Math 211
10-2-2020

Review Problems for the Final

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. Compute the following integrals.

(a) $\displaystyle\int e^x \cos 2x\,dx$.

(b) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$.

(c) $\displaystyle\int \frac{5x^2-6x-5}{(x-1)^2(x+2)}\,dx$.

(d) $\displaystyle\int (\sin 4x)^3(\cos 4x)^2\,dx$.

(e) $\displaystyle\int (\sin 4x)^2(\cos 4x)^2\,dx$.

(f) $\displaystyle\int \frac{1}{(-3-4x-x^2)^{3/2}}\,dx$.

(g) $\displaystyle\int \frac{-3x^2+7x+1}{(x+2)(x^2+1)}\,dx$.

(h) $\displaystyle\int (x+1)^2 e^{5x}\,dx$.

(i) $\displaystyle\int \frac{x^{1/2}-x^{1/4}}{x^{1/2}+x^{1/4}}\,dx$.

2. Determine whether the integral $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx$ converges or diverges. Find the value of the integral if it converges.

3. Prove by comparison that $\displaystyle\int_{1}^{\infty} \frac{1}{x^4+1}\,dx$ converges.

4. The region between the x-axis and $y = x^2$ from $x = -1$ to $x = 1$ is revolved about the line $x = -4$. Find the volume generated.

5. Let R be the region bounded above by $y = x + 2$, bounded below by $y = -x^2$, and bounded on the sides by $x = -2$ and by the y-axis. Find the volume of the solid generated by revolving R about the line $x = 1$.

6. Find the area of the region which lies between the graphs of $y = x^2$ and $y = x + 2$, from $x = 1$ to $x = 3$.

7. Find the area of the region between $y = x + 3$ and $y = 7 - x$ from $x = 0$ to $x = 3$.

8. The base of a solid is the region in the x-y-plane bounded above by the curve $y = e^x$, below by the x-axis, and on the sides by the lines $x = 0$ and $x = 1$. The cross-sections in planes perpendicular to the x-axis are squares with one side in the x-y-plane. Find the volume of the solid.

9. A tank built in the shape of the bottom half of a sphere of radius 2 feet is filled with water. Find the work done in pumping all the water out of the top of the tank.

10. Does the following series converge absolutely, converge conditionally, or diverge?

$$\frac{1}{2} - \frac{4}{2^3+1} + \frac{9}{3^3+1} - \frac{16}{4^3+1} + \cdots$$

11. Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{n}+1}{n^2+1}$ converges absolutely, converges conditionally, or diverges.

12. Does the series $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{\sqrt{n}+1}\right)^3$ converge absolutely, converge conditionally, or diverge?

13. Find the sum of the series

$$\frac{5}{9} - \frac{5}{27} + \frac{5}{81} - \frac{5}{243} + \cdots.$$

14. In each case, determine whether the series converges or diverges.

(a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{4/3}}$.

(b) $\frac{2}{1} + \frac{2\cdot 5}{1\cdot 5} + \cdots + \frac{2\cdot 5\cdot \cdots \cdot (3n-1)}{1\cdot 5\cdot \cdots \cdot (4n-3)}$.

(c) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$.

(d) $\frac{2}{3} - \frac{5}{8} + \frac{8}{13} - \frac{11}{18} + \cdots$.

(e) $\sum_{n=1}^{\infty}\frac{3n^2+4n+2}{\sqrt{n^5+16}}$.

(f) $\sum_{n=1}^{\infty}\frac{5+\cos(e^n)}{n}$.

(g) $\sum_{n=1}^{\infty} ne^{-2n}$.

(h) $\sum_{k=2}^{\infty}\frac{k+\sqrt{k}}{k^3+(-1)^k}$.

15. Find the values of x for which the following series converges absolutely.

$$\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}(x-5)^n$$

16. The series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^{1/3}+2}$ converges by the Alternating Series Test. Determine the smallest value of n for which the partial sum $\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k^{1/3}+2}$ approximates the actual sum to within 0.01.

17. (a) Find the Taylor expansion at $c = 1$ for e^{2x}.

(b) Find the Taylor expansion at $c = 1$ for $\dfrac{1}{3+x}$. What is the interval of convergence?

18. (a) Use the Binomial Series to write out the first three nonzero terms of the series for $\dfrac{1}{\sqrt{1-t^2}} = (1-t^2)^{-1/2}$.

(b) Find the first three terms of the Taylor series at $c = 0$ for $\sin^{-1} x$ by integrating the series you got in (a) from $t = 0$ to $x = x$.

19. Use the Remainder Term to find the minimum number of terms of the Taylor series at $c = 0$ for $f(x) = e^{5x}$ needed to approximate $f(x)$ on the interval $0 \le x \le 0.3$ to within $0.00001 = 10^{-5}$.

20. Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 7^n}$.

21. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n(2^n)^3}$.

22. If $x = t + e^t$ and $y = t + t^3$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $t = 1$.

23. Consider the parametric curve

$$x = t^2 + t + 1, \quad y = t^3 - 5t + 2.$$

(a) Find the equation of the tangent line at $t = 1$.

(b) Find $\dfrac{d^2y}{dx^2}$ at $t = 1$.

24. Find the length of the loop of the curve

$$x = t^2, \quad y = t - \frac{t^3}{3}.$$

25. Find the length of the curve $y = \dfrac{1}{2}x^2 + 2$ for $0 \le x \le 1$.

26. Let

$$x = \frac{\sqrt{3}}{2}t^2, \quad y = t - \frac{1}{4}t^3.$$

Find the length of the arc of the curve from $t = -2$ to $t = 2$.

27. Find the area of the surface generated by revolving $y = \dfrac{2}{3}x^{3/2} - \dfrac{1}{2}x^{1/2}$, $1 \le x \le 4$, about the x-axis.

28. Find the area of the surface generated by revolving $y = \dfrac{1}{3}x^3$, $0 \le x \le 2$, about the x-axis.

29. (a) Convert $(x-3)^2 + (y+4)^2 = 25$ to polar and simplify.

(b) Convert $r = 4\cos\theta - 6\sin\theta$ to rectangular and describe the graph.

30. Find the slope of the tangent line to the polar curve $r = \sin 2\theta$ at $\theta = \dfrac{\pi}{6}$.

31. Find the slope of the tangent line to $r = 2 + \cos\theta$ at $\theta = \dfrac{\pi}{6}$.

32. Find the values of θ in the interval $[0, 2\pi]$ for which the polar curve $r = \cos 2\theta - \sin 2\theta$ passes through the origin.

33. Find the length of the cardioid $r = 1 + \sin\theta$.

34. Find the area of the intersection of the interiors of the circles

$$x^2 + (y-1)^2 = 1 \quad \text{and} \quad (x - \sqrt{3})^2 + y^2 = 3.$$

35. Find the area of the region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$.

36. Let A be the region inside $r = \sin\theta$ and let B be the region inside $r = \sqrt{3}\cos\theta$. Find the area of the intersection of A and B — that is, the area of the region common to A and B.

Solutions to the Review Problems for the Final

1. Compute the following integrals.

(a) $\displaystyle\int e^x \cos 2x\, dx$.

(b) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\, dx$.

(c) $\displaystyle\int \frac{5x^2 - 6x - 5}{(x-1)^2(x+2)}\, dx$.

(d) $\displaystyle\int (\sin 4x)^3 (\cos 4x)^2\, dx$.

(e) $\displaystyle\int (\sin 4x)^2 (\cos 4x)^2\, dx$.

(f) $\displaystyle\int \frac{1}{(-3 - 4x - x^2)^{3/2}}\, dx$.

(g) $\displaystyle\int \frac{-3x^2 + 7x + 1}{(x+2)(x^2+1)}\, dx$.

(h) $\displaystyle\int (x+1)^2 e^{5x}\, dx$.

(i) $\displaystyle\int \frac{x^{1/2} - x^{1/4}}{x^{1/2} + x^{1/4}}\, dx$.

