]

7. The base of a rectangular tank is 2 feet long and 3 feet wide; the tank is 6 feet high. Find the work done in pumping all the water out of the top of the tank.

8. Write a formula for the \( n \)-th term of the sequence, assuming that the terms continue in the “obvious” way.
   (a) 7, 11, 15, 19, 23, 27, . . .
   (b) \( \frac{2}{8}, \frac{4}{13}, \frac{6}{18}, \frac{8}{23}, \ldots \).

9. Determine whether the sequence converges or diverges; if it converges, find the limit.
   (a) \( \{1.0001^n\} \).
   (b) \( \left\{ \frac{e^n + 3^n}{2^n + \pi^n} \right\} \).
   (c) \( \left\{ \frac{2n^3 - 5n + 7}{7n^2 - 13n^3} \right\} \).
   (d) \( \{\arctan n\}^2 \).
10. A sequence is defined recursively by

\[ a_1 = 5, \quad a_{n+1} = \sqrt{6a_n + 27} \quad \text{for} \quad n \geq 1. \]

Find \( \lim_{n \to \infty} a_n \).

11. In each case, determine whether the series converges or diverges. You should cite the test you’re using by name (to avoid ambiguity); be sure you verify that the hypotheses of the test apply.

(a) \( \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} \).

(b) \( \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} \).

(c) \( \sum_{k=3}^{\infty} \frac{k^2 - 3k + 2}{k^4} \).

(d) \( \sum_{k=2}^{\infty} \frac{2}{k^2 - 1} \).

(e) \( \sum_{k=1}^{\infty} \frac{2}{3k + 5} \).

(f) \( \sum_{k=1}^{\infty} \left( \frac{k + 1}{k^2 + 2} \right)^2 \).

(g) \( \sum_{k=1}^{\infty} \left( \sin \frac{1}{k} \right)^2 \).

(h) \( \sum_{k=1}^{\infty} \frac{2^k}{3^k + 2} \).

(i) \( \sum_{i=1}^{\infty} 3^i i^3 \).

(j) \( \sum_{k=1}^{\infty} \left( \frac{2k + 2}{2k - 1} \right)^{k^2} \).

(k) \( \sum_{k=1}^{\infty} \sqrt{\arctan k} \).

(l) \( \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k + 1)!} \).

12. (a) Find the partial fractions decomposition of \( \frac{2}{(2k + 1)(2k + 3)} \).

(b) Use (a) to find the sum of the series

\[ \sum_{k=1}^{\infty} \frac{2}{(2k + 1)(2k + 3)}. \]
13. If the series \( \sum_{k=17}^{\infty} a_k \), converges, does the series \( \sum_{k=1}^{\infty} a_k \) converge?

14. Does the series \( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k + 1}{4k + 3} \) converge?

15. Show that the Ratio Test always fails for a \( p \)-series

\[
\sum_{k=1}^{\infty} \frac{1}{k^p}, \quad \text{where} \quad p > 0.
\]

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**Solutions to the Review Problems for Test 2**

1. Find the area of the region bounded by the graphs of \( y = x^2 - 3x \) and \( y = 15 - x \).

![Graph of the region bounded by the curves](image)

The curves intersect at \( x = -3 \) and at \( x = 5 \):

\[
x^2 - 3x = 15 - x, \quad x^2 - 2x - 15 = 0, \quad (x - 5)(x + 3) = 0, \quad x = 5 \quad \text{or} \quad x = -3.
\]

\( y = 15 - x \) is the top curve and \( y = x^2 - 3x \) is the bottom curve. Hence, the area is

\[
\int_{-3}^{5} ((15 - x) - (x^2 - 3x)) \, dx = \int_{-3}^{5} (15 + 2x - x^2) \, dx = \left[ 15x + x^2 - \frac{1}{3}x^3 \right]_{-3}^{5} = \frac{256}{3}.
\]

