

Review Problems for Test 1

These problems are provided to help you study. The presence of a problem on this sheet does not imply that a similar problem will appear on the test. And the absence of a problem from this sheet does not imply that the test will not have a similar problem.

1. Compute $\int x^3 e^{4x} dx$.
2. Compute $\int e^{4x} \cos 2x dx$.
3. Compute $\int (x^2 + x + 1) \cos x dx$.
4. Compute $\int \cos 3x \sin 2x dx$.
5. Compute $\int \ln(x^2 + 5) dx$.
6. Compute $\int_0^{\pi/2} (\cos x)^3 (1 + (\sin x)^{1/2}) dx$.
7. Compute $\int (\cos 7x)^4 dx$.
8. Compute $\int [\sin(x + 3)]^2 [\cos(x + 3)]^2 dx$.
9. Compute $\int (\sec 3x)^5 \tan 3x dx$.
10. Compute $\int (\tan x)^4 dx$.
11. Compute $\int (\csc 2x)^3 (\cot 2x)^3 dx$.
12. Compute $\int \sqrt{x^2 - 1} dx$.
13. Compute $\int x^2 \sqrt{25 - x^2} dx$.
14. Compute $\int x \sqrt{25 - x^2} dx$.
15. Compute $\int \frac{x^2}{(x^2 - 9)^{3/2}} dx$.
16. Compute $\int \frac{x^2}{\sqrt{x^2 + 1}} dx$.
17. Compute $\int \frac{-x + 18}{x^2 - 8x + 12} dx$.

18. Compute $\int \frac{-3x^2 + 47x - 150}{(x-6)^2(x+2)} dx$.
19. Compute $\int \frac{3 + 4x + 5x^2 + 3x^3}{x^2(x+3)} dx$.
20. Compute $\int \frac{1}{x^{1/2}(x^{1/3} + x^{1/4})} dx$.
21. Compute $\int \frac{1}{x^{7/8} + x^{5/8}} dx$.
22. Compute $\int \frac{1}{\sqrt{7 + 6x - x^2}} dx$.
23. Compute $\int \frac{1}{x^2 + 10x + 74} dx$.
24. Compute $\int \frac{x-2}{x^2 - 8x + 25} dx$.
25. Compute $\int \frac{x+3}{\sqrt{-x^2 - 6x - 8}} dx$.
26. Compute $\int \frac{1}{2x^2 + 8x + 10} dx$.
27. Compute $\int \frac{6x^3 - 24x^2 + 16x + 4}{x^4 - 4x^3 + 4x^2} dx$.
28. Compute $\int \frac{4x^3 + 2x^2 + 16x + 11}{(x^2 + 1)(x^2 + 4)} dx$.
29. Compute $\int \frac{2x^2 - 9x + 11}{(x-1)^3} dx$.
30. How would you try to decompose $\frac{2(x-2)^2}{x^4(x^2+4)^3}$ using partial fractions? (Just write out the fractions — you don't need to solve for the parameters.)
31. What is wrong with the following “partial fractions decomposition”?
- $$\frac{5x}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x+1}?$$
32. What is wrong with the following “partial fractions decomposition”?
- $$\frac{7}{x(x-1)} = \frac{A}{x} + \frac{B}{x} + \frac{C}{x-1}.$$
33. Find the partial fractions decomposition of
- $$\frac{-3x^4 + x^3 - 6x^2 - 3}{x(x^2 + 1)^2}.$$
34. (a) Compute $\int \frac{-x^2 + 8x - 4}{x(x^2 + x - 2)} dx$.

(b) Calvin Butterball tries to use the antiderivative from (a) to compute

$$\int_{-1}^{1/2} \frac{-x^2 + 8x - 4}{x(x^2 + x - 2)} dx.$$

He gets

$$[2 \ln |x| + \ln |x - 1| - 4 \ln |x + 2|]_{-1}^{1/2} = \left(2 \ln \frac{1}{2} + \ln \frac{1}{2} - 4 \ln 3/2 \right) - (2 \ln 1 + \ln 2 - 4 \ln 1) \approx -4.39445.$$

Does this computation make sense? Why or why not?