(a)

$$\begin{array}{cccc}
 & \dfrac{d}{dx} & & \displaystyle\int dx \\
+ & e^x & & \cos 2x \\
 & & \searrow & \\
- & e^x & & \dfrac{1}{2}\sin 2x \\
 & & \searrow & \\
+ & e^x & \rightarrow & -\dfrac{1}{4}\cos 2x
\end{array}$$

$$\int e^x \cos 2x\,dx = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}\int e^x \cos 2x\,dx,$$

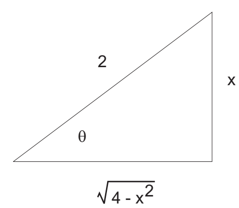
$$\frac{5}{4}\int e^x \cos 2x\,dx = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x,$$

$$\int e^x \cos 2x\,dx = \frac{2}{5}e^x \sin 2x + \frac{1}{5}e^x \cos 2x + C. \quad \square$$

(b)

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4(\sin\theta)^2}{\sqrt{4-4(\sin\theta)^2}} 2\cos\theta\,d\theta = \int \frac{4(\sin\theta)^2}{\sqrt{4(\cos\theta)^2}} 2\cos\theta\,d\theta = 4\int (\sin\theta)^2\,d\theta =$$

$$[x = 2\sin\theta, \quad dx = 2\cos\theta\,d\theta]$$



$$2\int (1-\cos 2\theta)\,d\theta = 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = 2\left(\theta - \sin\theta\cos\theta\right) + C = 2\sin^{-1}\frac{x}{2} - \frac{1}{2}x\sqrt{4-x^2} + C. \quad \square$$

(c)

$$\frac{5x^2-6x-5}{(x-1)^2(x+2)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+2},$$

$$5x^2-6x-5 = a(x-1)(x+2) + b(x+2) + c(x-1)^2.$$

Setting $x = 1$ gives $-6 = 3b$, so $b = -2$.
Setting $x = -2$ gives $27 = 9c$, so $c = 3$.
Therefore,

$$5x^2-6x-5 = a(x-1)(x+2) - 2(x+2) + 3(x-1)^2.$$

Setting $x = 0$ gives $-5 = -2a - 4 + 3$, so $a = 2$.
Thus,

$$\int \frac{5x^2-6x-5}{(x-1)^2(x+2)}\,dx = \int \left(\frac{2}{x-1} - \frac{2}{(x-1)^2} + \frac{3}{x+2}\right) dx = 2\ln|x-1| + \frac{2}{x-1} + 3\ln|x+2| + C. \quad \square$$

(d)

$$\int (\sin 4x)^3(\cos 4x)^2\,dx = \int (\sin 4x)^2(\cos 4x)^2(\sin 4x\,dx) = \int \left(1-(\cos 4x)^2\right)(\cos 4x)^2(\sin 4x\,dx) =$$

$$\left[u = \cos 4x, \quad du = -4\sin 4x\,dx, \quad dx = \frac{du}{-4\sin 4x}\right]$$

$$\int (1-u^2)u^2(\sin 4x)\left(\frac{du}{-4\sin 4x}\right) = \frac{1}{4}\int (u^4-u^2)\,du = \frac{1}{4}\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) + C =$$

$$\frac{1}{4}\left(\frac{1}{5}(\cos 4x)^5 - \frac{1}{3}(\cos 4x)^3\right) + C. \quad \square$$

(e)

$$\int (\sin 4x)^2(\cos 4x)^2\,dx = \int \frac{1}{2}(1-\cos 8x)\cdot\frac{1}{2}(1+\cos 8x)\,dx = \frac{1}{4}\int \left(1-(\cos 8x)^2\right)\,dx = \frac{1}{4}\int (\sin 8x)^2\,dx =$$

$$\frac{1}{8}\int (1-\cos 16x)\,dx = \frac{1}{8}\left(x - \frac{1}{16}\sin 16x\right) + C. \quad \square$$

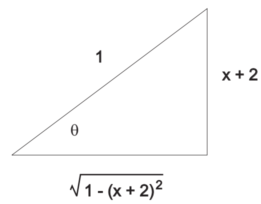
(f) I need to complete the square. Note that $\dfrac{-4}{2} = -2$ and $(-2)^2 = 4$. Then

$$-3-4x-x^2 = -(x^2+4x+3) = -(x^2+4x+4-1) = -\left[(x+2)^2-1\right] = 1-(x+2)^2.$$

So

$$\int \frac{1}{(-3-4x-x^2)^{3/2}}\,dx = \int \frac{1}{(1-(x+2)^2)^{3/2}}\,dx = \int \frac{1}{(1-(\sin\theta)^2)^{3/2}}(\cos\theta\,d\theta) = \int \frac{1}{(\cos\theta)^3}(\cos\theta\,d\theta) =$$

$$[x+2=\sin\theta, \quad dx = \cos\theta\,d\theta]$$



$$\int \frac{1}{(\cos\theta)^2}\,d\theta = \int (\sec\theta)^2\,d\theta = \tan\theta + C = \frac{x+2}{\sqrt{-3-4x-x^2}} + C. \quad \square$$

(g)

$$\frac{-3x^2+7x+1}{(x+2)(x^2+1)} = \frac{a}{x+2} + \frac{bx+c}{x^2+1}$$
$$-3x^2+7x+1 = a(x^2+1) + (bx+c)(x+2)$$

Let $x=-2$. I get

$$-12-14+1 = 5a, \quad -25 = 5a, \quad \text{so} \quad a = -5.$$

Then

$$-3x^2+7x+1 = -5(x^2+1) + (bx+c)(x+2).$$

Let $x=0$. I get

$$0+0+1 = -5+2c, \quad 6 = 2c, \quad \text{so} \quad c = 3.$$

Then

$$-3x^2+7x+1 = -5(x^2+1) + (bx+3)(x+2).$$

Let $x=1$. I get

$$-3+7+1 = -10+(b+3)(3), \quad 5 = -10+3b+9, \quad 6 = 3b, \quad \text{so} \quad b = 2.$$

Thus,

$$\int \frac{-3x^2+7x+1}{(x+2)(x^2+1)}\,dx = \int \left(\frac{-5}{x+2} + \frac{2x}{x^2+1} + \frac{3}{x^2+1}\right)dx = -5\ln|x+2| + \ln(x^2+1) + 3\tan^{-1}x + C. \quad \square$$

(h)

	$\dfrac{d}{dx}$	$\int dx$
$+$	$(x+1)^2$	e^{5x}
$-$	$2(x+1)$	$\frac{1}{5}e^{5x}$
$+$	2	$\frac{1}{25}e^{5x}$
$-$	0	$\frac{1}{125}e^{5x}$

$$\int (x+2)^2 e^{5x}\,dx = \frac{1}{5}(x+1)^2 e^{5x} - \frac{2}{25}(x+1)e^{5x} + \frac{2}{125}e^{5x} + C. \quad \square$$

(i)

$$\int \frac{x^{1/2} - x^{1/4}}{x^{1/2} + x^{1/4}}\,dx = \int \frac{u^2 - u}{u^2 + u} \cdot (4u^3\,du) = 4\int \frac{u-1}{u+1} \cdot u^3\,du = 4\int \frac{u^4 - u^3}{u+1}\,du.$$

$$\left[x = u^4, \quad dx = 4u^3\,du\right]$$

Use long division to divide $u^4 - u^3$ by $u + 1$:

```
        u³ - 2u² + 2u - 2
u+1 ⟌ u⁴ - u³
    - u⁴ + u³
        -2u³
      - -2u³ - 2u²
              2u²
            - 2u² + 2u
                   -2u
                 - -2u - 2
                        2
```

Thus,

$$4\int \frac{u^4 - u^3}{u+1}\,du = 4\int \left(u^3 - 2u^2 + 2u - 2 + \frac{2}{u+1}\right)du =$$

$$4\left(\frac{1}{4}u^4 - \frac{2}{3}u^3 + u^2 - 2u + 2\ln|u+1|\right) + C =$$

$$4\left(\frac{1}{4}x - \frac{2}{3}x^{3/4} + x^{1/2} - 2x^{1/4} + 2\ln|x^{1/4} + 1|\right) + C. \quad \square$$

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2. Determine whether the integral $\int_{-1}^{1} \frac{1}{x}\,dx$ converges or diverges. Find the value of the integral if it converges.

$$\int_{-1}^{1} \frac{1}{x}\,dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx = \lim_{a\to 0^-} \int_{-1}^{a} \frac{1}{x}\,dx + \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{x}\,dx =$$

$$\lim_{a\to 0^-} \left[\ln|x|\right]_{-1}^{a} + \lim_{b\to 0^+} \left[\ln|x|\right]_{b}^{1} = \lim_{a\to 0^-} (\ln|a| - \ln 1) + \lim_{b\to 0^+} (\ln 1 - \ln|b|) = \lim_{a\to 0^-} \ln|a| + \lim_{b\to 0^+} (-\ln|b|).$$

$\lim_{a\to 0^-} \ln|a| = -\infty$ and $\lim_{b\to 0^+} (-\ln|b|) = +\infty$, so the integral diverges. $\square$

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3. Prove by comparison that $\int_{1}^{\infty} \frac{1}{x^4+1}\,dx$ converges.