2. Find the area of the region between \( y = x^2 - x \) and \( y = x + 8 \) from \( x = 0 \) to \( x = 5 \).

![Graph of the region bounded by the curves](image)

The curves intersect at \( x = 4 \) and \( x = -2 \):

\[
x^2 - x = x + 8, \quad x^2 - 2x - 8 = 0, \quad (x - 4)(x + 2) = 0, \quad x = 4 \quad \text{or} \quad x = -2.
\]
Since the curves cross between 0 and 5, I will need two integrals. On the left-hand piece, the top curve is \( y = x + 8 \) and the bottom curve is \( y = x^2 - x \). On the right-hand piece, the top curve is \( y = x^2 - x \) and the bottom curve is \( y = x + 8 \).

The area is

\[
\int_0^4 (x + 8 - (x^2 - x)) \, dx + \int_4^5 ((x^2 - x) - (x + 8)) \, dx = \int_0^4 (-x^2 + 2x + 8) \, dx + \int_4^5 (x^2 - 2x - 8) \, dx = \\
\left[ \frac{1}{3}x^3 + x^2 + 8x \right]_0^4 + \left[ \frac{1}{3}x^3 - x^2 - 8x \right]_4^5 = 30.
\]

3. The region bounded by \( y = 4x - x^2 \) and the \( x \)-axis is revolved about the \( x \)-axis. Find the volume of the solid that is generated.

![Diagram of a region bounded by a curve and the x-axis, with a slice labeled r=e^x-1.](image)

The region extends from \( x = 0 \) to \( x = 4 \). I’ll use circular slices. The radius of a typical slice is \( r = y = 4x - x^2 \). The area of a typical slice is

\[
\pi r^2 = \pi (4x - x^2)^2 = \pi (16x^2 - 8x^3 + x^4).
\]

The volume generated is

\[
V = \int_0^4 \pi (16x^2 - 8x^3 + x^4) \, dx = \pi \left[ \frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5 \right]_0^4 = \frac{512\pi}{15} \approx 107.23303.
\]

4. Consider the region in the \( x \)-\( y \) plane bounded by \( y = e^x \), the line \( y = 1 \), and the line \( x = 1 \). Find the volume generated by revolving the region:

(a) About the line \( y = 1 \).

Since the solid has no “holes” or “gaps” in its interior, I can use circular slices. The radius of a slice is \( r = e^x - 1 \), so the volume is

\[
V = \int_0^1 \pi (e^x - 1)^2 \, dx = \pi \int_0^1 (e^{2x} - 2e^x + 1) \, dx = \pi \left[ \frac{1}{2}e^{2x} - 2e^x + x \right]_0^1 = \frac{\pi e^2}{2} - 2\pi + \frac{5\pi}{2} \approx 2.38122.
\]
(b) About the line \( x = 2 \).

I’ll use cylindrical shells. The height is \( h = e^x - 1 \), and the radius is \( r = 2 - x \). The volume is

\[
V = \int_0^1 2\pi(e^x - 1)(2 - x) \, dx = 2\pi \int_0^1 (2e^x - 2 - xe^x + x) \, dx = 2\pi \left[ 2e^x - 2x - xe^x + e^x + \frac{1}{2}x^2 \right]_0^1 = 4\pi e - 9\pi \approx 5.88460.
\]

Here’s the work for part of the integral:

\[
\frac{d}{dx} \int dx + \begin{array}{c} x \\ \vdash \end{array} e^x - 1 \quad e^x
\]

\[
\int xe^x \, dx = xe^x - e^x + C.
\]

(c) About the line \( y = e \).