35. Find the area of the region under $y = \frac{x}{(x^2 + 1)^2}$ from $x = 0$ to ∞ .

36. Compute $\int_4^6 \frac{1}{\sqrt{x-4}} dx$.

37. Compute $\int_{-\infty}^0 x e^{x^2} dx$.

38. Compute $\int_0^{\infty} x e^{-3x} dx$.

39. Compute $\int_0^{\infty} \cos 3x dx$.

40. Compute $\int_1^{\infty} \frac{x^2 + 4}{x^3 + 12x + 1} dx$.

41. Compute $\int_1^{\infty} \frac{x^2 + 4}{(x^3 + 12x + 1)^2} dx$.

42. Compute $\int_2^{11} \frac{1}{\sqrt[3]{x-3}} dx$.

43. Prove that $\int_0^{\infty} e^{-x^4} dx$ converges. [Hint: Compare the integral to $\int_0^{\infty} e^{-x} dx$.]

44. Prove that $\int_0^{\infty} \frac{(\sin x)^2}{x^2 + 1} dx$ converges. [Hint: Use comparison, starting with the fact that $(\sin x)^2 \leq 1$.]

45. (a) Show that the following integrals both diverge:

$$\int_0^{\infty} x dx \quad \text{and} \quad \int_{-\infty}^0 x dx.$$

(It follows that $\int_{-\infty}^{\infty} x dx$ diverges as well.)

(b) Show that $\lim_{b \rightarrow \infty} \int_{-b}^b x dx$ converges. (This is called the **Cauchy principal value** of the integral; this problem shows that $\lim_{b \rightarrow \infty} \int_{-b}^b x dx$ is not the same as $\int_{-\infty}^{\infty} x dx$.)

Solutions to the Review Problems for Test 1

1. Compute $\int x^3 e^{4x} dx$.

$$\begin{array}{rcl}
 & \frac{d}{dx} & \int dx \\
 + & x^3 & e^{4x} \\
 & \searrow & \\
 - & 3x^2 & \frac{1}{4}e^{4x} \\
 & \searrow & \\
 + & 6x & \frac{1}{16}e^{4x} \\
 & \searrow & \\
 - & 6 & \frac{1}{64}e^{4x} \\
 & \searrow & \\
 + & 0 & \frac{1}{256}e^{4x}
 \end{array}$$

$$\int x^3 e^{4x} dx = \frac{1}{4}x^3 e^{4x} - \frac{3}{16}x^2 e^{4x} + \frac{6}{64}x e^{4x} - \frac{6}{256}e^{4x} + C. \quad \square$$

2. Compute $\int (x^2 + x + 1) \cos x dx$.

$$\begin{array}{rcl}
 & \frac{d}{dx} & \int dx \\
 + & x^2 + x + 1 & \cos x \\
 - & 2x + 1 & \sin x \\
 + & 2 & -\cos x \\
 - & 0 & -\sin x
 \end{array}$$

$$\int (x^2 + x + 1) \cos x dx = (x^2 + x + 1) \sin x + (2x + 1) \cos x - 2 \sin x + C. \quad \square$$

3. Compute $\int e^{4x} \cos 2x dx$.

$$\begin{array}{rcl}
 & \frac{d}{dx} & \int dx \\
 + & \cos 2x & e^{4x} \\
 & \searrow & \\
 - & -2 \sin 2x & \frac{1}{4}e^{4x} \\
 & \searrow & \\
 + & -4 \cos 2x & \frac{1}{16}e^{4x}
 \end{array}$$

$$\int e^{4x} \cos 2x dx = \frac{1}{4}e^{4x} \cos 2x + \frac{1}{8}e^{4x} \sin 2x - \frac{1}{4} \int e^{4x} \cos 2x dx,$$

$$\int e^{4x} \cos 2x dx + \frac{1}{4} \int e^{4x} \cos 2x dx = \frac{1}{4}e^{4x} \cos 2x + \frac{1}{8}e^{4x} \sin 2x - \frac{1}{4} \int e^{4x} \cos 2x dx + \frac{1}{4} \int e^{4x} \cos 2x dx,$$