Since $0 \le \frac{1}{x^4+1} \le \frac{1}{x^4}$ for $x \ge 1$,

$$0 \le \int_{1}^{\infty} \frac{1}{x^4+1}\,dx \le \int_{1}^{\infty} \frac{1}{x^4}\,dx = \lim_{a\to\infty} \int_{1}^{a} \frac{1}{x^4}\,dx = \lim_{a\to\infty} \left[-\frac{1}{3x^3}\right]_{1}^{a} = \lim_{a\to\infty} \left(-\frac{1}{3a^3} + \frac{1}{3}\right) = \frac{1}{3}.$$

Since $\int_{1}^{\infty} \frac{1}{x^4}\,dx$ converges, the original integral converges as well. $\square$

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4. The region between the x-axis and $y = x^2$ from $x = -1$ to $x = 1$ is revolved about the line $x = -4$. Find the volume generated.

Use shells. The height of a shell is $h = x^2$, and the radius is $r = 4 + x$. The volume is

$$\int_{-1}^{1} 2\pi(4+x)(x^2)\,dx = 2\pi \int_{-1}^{1} (4x^2 + x^3)\,dx = 2\pi \left[\frac{4}{3}x^3 + \frac{1}{4}x^4\right]_{-1}^{1} = \frac{16\pi}{3} \approx 16.75516. \quad \square$$

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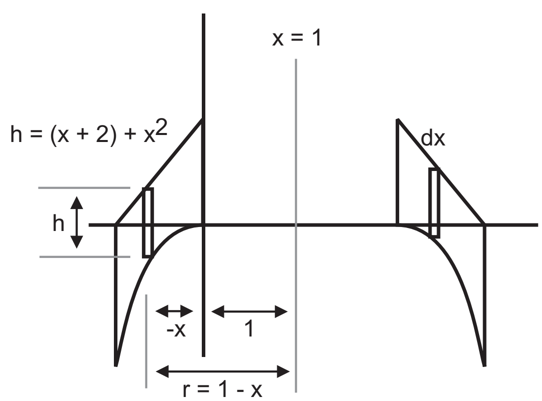
5. Let R be the region bounded above by $y = x + 2$, bounded below by $y = -x^2$, and bounded on the sides by $x = -2$ and by the y-axis. Find the volume of the solid generated by revolving R about the line $x = 1$.



Most of the things in the picture are easy to understand — but why is $r = 1 - x$?

Notice that the *distance* from the y-axis to the side of the shell is $-x$, not x. Reason: x-values to the left of the y-axis are *negative*, but distances are always *positive*. Thus, I must use $-x$ to get a positive value for the distance.

As usual, r is the distance from the axis of revolution $x = 1$ to the side of the shell, which is $1 + (-x) = 1 - x$.

The left-hand cross-section extends from $x = -2$ to $x = 0$. You can check that if you plug x's between -2 and 0 into $r = 1 - x$, you get the correct distance from the side of the shell to the axis $x = 1$.

The volume is

$$V = \int_{-2}^{0} 2\pi(1-x)((x+2)+x^2)\,dx = 4\pi = \int_{-2}^{0} 2\pi\left(2 - x - x^3\right)\,dx = 2\pi\left[2x - \frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{-2}^{0} =$$

$$20\pi \approx 62.83185. \quad \square$$

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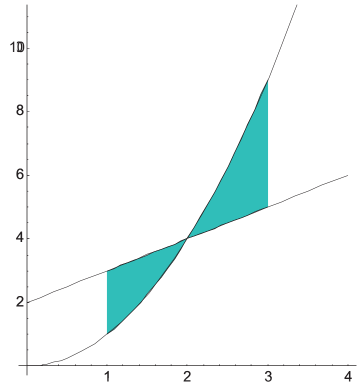
6. Find the area of the region which lies between the graphs of $y = x^2$ and $y = x + 2$, from $x = 1$ to $x = 3$.



As the picture shows, the curves intersect. Find the intersection point:

$$x^2 = x + 2, \quad x^2 - x - 2 = 0, \quad (x-2)(x+1) = 0, \quad x = 2 \quad \text{or} \quad x = -1.$$

On the interval $1 \le x \le 3$, the curves cross at $x = 2$. I'll use vertical rectangles. From $x = 1$ to $x = 2$, the top curve is $y = x + 2$ and the bottom curve is $y = x^2$. From $x = 2$ to $x = 3$, the top curve is $y = x^2$ and the bottom curve is $y = x + 2$. The area is

$$A = \int_1^2 \left((x+2) - x^2\right)\, dx + \int_2^3 \left(x^2 - (x+2)\right)\, dx = 3. \quad \square$$

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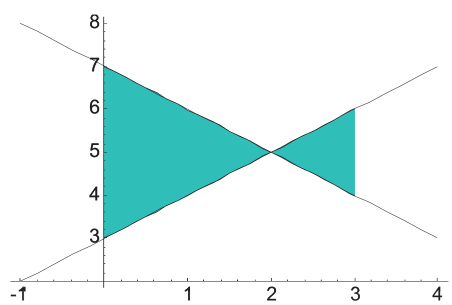
7. Find the area of the region between $y = x + 3$ and $y = 7 - x$ from $x = 0$ to $x = 3$.



As the picture shows, the curves intersect. Find the intersection point:

$$x + 3 = 7 - x, \quad 2x = 4, \quad x = 2.$$

I'll use vertical rectangles. From $x = 0$ to $x = 2$, the top curve is $y = 7 - x$ and the bottom curve is $y = x + 3$. From $x = 2$ to $x = 3$, the top curve is $y = x + 3$ and the bottom curve is $y = 7 - x$. The area is

$$\int_0^2 ((7-x) - (x+3))\, dx + \int_2^3 ((x+3) - (7-x))\, dx = \int_0^2 (4 - 2x)\, dx + \int_2^3 (2x - 4)\, dx =$$

$$\left[4x - x^2\right]_0^2 + \left[x^2 - 4x\right]_2^3 = 4 + 1 = 5. \quad \square$$

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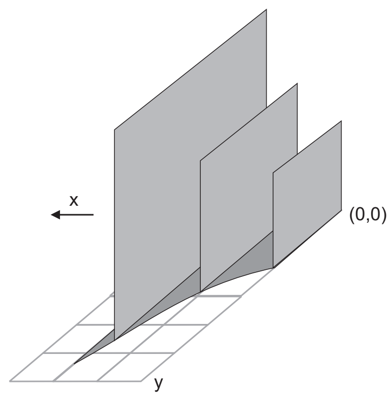
8. The base of a solid is the region in the x-y-plane bounded above by the curve $y = e^x$, below by the x-axis, and on the sides by the lines $x = 0$ and $x = 1$. The cross-sections in planes perpendicular to the x-axis are squares with one side in the x-y-plane. Find the volume of the solid.



The volume is

$$V = \int_0^1 (e^x)^2 \, dx = \int_0^1 e^{2x} \, dx = \left[\frac{1}{2} e^{2x}\right]_0^1 = \frac{1}{2}(e^2 - 1) \approx 3.19453. \quad \square$$

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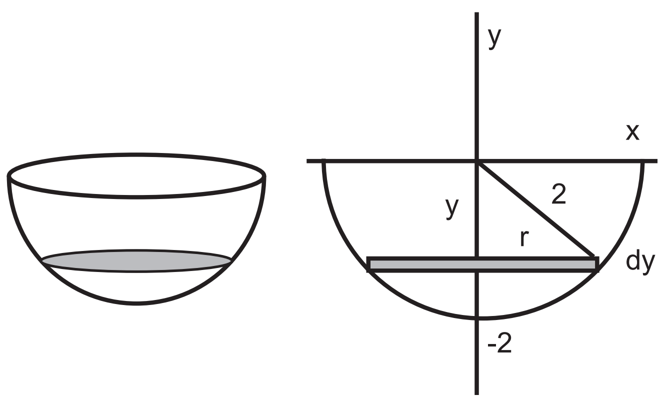
9. A tank built in the shape of the bottom half of a sphere of radius 2 feet is filled with water. Find the work done in pumping all the water out of the top of the tank.