I’ll use cylindrical shells. Since \( y = e^x \) gives \( x = \ln y \), the height is \( h = 1 - x = 1 - \ln y \), and the radius is \( r = e - y \). The vertical limits on the region are \( y = 1 \) and \( y = e \). The volume is

\[
V = \int_1^e 2\pi(1 - \ln y)(e - y) \, dy = 2\pi \int_1^e (e - e \ln y - y + y \ln y) \, dy = 2\pi \left[ ey - ey \ln y + ey - \frac{1}{2}y^2 + \frac{1}{2}y^2 \ln y - \frac{1}{4}y^2 \right]_1^e = \frac{3\pi e^2}{2} - 4\pi e + \frac{3\pi}{2} \approx 5.37356.
\]
Here is how I did two of the pieces of the integral:

\[
\frac{d}{dy} \int dy = \ln y \quad 1
\]

\[
- \frac{1}{y} \rightarrow y
\]

\[
\int \ln y \, dy = y \ln y - \int dy = y \ln y - y + C.
\]

5. The base of a solid is the region in the x-y plane bounded by the curves \( y = x^2 \) and \( y = x + 2 \). The cross-sections of the solid perpendicular to the x-y plane and the x-axis are isosceles right triangles with one leg in the x-y plane. Find the volume of the solid.

The first picture shows the base of the solid. The second picture shows three typical triangular slices standing on the base.

\( x^2 = x + 2 \) gives \( x^2 - x - 2 = 0 \), so \( (x - 2)(x + 1) = 0 \) and \( x = 2 \) or \( x = -1 \). Therefore, the base of the solid extends from \( x = -1 \) to \( x = 2 \).

The leg of a triangular slice has length \( x + 2 - x^2 \). Hence, the area of a triangular slice is \( \frac{1}{2} (x + 2 - x^2)^2 \).

The volume is

\[
V = \int_{-1}^{2} \frac{1}{2} (x + 2 - x^2)^2 \, dx = \frac{1}{2} \int_{-1}^{2} (x^4 - 2x^3 - 3x^2 + 4x + 4) \, dx = \frac{1}{2} \left[ \frac{1}{5} x^5 - \frac{1}{2} x^4 - x^3 + 2x^2 + 4x \right]_{-1}^{2} = \frac{81}{20} = 4.05. \]
6. The region which lies above the $x$-axis and below the graph of $y = \frac{1}{x^2 + 1}$, $-\infty < x < \infty$, is revolved about the $x$-axis. Find the volume of the solid which is generated.

![Graph of the region](image)

Chop the solid up into circular slices perpendicular to the $x$-axis. The thickness of a typical slice is $dx$. The radius of a slice is $r = \frac{1}{x^2 + 1}$. The volume is

$$V = \int_{-\infty}^{\infty} \pi \cdot \frac{1}{(x^2 + 1)^2} \, dx = \int_{0}^{\infty} \pi \cdot \frac{1}{(x^2 + 1)^2} \, dx + \int_{-\infty}^{0} \pi \cdot \frac{1}{(x^2 + 1)^2} \, dx.$$  

Compute the first integral:

$$\int_{0}^{\infty} \pi \cdot \frac{1}{(x^2 + 1)^2} \, dx = \lim_{a \to +\infty} \int_{0}^{a} \pi \cdot \frac{1}{(x^2 + 1)^2} \, dx = \pi \cdot \lim_{a \to +\infty} \left[ \frac{1}{2} \frac{x}{x^2 + 1} + \frac{1}{2} \tan^{-1} x \right]_{0}^{a} = \pi \cdot \frac{1}{2} \lim_{a \to +\infty} \left( \frac{a}{a^2 + 1} + \tan^{-1} a \right) = \frac{\pi^2}{4}.$$  

(I used the fact that $\lim_{a \to +\infty} \tan^{-1} a = \frac{\pi}{2}$.)

Similarly,

$$\int_{-\infty}^{0} \pi \cdot \frac{1}{(x^2 + 1)^2} \, dx = \frac{\pi^2}{4}.$$  

The volume is $\frac{\pi^2}{4} + \frac{\pi^2}{4} = \frac{\pi^2}{2}$. □

7. The base of a rectangular tank is 2 feet long and 3 feet wide; the tank is 6 feet high. Find the work done in pumping all the water out of the top of the tank.