$$\frac{5}{4} \int e^{4x} \cos 2x dx = \frac{1}{4}e^{4x} \cos 2x + \frac{1}{8}e^{4x} \sin 2x,$$

$$\begin{aligned} \frac{4}{5} \cdot \frac{5}{4} \int e^{4x} \cos 2x \, dx &= \frac{4}{5} \cdot \frac{1}{4} e^{4x} \cos 2x + \frac{4}{5} \cdot \frac{1}{8} e^{4x} \sin 2x, \\ \int e^{4x} \cos 2x \, dx &= \frac{1}{5} e^{4x} \cos 2x + \frac{1}{10} e^{4x} \sin 2x + C. \quad \square \end{aligned}$$

4. Compute $\int \cos 3x \sin 2x \, dx$.

$$\begin{array}{r} \frac{d}{dx} \quad \int dx \\ + \cos 3x \quad \sin 2x \\ - -3 \sin 3x \quad \searrow \quad -\frac{1}{2} \cos 2x \\ + -9 \cos 3x \quad \rightarrow \quad -\frac{1}{4} \sin 2x \end{array}$$

$$\begin{aligned} \int \cos 3x \sin 2x \, dx &= -\frac{1}{2} \cos 3x \cos 2x - \frac{3}{4} \sin 3x \sin 2x + \frac{9}{4} \int \cos 3x \sin 2x \, dx, \\ -\frac{5}{4} \int \cos 3x \sin 2x \, dx &= -\frac{1}{2} \cos 3x \cos 2x - \frac{3}{4} \sin 3x \sin 2x, \\ \int \cos 3x \sin 2x \, dx &= \frac{2}{5} \cos 3x \cos 2x + \frac{3}{5} \sin 3x \sin 2x + C. \quad \square \end{aligned}$$

5. Compute $\int \ln(x^2 + 5) \, dx$.

$$\begin{array}{r} \frac{d}{dx} \quad \int dx \\ + \ln(x^2 + 5) \quad 1 \\ - \frac{2x}{x^2 + 5} \quad \searrow \quad x \end{array}$$

$$\begin{aligned} \int \ln(x^2 + 5) \, dx &= x \ln(x^2 + 5) - \int \frac{2x^2}{x^2 + 5} \, dx = x \ln(x^2 + 5) - \int \left(2 - \frac{10}{x^2 + 5} \right) \, dx = \\ &= x \ln(x^2 + 5) - 2x + \frac{10}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} + C. \end{aligned}$$

The second equality comes from dividing $2x^2$ by $x^2 + 5$ (long division). Alternatively, you can do this:

$$\frac{2x^2}{x^2 + 5} = \frac{2x^2 + 10 - 10}{x^2 + 5} = \frac{2x^2 + 10}{x^2 + 5} - \frac{10}{x^2 + 5} = \frac{2(x^2 + 5)}{x^2 + 5} - \frac{10}{x^2 + 5} = 2 - \frac{10}{x^2 + 5}. \quad \square$$

6. Compute $\int_0^{\pi/2} (\cos x)^3 (1 + (\sin x)^{1/2}) \, dx$.

I'll do the antiderivative first:

$$\int (\cos x)^3 (1 + (\sin x)^{1/2}) \, dx = \int (\cos x)^2 (1 + (\sin x)^{1/2}) \cos x \, dx =$$

$$\int (1 - (\sin x)^2) (1 + (\sin x)^{1/2}) \cos x \, dx =$$

$$\left[u = \sin x, \quad du = \cos x \, dx, \quad dx = \frac{du}{\cos x} \right]$$

$$\int (1 - u^2)(1 + u^{1/2}) \, du = \int (1 + u^{1/2} - u^2 - u^{5/2}) \, du = u + \frac{2}{3}u^{3/2} - \frac{1}{3}u^3 - \frac{2}{7}u^{7/2} + C =$$

$$\sin x + \frac{2}{3}(\sin x)^{3/2} - \frac{1}{3}(\sin x)^3 - \frac{2}{7}(\sin x)^{7/2} + C.$$