I've drawn the tank in cross-section as a semicircle of radius 2 extending from $y = -2$ to $y = 0$.

Divide the volume of water up into circular slices. The radius of a slice is $r = \sqrt{4 - y^2}$, so the volume of a slice is $dV = \pi r^2 \, dy = \pi(4 - y^2) \, dy$. The weight of a slice is $62.4\pi(4 - y^2) \, dy$, where I'm using 62.4 pounds per cubic foot as the density of water.

To pump a slice out of the top of the tank, it must be raised a distance of $-y$ feet. (The "-" is necessary to make y positive, since y is going from -2 to 0.)

The work done is

$$W = \int_{-2}^0 62.4\pi(-y)(4 - y^2) \, dy = 62.4\pi \int_{-2}^0 (y^3 - 4y) \, dy = 62.4\pi \left[\frac{1}{4} y^4 - 2y^2\right]_{-2}^0 =$$

$$249.6\pi \approx 784.14153 \text{ foot} - \text{pounds}. \quad \square$$

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10. Does the following series converge absolutely, converge conditionally, or diverge?

$$\frac{1}{2} - \frac{4}{2^3+1} + \frac{9}{3^3+1} - \frac{16}{4^3+1} + \cdots$$

$$\frac{1}{2} - \frac{4}{2^3+1} + \frac{9}{3^3+1} - \frac{16}{4^3+1} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}.$$

The absolute value series is

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}.$$

Note that when n is large,

$$\frac{n^2}{n^3+1} \approx \frac{1}{n}.$$

Hence, I'll compare the series to $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1.$$

The limit is finite ($\neq \infty$) and positive (> 0). The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. By Limit Comparison, the series $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ diverges. Hence, the original series does not converge absolutely.

Returning to the original series, note that it alternates, and

$$\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0.$$

Let $f(n) = \dfrac{n^2}{n^3+1}$. Then

$$f'(n) = \frac{n(2-n^3)}{(1+n^3)^2} < 0 \quad \text{for} \quad n > 1.$$

Therefore, the terms of the series decrease for $n \geq 2$, and I can apply the Alternating Series Rule to conclude that the series converges. Since it doesn't converge absolutely, but it *does* converge, it converges conditionally. □

---

11. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n^2+1}$ converges absolutely, converges conditionally, or diverges.

The absolute value series is $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{n^2+1}$.

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}+1}{n^2+1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^2+n^{3/2}}{n^2+1} = 1.$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges because it's a p-series with $p = \frac{3}{2} > 1$.
The absolute value series converges by limit comparison.
The original series converges absolutely. □

---

12. Does the series $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{\sqrt{n}+1}\right)^3$ converge absolutely, converge conditionally, or diverge?

Consider the absolute value series $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}+1}\right)^3$.

$$\lim_{n\to\infty} \frac{\left(\frac{1}{\sqrt{n}+1}\right)^3}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} n^{3/2} \cdot \left(\frac{1}{\sqrt{n}+1}\right)^3 = \lim_{n\to\infty} (n^{1/2})^3 \cdot \left(\frac{1}{\sqrt{n}+1}\right)^3 = \lim_{n\to\infty} \left(\frac{n^{1/2}}{n^{1/2}+1}\right)^3 = 1^3 = 1.$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, because it's a p-series with $p = \frac{3}{2} > 1$. Hence, the absolute value series converges by Limit Comparison.

Therefore, the original series converges absolutely. □

---

13. Find the sum of the series

$$\frac{5}{9} - \frac{5}{27} + \frac{5}{81} - \frac{5}{243} + \cdots.$$

$$\frac{5}{9} - \frac{5}{27} + \frac{5}{81} - \frac{5}{243} + \cdots = \frac{5}{9}\left(1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots\right) = \frac{5}{9} \cdot \frac{1}{1 - \left(-\frac{1}{3}\right)} = \frac{5}{9} \cdot \frac{3}{4} = \frac{5}{12}. \quad \square$$

---

14. In each case, determine whether the series converges or diverges.

(a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{4/3}}$.

(b) $\frac{2}{1} + \frac{2 \cdot 5}{1 \cdot 5} + \cdots + \frac{2 \cdot 5 \cdot \cdots \cdot (3n-1)}{1 \cdot 5 \cdot \cdots \cdot (4n-3)}$.

(c) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$.

(d) $\frac{2}{3} - \frac{5}{8} + \frac{8}{13} - \frac{11}{18} + \cdots$.

(e) $\sum_{n=1}^{\infty} \frac{3n^2 + 4n + 2}{\sqrt{n^5 + 16}}$.

(f) $\sum_{n=1}^{\infty} \frac{5 + \cos(e^n)}{n}$.

(g) $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$.

(h) $\displaystyle\sum_{k=2}^{\infty} \frac{k+\sqrt{k}}{k^3+(-1)^k}$.

(a) Apply the Integral Test. The function $f(n) = \dfrac{1}{n(\ln n)^{4/3}}$ is positive and continuous on the interval $[2, +\infty)$.

Note that

$$f'(n) = -\frac{4}{3n^2(\ln n)^{7/3}} - \frac{1}{n^2(\ln n)^{4/3}}.$$

It follows that $f'(n) < 0$ for $n \geq 2$. Hence, f decreases on the interval $[2, +\infty)$. The hypotheses of the Integral Test are satisfied.

Compute the integral:

$$\int_2^{\infty} \frac{1}{n(\ln n)^{4/3}}\,dn = \lim_{p\to\infty} \int_2^p \frac{1}{n(\ln n)^{4/3}}\,dn =$$

$$\lim_{p\to\infty} \left[-3\frac{1}{(\ln n)^{1/3}}\right]_2^p = -3 \lim_{p\to\infty} \left(\frac{1}{(\ln p)^{1/3}} - \frac{1}{(\ln 2)^{1/3}}\right) = \frac{3}{(\ln 2)^{1/3}}.$$

(To do the integral, I substituted $u = \ln n$, so $du = \dfrac{1}{n}\,dn$.)

Since the integral converges, the series converges by the Integral Test. □

(b) Apply the Ratio Test. The n^{th} term of the series is

$$a_n = \frac{2 \cdot 5 \cdot \cdots \cdot (3n-1)}{1 \cdot 5 \cdot \cdots \cdot (4n-3)}.$$

Hence, the $(n+1)$-st term is

$$a_{n+1} = \frac{2 \cdot 5 \cdot \cdots \cdot (3n-1) \cdot (3(n+1)-1)}{1 \cdot 5 \cdot \cdots \cdot (4n-3) \cdot (4(n+1)-3)}.$$

Hence,

$$\frac{a_{n+1}}{a_n} = \frac{2 \cdot 5 \cdot \cdots \cdot (3n-1) \cdot (3(n+1)-1)}{1 \cdot 5 \cdot \cdots \cdot (4n-3) \cdot (4(n+1)-3)} \cdot \frac{1 \cdot 5 \cdot \cdots \cdot (4n-3)}{2 \cdot 5 \cdot \cdots \cdot (3n-1)} = \frac{3(n+1)-1}{4(n+1)-3} = \frac{3n+2}{4n+1}.$$

The limiting ratio is

$$\lim_{n\to\infty} \frac{3n+2}{4n+1} = \frac{3}{4}.$$

The limit is less than 1, so the series converges, by the Ratio Test. □

(c) Apply the Root Test.

$$a_n^{1/n} = \left(1 + \frac{1}{n}\right)^{-n}.$$

The limit is

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \lim_{n\to\infty} \left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{-1} = \left\{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n}\right\}^{-1} = e^{-1}.$$