Divider the water up into rectangular slabs parallel to the base. Let $y$ denote the height of a slab above the base.

![Diagram of the tank](image)
The volume of a typical slab is \((2)(3)\, dy = 6\, dy\), so the weight is \(62.4 \cdot 6\, dy\). (The density of water is 62.4 pounds per cubic foot.) A slab at height \(y\) must be lifted a distance of \(6 - y\) to get to the top of the tank. Therefore, the work done in lifting the slab is \(62.4 \cdot 6(6 - y)\, dy\). The total work is

\[
\int_{0}^{6} 62.4 \cdot 6(6 - y)\, dy = 62.4 \cdot 6 \left[6y - \frac{1}{2}y^2\right]_{0}^{6} = 6739.2 \text{ foot-pounds.}
\]

8. Write a formula for the \(n\)-th term of the sequence, assuming that the terms continue in the “obvious” way.

(a) \(7, 11, 15, 19, 23, 27, \ldots\)

\[a_n = 7 + 4n \quad \text{for} \quad n = 0, 1, 2, \ldots \]

(b) \(\frac{2}{5}, \frac{4}{13}, \frac{6}{18}, \frac{8}{25}, \ldots\)

\[a_n = \frac{2n}{3 + 5n} \quad \text{for} \quad n = 1, 2, 3, \ldots \]

9. Determine whether the sequence converges or diverges; if it converges, find the limit.

(a) \(\{1.0001^n\}\)

Since \(\{1.0001^n\}\) is a geometric sequence with ratio \(r = 1.0001 > 1\),

\[
\lim_{n \to \infty} 1.0001^n = +\infty.
\]

(b) \(\left\{ \frac{e^n + 3^n}{2^n + \pi^n} \right\}\)

Divide the top and bottom by \(\pi^n\) (since \(\pi^n\) is the biggest exponential in the fraction):

\[
\lim_{n \to \infty} \frac{e^n + 3^n}{2^n + \pi^n} = \lim_{n \to \infty} \frac{\frac{e^n}{\pi^n} + \frac{3^n}{\pi^n}}{\frac{2^n}{\pi^n} + 1} = \frac{0 + 0}{0 + 1} = 0.
\]

I computed the limit using the fact that

\[
\frac{e^n}{\pi^n} = \left(\frac{e}{\pi}\right)^n, \quad \frac{3^n}{\pi^n} = \left(\frac{3}{\pi}\right)^n, \quad \text{and} \quad \frac{2^n}{\pi^n} = \left(\frac{2}{\pi}\right)^n
\]

are geometric sequences and their ratios are all less than 1. Therefore, they go to 0 as \(n \to \infty\).

(c) \(\left\{ \frac{2n^3 - 5n + 7}{7n^2 - 13n^3} \right\}\)

\[
\lim_{n \to \infty} \frac{2n^3 - 5n + 7}{7n^2 - 13n^3} = -\frac{2}{13}.
\]

I did this by considering the highest powers on the top and bottom; they’re both \(x^3\), so I just looked at their coefficients. You could also do this by using L’Hôpital’s rule, or by dividing the top and the bottom by \(x^3\).

(d) \(\{\arctan n)^2\}\)

\[
\lim_{n \to \infty} (\arctan n)^2 = \left(\lim_{n \to \infty} \arctan n\right)^2 = \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}.
\]
10. A sequence is defined recursively by
\[ a_1 = 5, \quad a_{n+1} = \sqrt{6a_n + 27} \quad \text{for} \quad n \geq 1. \]

Find \( \lim_{n \to \infty} a_n \).

Taking the limit on both sides of the recursion equation, I get
\[ \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{6a_n + 27} = \sqrt{6 \lim_{n \to \infty} a_n + 27}. \]

I'm allowed to move the limit inside the square root by a standard rule for limits.