Therefore,

$$\int_0^{\pi/2} (\cos x)^3 (1 + (\sin x)^{1/2}) \, dx = \left[\sin x + \frac{2}{3}(\sin x)^{3/2} - \frac{1}{3}(\sin x)^3 - \frac{2}{7}(\sin x)^{7/2} \right]_0^{\pi/2} = \frac{22}{21}. \quad \square$$

7. Compute $\int (\cos 7x)^4 \, dx$.

$$\int (\cos 7x)^4 \, dx = \int (\cos 7x)^2 (\cos 7x)^2 \, dx = \int \frac{1}{2}(1 + \cos 14x) \cdot \frac{1}{2}(1 + \cos 14x) \, dx =$$

$$\frac{1}{4} \int (1 + 2 \cos 14x + (\cos 14x)^2) \, dx = \frac{1}{4} \int \left(1 + 2 \cos 14x + \frac{1}{2}(1 + \cos 28x) \right) \, dx =$$

$$\frac{1}{4} \left(x + \frac{1}{7} \sin 14x + \frac{1}{2} \left(x + \frac{1}{28} \sin 28x \right) \right) + C. \quad \square$$

8. Compute $\int [\sin(x+3)]^2 [\cos(x+3)]^2 \, dx$.

$$\int [\sin(x+3)]^2 [\cos(x+3)]^2 \, dx = \int \left(\frac{1}{2}[1 - \cos 2(x+3)] \right) \left(\frac{1}{2}[1 + \cos 2(x+3)] \right) \, dx =$$

$$\frac{1}{4} \int (1 - [\cos 2(x+3)]^2) \, dx = \frac{1}{4} \int [\sin 2(x+3)]^2 \, dx = \frac{1}{4} \int \frac{1}{2}[1 - \cos 4(x+3)] \, dx =$$

$$\frac{1}{8} \left(x - \frac{1}{4} \sin 4(x+3) \right) + C. \quad \square$$

9. Compute $\int (\sec 3x)^5 \tan 3x \, dx$.

$$\int (\sec 3x)^5 \tan 3x \, dx = \int (\sec 3x)^4 (\sec 3x \tan 3x \, dx) = \frac{1}{3} \int u^4 \, du = \frac{1}{15} u^5 + C = \frac{1}{15} (\sec 3x)^5 + C.$$

$$\left[u = \sec 3x, \quad du = 3 \sec 3x \tan 3x \, dx, \quad dx = \frac{du}{3 \sec 3x \tan 3x} \right] \quad \square$$

10. Compute $\int (\tan x)^4 \, dx$.

$$\begin{aligned}
\int (\tan x)^4 dx &= \int (\tan x)^2 (\tan x)^2 dx = \int (\tan x)^2 ((\sec x)^2 - 1) dx = \\
&= \int (\tan x)^2 (\sec x)^2 dx - \int (\tan x)^2 dx = \int (\tan x)^2 (\sec x)^2 dx - \int ((\sec x)^2 - 1) dx = \\
&= \int (\tan x)^2 (\sec x)^2 dx - \int (\sec x)^2 dx + \int dx = \\
&= \left[u = \tan x, \quad du = (\sec x)^2 dx, \quad dx = \frac{du}{(\sec x)^2} \right] \\
&= \int u^2 du - \tan x + x + C = \frac{1}{3}u^3 - \tan x + x + C = \frac{1}{3}(\tan x)^3 - \tan x + x + C. \quad \square
\end{aligned}$$

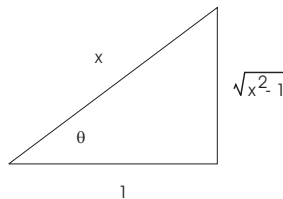
11. Compute $\int (\csc 2x)^3 (\cot 2x)^3 dx$.