Since $e^{-1} = \dfrac{1}{e} < 1$, the series converges, by the Root Test. □

(d) Note that

$$\lim_{n\to\infty} \frac{2+3n}{3+5n} = \frac{3}{5}.$$

It follows that $\lim_{n\to\infty} a_n$ is undefined — the values oscillate, approaching $\pm\dfrac{3}{5}$. Since, in particular, the limit is nonzero, the series diverges, by the Zero Limit Test. $\square$

(e) Apply Limit Comparison:

$$\lim_{n\to\infty} \frac{\dfrac{3n^2+4n+2}{\sqrt{n^5+16}}}{\dfrac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{3n^{5/2}+4n^{3/2}+2n^{1/2}}{\sqrt{n^5+16}} = 3.$$

The limit is finite and positive. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, because it's a p-series with $p = \dfrac{1}{2} < 1$. Therefore, the original series diverges by Limit Comparison. $\square$

(f)

$$\begin{array}{ccccc} -1 & \le & \cos(e^n) & \le & 1 \\ 4 & \le & 5+\cos(e^n) & \le & 6 \\ \dfrac{4}{n} & \le & \dfrac{5+\cos(e^n)}{n} & \le & \dfrac{6}{n} \end{array}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{4}{n}$ diverges, because it's 4 times the harmonic series. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{5+\cos(e^n)}{n}$ diverges by Direct Comparison. $\square$

(g) The series has positive terms.

$f(x) = xe^{-2x}$ is continuous for $x \ge 1$.

Compute the derivative:

$$f'(x) = -2xe^{-2x} + e^{-2x} = (1-2x)e^{-2x}.$$

$e^{-2x} > 0$ for all x, and $1-2x < 0$ for $x \ge 1$. Therefore, $f'(x) < 0$ for $x \ge 1$. Hence, $f(x)$ decreases for $x \ge 1$.

The three conditions for applying the Integral Test are satisfied. Compute the integral:

$$\int_1^{\infty} xe^{-2x}\,dx = \lim_{a\to\infty} \int_1^a xe^{-2x}\,dx = \lim_{a\to\infty} \left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right]_1^a =$$

$$\lim_{a\to\infty} \left(-\frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a} + \frac{1}{2}e^{-2} + \frac{1}{4}e^{-2}\right) = 0 - 0 + \frac{3}{4}e^{-2} = \frac{3}{4}e^{-2}.$$

Here's the work for the integral:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \displaystyle\int dx \\ + & x & & e^{-2x} \\ & & \searrow & \\ - & 1 & & -\dfrac{1}{2}e^{-2x} \\ & & \searrow & \\ + & 0 & & \dfrac{1}{4}e^{-2x} \end{array}$$

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C.$$

Here's the work for the two limits. I used L'Hôpital's Rule to compute the first limit.

$$\lim_{a\to\infty} -\frac{1}{2}ae^{-2a} = -\frac{1}{2}\lim_{a\to\infty}\frac{a}{e^{2a}} = -\frac{1}{2}\lim_{a\to\infty}\frac{1}{2e^{2a}} = 0.$$

$$\lim_{a\to\infty}\frac{1}{4}e^{-2a} = \frac{1}{4}\lim_{a\to\infty}\frac{1}{e^{2a}} = 0.$$

Since the integral converges, the series converges by the Integral Test. □

(h) This is *not* an alternating series, even though it contains a $(-1)^k$!

The series looks like $\dfrac{1}{k^2}$ for large k; use Limit Comparison. The limiting ratio is

$$\lim_{k\to\infty}\frac{\dfrac{k+\sqrt{k}}{k^3+(-1)^k}}{\dfrac{1}{k^2}} = \lim_{k\to\infty}\frac{k^3+k^{5/2}}{k^3+(-1)^k} = 1.$$

The limit is nonzero and finite. $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2}$ converges, because it's a p-series with $p = 2 > 1$. Therefore, $\displaystyle\sum_{k=2}^{\infty}\frac{k+\sqrt{k}}{k^3+(-1)^k}$ converges by Limit Comparison. □

---

15. Find the values of x for which the following series converges absolutely.

$$\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}(x-5)^n$$

Apply the Ratio Test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{((n+1)!)^2}{(2(n+1))!}|x-5|^{n+1}}{\dfrac{(n!)^2}{(2n)!}|x-5|^n} = \left(\frac{(n+1)!}{n!}\right)^2\frac{(2n)!}{(2n+2)!}\frac{|x-5|^{n+1}}{|x-5|^n} = \frac{(n+1)^2}{(2n+1)(2n+2)}|x-5|.$$

The limiting ratio is

$$\lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n+2)}|x-5| = \frac{1}{4}|x-5|.$$

The series converges absolutely for $\dfrac{1}{4}|x-5| < 1$, i.e. for $1 < x < 9$. The series diverges for $x < 1$ and for $x > 9$.

You'll probably find it difficult to determine what is happening at the endpoints! However, if you experiment — compute some terms of the series for $x = 9$, for instance — you'll see that the individual terms are growing larger, so the series at $x = 1$ and at $x = 9$ diverge, by the Zero Limit Test. □

---

16. The series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k^{1/3}+2}$ converges by the Alternating Series Test. Determine the smallest value of n for which the partial sum $\displaystyle\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k^{1/3}+2}$ approximates the actual sum to within 0.01.

The error in approximating the exact value of the sum by $\sum_{k=1}^{n}(-1)^{k+1}\frac{1}{k^{1/3}+2}$ is less than the $(n+1)^{\text{st}}$ term, which is $\frac{1}{(n+1)^{1/3}+2}$. So I want

$$\begin{aligned}\frac{1}{(n+1)^{1/3}+2} &< 0.01\\ 100 &< (n+1)^{1/3}+2\\ 98 &< (n+1)^{1/3}\\ 941192 &< n+1\\ 941191 &< n\end{aligned}$$

Take $n = 941192$. □

---

17. (a) Find the Taylor expansion at $c = 1$ for e^{2x}.

(b) Find the Taylor expansion at $c = 1$ for $\frac{1}{3+x}$. What is the interval of convergence?

(a)

$$e^{2x} = e^{2(x-1)+2} = e^2 e^{2(x-1)} = e^2\left(1 + 2(x-1) + \frac{2^2(x-1)^2}{2!} + \frac{2^3(x-1)^3}{3!} + \cdots\right). \quad \square$$

(b)

$$\frac{1}{3+x} = \frac{1}{4+(x-1)} = \frac{1}{4}\cdot\frac{1}{1+\frac{x-1}{4}} = \frac{1}{4}\cdot\frac{1}{1-\left(-\frac{x-1}{4}\right)} =$$

$$\frac{1}{4}\left(1 - \frac{x-1}{4} + \left(\frac{x-1}{4}\right)^2 - \left(\frac{x-1}{4}\right)^3 + \cdots\right).$$

The series converges for $-1 < \frac{x-1}{4} < 1$, i.e. for $-3 < x < 5$. □

---

18. (a) Use the Binomial Series to write out the first three nonzero terms of the series for $\frac{1}{\sqrt{1-t^2}} = (1-t^2)^{-1/2}$.

(b) Find the first three terms of the Taylor series at $c = 0$ for $\sin^{-1} x$ by integrating the series you got in (a) from $t = 0$ to $x = x$.

(a)

$$(1-t^2)^{-1/2} = 1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots.$$

(b)

$$\sin^{-1} x = \int_0^x (1-t^2)^{-1/2}\,dt = \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots\right) dt = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots. \quad \square$$

---

19. Use the Remainder Term to find the minimum number of terms of the Taylor series at $c = 0$ for $f(x) = e^{5x}$ needed to approximate $f(x)$ on the interval $0 \le x \le 0.3$ to within $0.00001 = 10^{-5}$.