Now \( \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n \) because both limits represent what the sequence \( \{a_n\} \) is approaching. So let
\[ L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} a_n. \]

Then
\[ L = \sqrt{6L + 27}, \quad L^2 = 6L + 27, \quad L^2 - 6L - 27 = 0, \quad (L - 9)(L + 3) = 0, \quad L = 9 \quad \text{or} \quad L = -3. \]

Since the sequence consists of positive numbers, it can’t have a negative limit. This rules out \(-3\). Therefore,
\[ \lim_{n \to \infty} a_n = 9. \quad \square \]

11. In each case, determine whether the series converges or diverges. You should cite the test you’re using by name (to avoid ambiguity); be sure you verify that the hypotheses of the test apply.

(a) \( \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} \).

\[ \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} = \sum_{k=1}^{\infty} \frac{3^k}{6^k} + \sum_{k=1}^{\infty} \frac{2^k}{6^k} = \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^k + \sum_{k=1}^{\infty} \left( \frac{1}{3} \right)^k = \frac{1}{1 - \frac{1}{2}} + \frac{1}{1 - \frac{1}{3}} = 1 + \frac{1}{2} = \frac{3}{2}. \quad \square \]

(b) \( \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2} \).

Let \( f(x) = \frac{1}{x(\ln x)^2} \). It is positive and continuous for \( x \geq 2 \). The derivative is
\[ f'(x) = \frac{-2}{x^2(\ln x)^3} - \frac{1}{x^2(\ln x)^2}. \]

\( f'(x) < 0 \) for \( x \geq 2 \), so \( f \) decreases for \( x \geq 2 \). The hypotheses of the Integral Test are satisfied. Compute the integral:
\[ \int_{2}^{\infty} \frac{1}{x(\ln x)^2} \, dx = \lim_{p \to \infty} \left[ -\frac{1}{\ln x} \right]^p_2 = \lim_{p \to \infty} \left( -\frac{1}{\ln p} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2}. \]

Since the integral converges, the series converges by the Integral Test. \( \square \)
(c) \( \sum_{k=3}^{\infty} \frac{k^2 - 3k + 2}{k^4} \).

\[
\sum_{k=3}^{\infty} \frac{k^2 - 3k + 2}{k^4} = \sum_{k=3}^{\infty} \frac{1}{k^2} - 3 \sum_{k=3}^{\infty} \frac{1}{k^3} + 2 \sum_{k=3}^{\infty} \frac{1}{k^4}.
\]

The series on the right are convergent p-series. Hence, the original series converges. \( \square \)

(d) \( \sum_{k=2}^{\infty} \frac{2}{k^2 - 1} \).

\[
\lim_{k \to \infty} \frac{\frac{2}{k^2} - 1}{1} = \lim_{k \to \infty} \frac{2k^2}{k^2 - 1} = 2.
\]

The limit is a finite positive number. \( \sum_{k=2}^{\infty} \frac{1}{k^2} \) converges, because it’s a p-series with \( p = 2 > 1 \). Therefore, \( \sum_{k=2}^{\infty} \frac{2}{k^2 - 1} \) converges by Limit Comparison. \( \square \)

(e) \( \sum_{k=1}^{\infty} \frac{2}{3k + 5} \).

Let \( f(x) = \frac{2}{3x + 5} \). Then \( f \) is positive and continuous for \( x \geq 1 \). The derivative is

\[
f'(x) = \frac{-6}{(3x + 5)^2}.
\]

\( f'(x) < 0 \) for \( x \geq 1 \), so \( f \) decreases for \( x \geq 1 \). The hypotheses of the Integral Test are satisfied. Compute the integral:

\[
\int_{1}^{\infty} \frac{2}{3x + 5} \, dx = \lim_{p \to \infty} \left[ \frac{2}{3} \ln |3x + 5| \right]_{1}^{p} = \frac{2}{3} \lim_{p \to \infty} (\ln |3p + 5| - \ln 8) = +\infty.
\]

The limit diverges, so the integral diverges. Therefore, the series diverges, by the Integral Test. \( \square \)

(f) \( \sum_{k=1}^{\infty} \left( \frac{k + 1}{k^2 + 2} \right)^2 \).