$$\begin{aligned}
\int (\csc 2x)^3 (\cot 2x)^3 dx &= \int (\csc 2x)^2 (\cot 2x)^2 (\csc 2x \cot 2x) dx = \\
&= \int (\csc 2x)^2 [(\csc 2x)^2 - 1] (\csc 2x \cot 2x) dx = \int u^2 (u^2 - 1) (\csc 2x \cot 2x) \cdot \frac{du}{-2 \csc 2x \cot 2x} = \\
&= \left[u = \csc 2x, \quad du = -2 \csc 2x \cot 2x dx, \quad dx = \frac{du}{-2 \csc 2x \cot 2x} \right] \\
-\frac{1}{2} \int u^2 (u^2 - 1) du &= -\frac{1}{2} \int (u^4 - u^2) du = -\frac{1}{2} \left(\frac{1}{5}u^5 - \frac{1}{3}u^3 \right) + C = -\frac{1}{2} \left(\frac{1}{5}(\csc 2x)^5 - \frac{1}{3}(\csc 2x)^3 \right) + C. \quad \square
\end{aligned}$$

12. Compute $\int \sqrt{x^2 - 1} dx$.

$$\int \sqrt{x^2 - 1} dx = \int \sqrt{(\sec \theta)^2 - 1} \sec \theta \tan \theta d\theta = \int \sqrt{(\tan \theta)^2} \sec \theta \tan \theta d\theta = \int \sec \theta (\tan \theta)^2 d\theta =$$

$$[x = \sec \theta, \quad dx = \sec \theta \tan \theta d\theta]$$

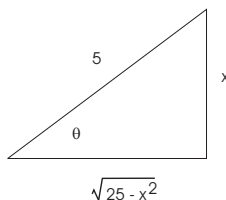


$$\begin{aligned}
\int \sec \theta ((\sec \theta)^2 - 1) d\theta &= \int ((\sec \theta)^3 - \sec \theta) d\theta = \frac{1}{2} \sec \theta \tan \theta + \frac{1}{2} \ln |\sec \theta + \tan \theta| - \ln |\sec \theta + \tan \theta| + C = \\
\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C &= \frac{1}{2} x \sqrt{x^2 - 1} - \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| + C. \quad \square
\end{aligned}$$

13. Compute $\int x^2 \sqrt{25 - x^2} dx$.

$$\int x^2 \sqrt{25 - x^2} dx = \int 25(\sin \theta)^2 \sqrt{25 - 25(\sin \theta)^2} (5 \cos \theta) d\theta = \int 25(\sin \theta)^2 \sqrt{25(\cos \theta)^2} (5 \cos \theta) d\theta =$$

$$[x = 5 \sin \theta, \quad dx = 5 \cos \theta d\theta]$$



$$625 \int (\sin \theta)^2 (\cos \theta)^2 d\theta = 625 \int \frac{1}{2}(1 - \cos 2\theta) \cdot \frac{1}{2}(1 + \cos 2\theta) d\theta = \frac{625}{4} \int (1 - (\cos 2\theta)^2) d\theta =$$

$$\frac{625}{4} \int (\sin 2\theta)^2 d\theta = \frac{625}{4} \int \frac{1}{2}(1 - \cos 4\theta) d\theta = \frac{625}{8} \left(\theta - \frac{1}{4} \sin 4\theta \right) + C =$$

$$\frac{625}{8} (\theta - \sin \theta \cos \theta (2(\cos \theta)^2 - 1)) + C = \frac{625}{8} \arcsin \frac{x}{5} - \frac{1}{8} x (\sqrt{25 - x^2}) (25 - 2x^2) + C. \quad \square$$

14. Compute $\int x \sqrt{25 - x^2} dx$.

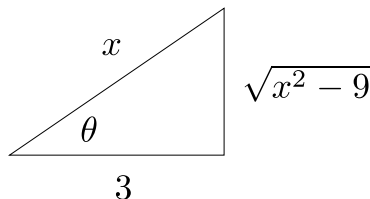
$$\int x \sqrt{25 - x^2} dx = \int x \sqrt{u} \cdot \left(\frac{du}{-2x} \right) = -\frac{1}{2} \int u^{1/2} du = -\frac{1}{3} u^{3/2} + C = -\frac{1}{3} (25 - x^2)^{3/2} + C.$$

$$\left[u = 25 - x^2, \quad du = -2x dx, \quad dx = \frac{du}{-2x} \right] \quad \square$$