First, I'll find the Remainder Term.

$$f'(x) = 5e^{5x}, \quad f''(x) = 5^2 e^{5x}, \quad \ldots, f^{(n)}(x) = 5^n e^{5x}.$$

Hence, for some z between 0 and x,

$$R_n(x; 0) = \frac{5^{n+1} e^{5z}}{(n+1)!} x^{n+1}.$$

I want $|R_n(x; 0)| < 10^{-5}$.
Since $0 \le x \le 0.3$, I have

$$|x|^{n+1} \le 0.3^{n+1}.$$

Moreover, since z is between 0 and x and $0 \le x \le 0.3$, I also have $0 \le z \le 0.3$. So

$$\begin{aligned} 0 &\le z \le 0.3 \\ 0 &\le 5z \le 1.5 \\ e^0 &\le e^{5z} \le e^{1.5} \end{aligned}$$

Therefore,

$$|R_n(x; 0)| = \frac{5^{n+1} e^{5z}}{(n+1)!} |x|^{n+1} \le \frac{5^{n+1} e^{1.5}}{(n+1)!} (0.3)^{n+1}.$$

Therefore, I want the smallest value of n such that

$$\frac{5^{n+1} e^{1.5}}{(n+1)!} (0.3)^{n+1} < 10^{-5}.$$

This inequality can't be solved algebraically, due to the factorial in the denominator. So I have to do this by trial and error.

n	$\frac{5^{n+1} e^{1.5}}{(n+1)!} (0.3)^{n+1}$
7	$.00284\ldots$
8	$4.74788\ldots \cdot 10^{-4}$
9	$7.12182\ldots \cdot 10^{-5}$
10	$9.71157\ldots \cdot 10^{-6}$

The smallest value of n that works is $n = 10$. □

---

20. Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n \cdot 7^n}$.

$$\lim_{n \to \infty} \frac{\dfrac{|x-3|^{n+1}}{(n+1) \cdot 7^{n+1}}}{\dfrac{|x-3|^n}{n \cdot 7^n}} = \lim_{n \to \infty} \frac{1}{7} \cdot \frac{n}{n+1} |x-3| = \frac{1}{7} |x-3|.$$

By the Ratio Test, the series converges for $\frac{1}{7}|x-3| < 1$. Hence, the base interval is : $-4 < x < 10$.

At $x = 10$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. It diverges, because it's harmonic.

At $x = -4$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. It converges, because it's alternating harmonic.
The interval of convergence is $-4 \le x < 10$. $\square$

---

21. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n(2^n)^3}$.

Apply the Ratio Test to the absolute value series:

$$\lim_{n\to\infty} \frac{\dfrac{|x-3|^{n+1}}{(n+1)(2^{n+1})^3}}{\dfrac{|x-3|^n}{n(2^n)^3}} = \lim_{n\to\infty} \frac{n}{n+1}\left(\frac{2^n}{2^{n+1}}\right)^3 \frac{|x-3|^{n+1}}{|x-3|^n} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1}{8}|x-3| = \frac{1}{8}|x-3|.$$

The series converges for $\dfrac{1}{8}|x-3| < 1$, i.e. for $-5 < x < 11$.
At $x = 11$, the series is

$$\sum_{n=1}^{\infty} \frac{8^n}{n(2^n)^3} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

It's harmonic, so it diverges.
At $x = -5$, the series is

$$\sum_{n=1}^{\infty} \frac{(-8)^n}{n(2^n)^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

This is the alternating harmonic series, so it converges.
Therefore, the power series converges for $-5 \le x < 11$, and diverges elsewhere. $\square$

---

22. If $x = t + e^t$ and $y = t + t^3$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $t = 1$.

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{1+3t^2}{1+e^t}.$$

When $t = 1$, $\dfrac{dy}{dx} = \dfrac{4}{1+e}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(\frac{dt}{dx}\right)\left(\frac{d}{dt}\left(\frac{dy}{dx}\right)\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{d}{dt}\dfrac{1+3t^2}{1+e^t}}{1+e^t} =$$

$$\frac{\dfrac{(1+e^t)(6t)-(1+3t^2)(e^t)}{(1+e^t)^2}}{1+e^t} = \frac{(1+e^t)(6t)-(1+3t^2)(e^t)}{(1+e^t)^3}.$$

When $t = 1$, $\dfrac{d^2y}{dx^2} = \dfrac{6+2e}{(1+e)^3}$. $\square$

---

23. Consider the parametric curve

$$x = t^2 + t + 1, \quad y = t^3 - 5t + 2.$$

(a) Find the equation of the tangent line at $t = 1$.

(b) Find $\dfrac{d^2y}{dx^2}$ at $t = 1$.

(a)

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{3t^2 - 5}{2t + 1}.$$

When $t = 1$, $x = 3$, $y = -2$, and $\dfrac{dy}{dx} = -\dfrac{2}{3}$. The equation of the tangent line is

$$y + 2 = -\frac{2}{3}(x - 3). \quad \square$$

(b)

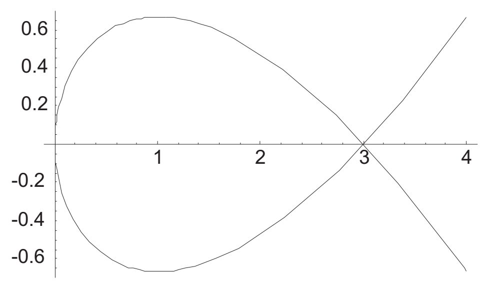
$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{d}{dt}\left(\dfrac{3t^2 - 5}{2t + 1}\right)}{2t + 1} = \frac{\dfrac{(2t + 1)(6t) - (3t^2 - 5)(2)}{(2t + 1)^2}}{2t + 1} = \frac{6t^2 + 6t + 10}{(2t + 1)^3}.$$

When $t = 1$, $\dfrac{d^2y}{dx^2} = \dfrac{22}{27}$. $\square$

---

24. Find the length of the loop of the curve

$$x = t^2, \quad y = t - \frac{t^3}{3}.$$



I'll do the easy part first, which is to find the integrand for the arc length. It is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(2t)^2 + (1 - t^2)^2} = \sqrt{4t^2 + (t^4 - 2t^2 + 1)} = \sqrt{1 + 2t^2 + t^4} = \sqrt{(1 + t^2)^2} = 1 + t^2.$$

(Note that since $(t^2 - 1)^2 = t^4 - 2t^2 + 1$, you know that $t^4 + 2t^2 + 1 = (t^2 + 1)^2$.)

To find the limits of integration, I have to find two values of t which give the same values of x and y. The loop is traced out between these limits.

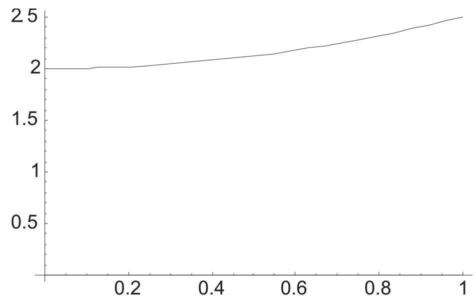
Note that $x = t^2$ is the same for t and $-t$, because of the square. Note also that $y = t\left(1 - \dfrac{t^2}{3}\right)$, so $y = 0$ for $t = 0$ and $t = \pm\sqrt{3}$. Therefore, the values $t = \pm\sqrt{3}$ make $y = 0$, *and* since they're negatives of one another they give the same x-value. In other words, they give the same point on the curve. Thus, the loop is traced out from $t = -\sqrt{3}$ to $t = \sqrt{3}$.

The length is

$$L = \int_{-\sqrt{3}}^{\sqrt{3}} (1+t^2)\, dt = \left[t + \frac{1}{3}t^3\right]_{-\sqrt{3}}^{\sqrt{3}} = 4\sqrt{3} \approx 6.92820. \quad \square$$

---

25. Find the length of the curve $y = \frac{1}{2}x^2 + 2$ for $0 \le x \le 1$.



$$\frac{dy}{dx} = x, \quad \left(\frac{dy}{dx}\right)^2 = x^2,$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + x^2, \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1+x^2}.$$

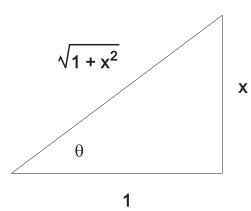
The length is

$$L = \int_0^1 \sqrt{1+x^2}\, dx = \left[\frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln|\sqrt{1+x^2}+x|\right]_0^1 = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1) \approx 1.14779.$$

Here's the work for the integral:

$$\int \sqrt{1+x^2}\, dx = \int \sqrt{1+(\tan\theta)^2}(\sec\theta)^2\, d\theta = \int \sqrt{(\sec\theta)^2}(\sec\theta)^2\, d\theta = \int (\sec\theta)^3\, d\theta =$$

$$\left[x = \tan\theta, \quad dx = (\sec\theta)^2\, d\theta\right]$$



$$\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln|\sqrt{1+x^2}+x| + C. \quad \square$$