Rewrite the series as

\[
\sum_{k=1}^{\infty} \frac{(k + 1)^2}{(k^2 + 2)^2}.
\]

For large \( k \),

\[
\frac{(k + 1)^2}{(k^2 + 2)^2} \approx \frac{k^2}{k^4} = \frac{1}{k^2}.
\]
Use Limit Comparison with the series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \). The limiting ratio is

\[
\lim_{k \to \infty} \frac{(k + 1)^2}{k^2} = \lim_{k \to \infty} \frac{k^2(k + 1)^2}{(k^2 + 2)^2} = 1.
\]

The limit is finite and positive. Since \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) is a convergent \( p \)-series \((p = 2)\), the series converges by Limit Comparison.

\[(g) \sum_{k=1}^{\infty} \left( \sin \frac{1}{k} \right)^2.
\]

I’ll use Limit Comparison with \( \sum_{k=1}^{\infty} \frac{1}{k^2} \). Rationale: For \( \theta \approx 0 \), \( \sin \theta \approx \theta \), so \( \left( \sin \frac{1}{k} \right)^2 \approx \frac{1}{k^2} \).

\[
\lim_{k \to \infty} \frac{\left( \sin \frac{1}{k} \right)^2}{\frac{1}{k^2}} = \left( \lim_{k \to \infty} \frac{\sin \frac{1}{k}}{\frac{1}{k}} \right)^2 = \left( \lim_{m \to 0} \frac{\sin m}{m} \right)^2 = 1.
\]

(I set \( m = \frac{1}{k} \). As \( k \to \infty \), \( m \to 0 \).)

The limit is a finite, positive number, and the series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) is a convergent \( p \)-series \((p = 2)\). Therefore, the series converges, by Limit Comparison.

\[(h) \sum_{k=1}^{\infty} \frac{2^k}{3^k + 2}.
\]

since making the bottom of a fraction smaller makes the fraction larger.

The series \( \sum_{k=1}^{\infty} \frac{2^k}{3^k} \) is geometric with ratio \( r = \frac{2}{3} \), so it converges. Therefore, the series \( \sum_{k=1}^{\infty} \frac{2^k}{3^k + 2} \) converges by comparison.

\[(i) \sum_{i=1}^{\infty} \frac{3^i}{i^3}.
\]

Apply the Ratio Test:

\[
\lim_{i \to \infty} \frac{a_{i+1}}{a_i} = \lim_{i \to \infty} \frac{\frac{3^{i+1}}{(i+1)^3}}{\frac{3^i}{i^3}} = \lim_{i \to \infty} \frac{3^{i+1}}{3^i} \cdot \frac{i^3}{(i+1)^3} = \lim_{i \to \infty} 3 \cdot \left( \frac{i}{i+1} \right)^3 = 3.
\]
Since the limit is larger than 1, the series diverges by the Ratio Test. □

\[ \sum_{k=1}^{\infty} \left( \frac{2k+2}{2k-1} \right)^k. \]

The \( k \)th root of the \( k \)th term is

\[ a_k^{1/k} = \left( \frac{2k+2}{2k-1} \right)^{k}. \]

To compute the limit as \( k \to \infty \), let \( y = \left( \frac{2k+2}{2k-1} \right)^k \). Then

\[ \ln y = k \ln \left( \frac{2k+2}{2k-1} \right) = k \ln \frac{2k+2}{2k-1}. \]

Therefore,

\[ \lim_{k \to \infty} \ln y = \lim_{k \to \infty} k \ln \frac{2k+2}{2k-1} = \lim_{k \to \infty} \frac{2k+2}{2k-1} \cdot \frac{-6}{k^2} = 6 \lim_{k \to \infty} \frac{(2k-1)(2k)}{(2k+2)(2k-1)^2} = \frac{6}{4} = \frac{3}{2}. \]

Therefore, \( \lim_{k \to \infty} y = e^{3/2} \).