---

26. Let

$$x = \frac{\sqrt{3}}{2}t^2, \quad y = t - \frac{1}{4}t^3.$$

Find the length of the arc of the curve from $t = -2$ to $t = 2$.

$$\frac{dx}{dt} = \sqrt{3}t \quad \text{and} \quad \frac{dy}{dt} = 1 - \frac{3}{4}t^2.$$

Hence,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 3t^2 + \left(1 - \frac{3}{4}t^2\right)^2 = 3t^2 + 1 - \frac{3}{2}t^2 + \frac{9}{16}t^4 = 1 + \frac{3}{2}t^2 + \frac{9}{16}t^4 = \left(1 + \frac{3}{4}t^2\right)^2.$$

Therefore,

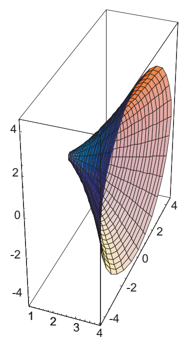
$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1 + \frac{3}{4}t^2.$$

The length is

$$\int_{-2}^{2} \left(1 + \frac{3}{4}t^2\right)\, dt = \left[t + \frac{1}{4}t^3\right]_{-2}^{2} = 8. \quad \square$$

---

27. Find the area of the surface generated by revolving $y = \frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}$, $1 \le x \le 4$, about the x-axis.



$$y' = x^{1/2} - \frac{1}{4x^{1/2}}, \quad \text{so} \quad (y')^2 = x - \frac{1}{2} + \frac{1}{16x}.$$

Hence,

$$1 + (y')^2 = x + \frac{1}{2} + \frac{1}{16x}.$$

Notice that this is just $(y')^2$ with the sign of the middle term changed. But $(y')^2$ was $x^{1/2} - \frac{1}{4x^{1/2}}$ squared, so $1 + (y')^2$ must be $x^{1/2} + \frac{1}{4x^{1/2}}$ squared:

$$1 + (y')^2 = \left(x^{1/2} + \frac{1}{4x^{1/2}}\right)^2.$$

Thus,

$$\sqrt{1 + (y')^2} = x^{1/2} + \frac{1}{4x^{1/2}}.$$

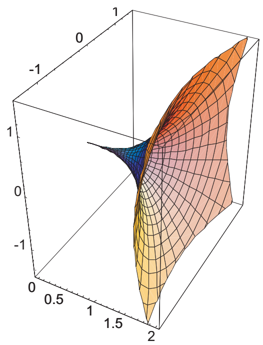
The area is

$$S = \int_1^4 2\pi \left(\frac{2}{3}x^{3/2} - \frac{1}{2}x^{1/2}\right)\left(x^{1/2} + \frac{1}{4x^{1/2}}\right) dx = 2\pi \int_1^4 \left(\frac{2}{3}x^2 - \frac{1}{3}x - \frac{1}{8}\right) dx = 2\pi \left[\frac{2x^3}{9} - \frac{x^2}{6} - \frac{x}{8}\right]_1^4 =$$

$$\frac{89\pi}{4} \approx 69.90044. \quad \square$$

---

28. Find the area of the surface generated by revolving $y = \frac{1}{3}x^3$, $0 \le x \le 2$, about the x-axis.



The derivative is

$$\frac{dy}{dx} = x^2, \quad \text{so} \quad \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} = \sqrt{x^4 + 1}.$$

The curve is being revolved about the x-axis, so the radius of revolution is $R = y = \frac{1}{3}x^3$. The area of the surface is

$$S = \int_0^2 2\pi \left(\frac{1}{3}x^3\right) \sqrt{x^4 + 1}\, dx = \frac{2\pi}{3} \int_1^{17} u^{1/2} \cdot x^3 \left(\frac{du}{4x^3}\right) = \frac{\pi}{6} \int_1^{17} u^{1/2}\, du = \frac{\pi}{6} \left[\frac{2}{3} u^{3/2}\right]_1^{17} =$$

$$\left[u = x^4 + 1, \quad du = 4x^3\, dx, \quad dx = \frac{du}{4x^3}; \quad x = 0, u = 1, \quad x = 2, u = 17\right]$$

$$\frac{\pi}{9}\left(17^{3/2} - 1\right) \approx 24.11794.a \quad \square$$

---

29. (a) Convert $(x - 3)^2 + (y + 4)^2 = 25$ to polar and simplify.

(b) Convert $r = 4\cos\theta - 6\sin\theta$ to rectangular and describe the graph.

(a)

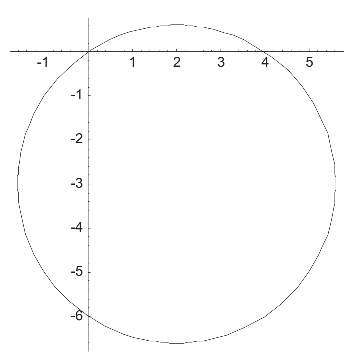
$$(x - 3)^2 + (y + 4)^2 = 25, \quad x^2 - 6x + 9 + y^2 + 8y + 16 = 25, \quad x^2 + y^2 = 6x - 8y,$$

$$r^2 = 6r\cos\theta - 8r\sin\theta, \quad r = 6\cos\theta - 8\sin\theta. \quad \square$$

(b)

$$r = 4\cos\theta - 6\sin\theta, \quad r^2 = 4r\cos\theta - 6r\sin\theta, \quad x^2 + y^2 = 4x - 6y, \quad x^2 - 4x + y^2 + 6y = 0,$$

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 13, \quad (x - 2)^2 + (y + 3)^2 = 13.$$



The graph is a circle of radius $\sqrt{13}$ centered at $(2, -3)$. □

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30. Find the slope of the tangent line to the polar curve $r = \sin 2\theta$ at $\theta = \dfrac{\pi}{6}$.

When $\theta = \dfrac{\pi}{6}$, $r = \sin \dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$. Since $\dfrac{dr}{d\theta} = 2\cos 2\theta$, when $\theta = \dfrac{\pi}{6}$, $\dfrac{dr}{d\theta} = 2\cos \dfrac{\pi}{3} = 1$.
The slope of the tangent line is

$$\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta \dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta \dfrac{dr}{d\theta}} = \frac{\left(\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) + \left(\dfrac{1}{2}\right)(1)}{\left(-\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{1}{2}\right) + \left(\dfrac{\sqrt{3}}{2}\right)(1)} = \frac{5\sqrt{3}}{3} \approx 2.88675. \quad \square$$

---

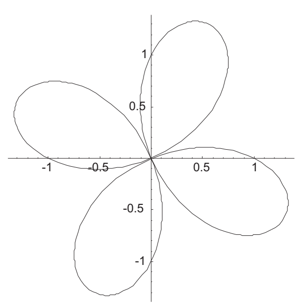
31. Find the slope of the tangent line to $r = 2 + \cos\theta$ at $\theta = \dfrac{\pi}{6}$.

First, $\dfrac{dr}{d\theta} = -\sin\theta$. When $\theta = \dfrac{\pi}{6}$, $r = 2 + \dfrac{\sqrt{3}}{2}$ and $\dfrac{dr}{d\theta} = -\dfrac{1}{2}$.
Therefore,

$$\frac{dy}{dx} = \frac{r\cos\theta + (\sin\theta)\dfrac{dr}{d\theta}}{-r\sin\theta + (\cos\theta)\dfrac{dr}{d\theta}} = \frac{\left(2 + \dfrac{\sqrt{3}}{2}\right)\left(\dfrac{\sqrt{3}}{2}\right) + \left(\dfrac{1}{2}\right)\left(-\dfrac{1}{2}\right)}{-\left(2 + \dfrac{\sqrt{3}}{2}\right)\left(\dfrac{1}{2}\right) + \left(\dfrac{\sqrt{3}}{2}\right)\left(-\dfrac{1}{2}\right)} = -\frac{2\sqrt{3} + 1}{\sqrt{3} + 2} \approx -1.19615. \quad \square$$

---

32. Find the values of θ in the interval $[0, 2\pi]$ for which the polar curve $r = \cos 2\theta - \sin 2\theta$ passes through the origin.



Set $r = 0$:

$$\cos 2\theta - \sin 2\theta = 0, \quad \cos 2\theta = \sin 2\theta, \quad 1 = \tan 2\theta.$$

I'll solve $\tan u = 1$. Since the argument of the equation above is 2θ, I need solutions in the range $0 \le u \le 2 \cdot 2\pi = 4\pi$. By basic trigonometry,

$$\tan \frac{\pi}{4} = 1, \quad \tan \frac{5\pi}{4} = 1, \quad \tan \frac{9\pi}{4} = 1, \quad \tan \frac{13\pi}{4} = 1.$$

Thus, the solutions are

$$u = \frac{\pi}{4}, \quad u = \frac{5\pi}{4}, \quad u = \frac{9\pi}{4}, \quad u = \frac{13\pi}{4}.$$

Set $u = 2\theta$ and solve for θ:

$$2\theta = \frac{\pi}{4}, \quad 2\theta = \frac{5\pi}{4}, \quad 2\theta = \frac{9\pi}{4}, \quad 2\theta = \frac{13\pi}{4},$$
$$\theta = \frac{\pi}{8}, \quad \theta = \frac{5\pi}{8}, \quad \theta = \frac{9\pi}{8}, \quad \theta = \frac{13\pi}{8}. \quad \square$$

---

33. Find the length of the cardioid $r = 1 + \sin\theta$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1+\sin\theta)^2 + (\cos\theta)^2 = 1 + 2\sin\theta + (\sin\theta)^2 + (\cos\theta)^2 = 2 + 2\sin\theta.$$

By the double angle formula,

$$\left(\sin\frac{\theta}{2}\right)^2 = \frac{1}{2}(1+\sin\theta)$$
$$4\left(\sin\frac{\theta}{2}\right)^2 = 2(1+\sin\theta)$$
$$4\left(\sin\frac{\theta}{2}\right)^2 = 2 + 2\sin\theta$$

Thus,

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4\left(\sin\frac{\theta}{2}\right)^2$$
$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\sin\frac{\theta}{2}$$

The length is

$$\int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta = \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 8. \quad \square$$

---

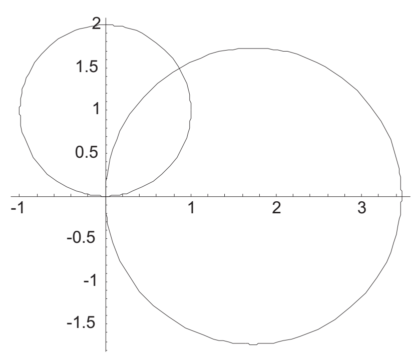
34. Find the area of the intersection of the interiors of the circles

$$x^2 + (y-1)^2 = 1 \quad \text{and} \quad (x-\sqrt{3})^2 + y^2 = 3.$$

Convert the two equations to polar:

$$x^2 + y^2 - 2y + 1 = 1, \quad x^2 + y^2 = 2y, \quad r^2 = 2r\sin\theta, \quad r = 2\sin\theta.$$

$$(x-\sqrt{3})^2 + y^2 = 3, \quad x^2 - 2\sqrt{3}x + 3 + y^2 = 3, \quad x^2 + y^2 = 2\sqrt{3}x, \quad r^2 = 2\sqrt{3}r\cos\theta, \quad r = 2\sqrt{3}\cos\theta.$$



Set the equations equal to solve for the line of intersection:

$$2\sin\theta = 2\sqrt{3}\cos\theta, \quad \tan\theta = \sqrt{3}, \quad \theta = \frac{\pi}{3}.$$

The region is "orange-slice"-shaped, with the bottom/right half bounded by $r = 2\sin\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{3}$ and the top/left half bounded by $r = 2\sqrt{3}\cos\theta$ from $\theta = \frac{\pi}{3}$ to $\theta = \frac{\pi}{2}$. Hence, the area is

$$A = \int_0^{\pi/3} \frac{1}{2}(2\sin\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(2\sqrt{3}\cos\theta)^2\,d\theta = 2\int_0^{\pi/3}(\sin\theta)^2\,d\theta + 6\int_{\pi/3}^{\pi/2}(\cos\theta)^2\,d\theta =$$

$$\int_0^{\pi/3}(1-\cos 2\theta)\,d\theta + 3\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta = \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/3} + 3\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2} =$$

$$\frac{5}{6}\pi - \sqrt{3} \approx 0.88594. \quad \square$$

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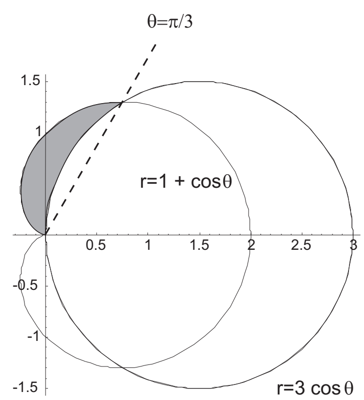
35. Find the area of the region inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 3\cos\theta$.



Find the intersection points:

$$3\cos\theta = 1 + \cos\theta, \quad 2\cos\theta = 1, \quad \cos\theta = \frac{1}{2}, \quad \theta = \pm\frac{\pi}{3}.$$

I'll find the area of the shaded region and double it to get the total. The shaded area is

$$\left(\text{cardioid area from } \frac{\pi}{3} \text{ to } \pi\right) - \left(\text{circle area from } \frac{\pi}{3} \text{ to } \frac{\pi}{2}\right).$$

The cardioid area is

$$\int_{\pi/3}^{\pi} \frac{1}{2}(1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_{\pi/3}^{\pi}\left(1 + 2\cos\theta + (\cos\theta)^2\right)\,d\theta = \frac{1}{2}\int_{\pi/3}^{\pi}\left(1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right)\,d\theta =$$

$$\frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_{\pi/3}^{\pi} = \frac{\pi}{2} - \frac{9}{16}\sqrt{3}.$$

The circle area is

$$\int_{\pi/3}^{\pi/2} \frac{1}{2}(3\cos\theta)^2\,d\theta = \frac{9}{4}\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{9}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2} = \frac{3\pi}{8} - \frac{9}{16}\sqrt{3}.$$

Thus, the shaded area is

$$\left(\frac{\pi}{2} - \frac{9}{16}\sqrt{3}\right) - \left(\frac{3\pi}{8} - \frac{9}{16}\sqrt{3}\right) = \frac{\pi}{8}.$$

The total area is $2 \cdot \frac{\pi}{8} = \frac{\pi}{4} \approx 0.78540$. □

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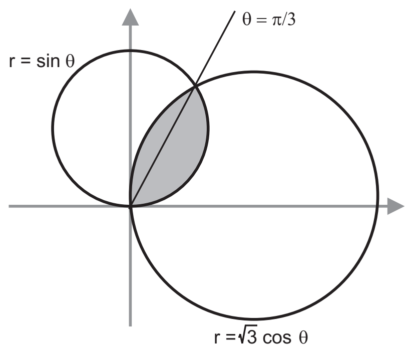
36. Let A be the region inside $r = \sin\theta$ and let B be the region inside $r = \sqrt{3}\cos\theta$. Find the area of the intersection of A and B — that is, the area of the region common to A and B.



Find the intersection point:

$$\sin\theta = \sqrt{3}\cos\theta, \quad \tan\theta = \sqrt{3}, \quad \theta = \frac{\pi}{3}.$$

(The circles also intersect at the origin, but they pass through the origin at different values of θ.)

The shaded area is the sum of the area inside $r = \sin\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{3}$ and the area inside $r = \sqrt{3}\cos\theta$ from $\theta = \frac{\pi}{3}$ to $\theta = \frac{\pi}{2}$:

$$A = \int_0^{\pi/3} \frac{1}{2}(\sin\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(\sqrt{3}\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^{\pi/3} \frac{1}{2}(1 - \cos 2\theta)\,d\theta + \frac{3}{2}\int_{\pi/3}^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\,d\theta =$$

$$\frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/3} + \frac{3}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2} = \frac{5}{24}\pi - \frac{1}{4}\sqrt{3} \approx 0.22149.$$ □

The best thing for being sad is to learn something. - Merlyn, in T. H. White's *The Once and Future King*