Since \( e^{3/2} > 1 \), the series diverges by the Root Test. □

\[ \sum_{k=1}^{\infty} \sqrt{\arctan k}. \]

\[ \lim_{k \to \infty} \sqrt{\arctan k} = \sqrt{\frac{\pi}{2}} \neq 0, \]

so the series diverges, by the Zero Limit Test. □

\[ \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k+1)!}. \]

Apply the Ratio Test:

\[ \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \frac{(2k+3)!}{(k!)^2} \cdot \frac{(k+1)!}{(2k+1)!} = \lim_{k \to \infty} \left( \frac{(k+1)!}{k!} \right)^2 \cdot \frac{(2k+1)!}{(2k+3)!} = \lim_{k \to \infty} \frac{(2k+2)!}{(k+1)!} \cdot \frac{(k+1)!}{(2k+2)!} = 1. \]

Since the limit is less than 1, the series converges by the Ratio Test. □
12. (a) Find the partial fractions decomposition of \( \frac{2}{(2k + 1)(2k + 3)} \).

\[
\frac{2}{(2k + 1)(2k + 3)} = \frac{A}{2k + 1} + \frac{B}{2k + 3},
\]

so

\[
2 = A(2k + 3) + B(2k + 1).
\]

Set \( x = -\frac{1}{2} \): I get \( 2 = 2A \), so \( A = 1 \).

Set \( x = -\frac{3}{2} \): I get \( 2 = -2B \), so \( B = -1 \).

Therefore,

\[
\frac{2}{(2k + 1)(2k + 3)} = \frac{1}{2k + 1} - \frac{1}{2k + 3}. \quad \Box
\]

(b) Use (a) to find the sum of the series

\[
\sum_{k=1}^{\infty} \frac{2}{(2k + 1)(2k + 3)}.
\]

The second fraction in each pair cancels with the first fraction in the next pair. The only one that isn’t cancelled is the very first one: \( \frac{1}{3} \). Therefore,

\[
\sum_{k=1}^{\infty} \frac{2}{(2k + 1)(2k + 3)} = \frac{1}{3}. \quad \Box
\]

13. If the series \( \sum_{k=17}^{\infty} a_k \), converges, does the series \( \sum_{k=1}^{\infty} a_k \) converge?

If the series \( \sum_{k=17}^{\infty} a_k \) converges, then the series \( \sum_{k=1}^{\infty} a_k \) converges. They only differ in the first 16 terms, and a finite number of terms cannot affect the convergence or divergence of an infinite series. \( \Box \)

14. Does the series \( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k + 1}{4k + 3} \) converge?

The series alternates, but

\[
\lim_{k \to \infty} \frac{2k + 1}{4k + 3} = \frac{1}{2}.
\]

The \((-1)^{k+1}\) causes the terms to oscillate in sign, so

\[
\lim_{k \to \infty} (-1)^{k+1} \frac{2k + 1}{4k + 3}
\]

is undefined.

The series diverges by the Zero Limit Test. \( \Box \)
15. Show that the Ratio Test always fails for a $p$-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}, \quad \text{where} \quad p > 0.$$ 

The ratio of successive terms is

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \frac{1}{(k+1)^p} \frac{k^p}{1} = \lim_{k \to \infty} \left( \frac{k}{k+1} \right)^p = 1.$$ 

Since the limit is 1, the Ratio Test fails.  □

**Remark.** This problem also shows that it’s useless to apply the Ratio Test to series where the $k$-th term is a rational function of $k$.

For example, it’s useless to apply the Ratio Test to

$$\sum_{k=1}^{\infty} \frac{k^2 + 5}{k^4 + 3},$$

since for large $k$, \( \frac{k^2 + 5}{k^4 + 3} \approx \frac{1}{k^2} \), and the series is essentially a $p$-series.  □

*Happiness depends upon ourselves.* - ARISTOTLE