15. Compute $\int \frac{x^2}{(x^2 - 9)^{3/2}} dx$.

$$\int \frac{x^2}{(x^2 - 9)^{3/2}} dx = \int \frac{9(\sec \theta)^2}{(9(\sec \theta)^2 - 9)^{3/2}} \cdot 3 \sec \theta \tan \theta d\theta = \int \frac{27(\sec \theta)^3 \tan \theta}{27((\sec \theta)^2 - 1)^{3/2}} d\theta = \int \frac{(\sec \theta)^3 \tan \theta}{((\tan \theta)^2)^{3/2}} d\theta =$$

$$[x = 3 \sec \theta, \quad dx = 3 \sec \theta \tan \theta d\theta]$$



$$\int \frac{(\sec \theta)^3 \tan \theta}{(\tan \theta)^3} d\theta = \int \frac{(\sec \theta)^3}{(\tan \theta)^2} d\theta = \int \frac{1}{(\cos \theta)^3} \cdot \frac{(\cos \theta)^2}{(\sin \theta)^2} d\theta = \int \frac{1}{(\cos \theta)(\sin \theta)^2} d\theta =$$

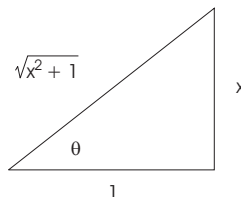
$$\int \frac{(\sin \theta)^2 + (\cos \theta)^2}{(\cos \theta)(\sin \theta)^2} d\theta = \int \left(\frac{(\sin \theta)^2}{(\cos \theta)(\sin \theta)^2} + \frac{(\cos \theta)^2}{(\cos \theta)(\sin \theta)^2} \right) d\theta = \int \left(\frac{1}{\cos \theta} + \frac{\cos \theta}{(\sin \theta)^2} \right) d\theta =$$

$$\int (\sec \theta + \csc \theta \cot \theta) d\theta = \ln |\sec \theta + \tan \theta| - \csc \theta + c = \ln \left| \frac{x}{3} + \frac{\sqrt{x^2 - 9}}{3} \right| - \frac{3}{\sqrt{x^2 - 9}} + c. \quad \square$$

16. Compute $\int \frac{x^2}{\sqrt{x^2 + 1}} dx$.

$$\int \frac{x^2}{\sqrt{x^2 + 1}} dx = \int \frac{(\tan \theta)^2}{\sqrt{(\tan \theta)^2 + 1}} (\sec \theta)^2 d\theta = \int \frac{(\tan \theta)^2}{\sqrt{(\sec \theta)^2}} (\sec \theta)^2 d\theta = \int \frac{(\tan \theta)^2}{\sec \theta} (\sec \theta)^2 d\theta =$$

$$[x = \tan \theta, \quad dx = (\sec \theta)^2 d\theta]$$



$$\int \sec \theta (\tan \theta)^2 d\theta = \int \sec \theta ((\sec \theta)^2 - 1) d\theta = \int ((\sec \theta)^3 - \sec \theta) d\theta =$$

$$\frac{1}{2} \sec \theta \tan \theta - \frac{1}{2} \ln |\sec \theta + \tan \theta| + C = \frac{1}{2} x \sqrt{x^2 + 1} - \frac{1}{2} \ln |\sqrt{x^2 + 1} + x| + C. \quad \square$$

17. Compute $\int \frac{-x + 18}{x^2 - 8x + 12} dx$.

Since $x^2 - 8x + 12 = (x - 6)(x - 2)$, I have

$$\frac{-x + 18}{(x - 6)(x - 2)} = \frac{a}{x - 6} + \frac{b}{x - 2}$$

$$-x + 18 = a(x - 2) + b(x - 6)$$

Set $x = 6$. This gives

$$12 = a \cdot 4, \quad \text{so} \quad a = 3.$$

Set $x = 2$. This gives

$$16 = b \cdot (-4), \quad \text{so} \quad b = -4.$$

Therefore,

$$\int \frac{-x + 18}{(x - 6)(x - 2)} dx = \int \left(\frac{3}{x - 6} + \frac{-4}{x - 2} \right) dx = 3 \ln |x - 6| - 4 \ln |x - 2| + C. \quad \square$$

18. Compute $\int \frac{-3x^2 + 47x - 150}{(x - 6)^2(x + 2)} dx$.

$$\frac{-3x^2 + 47x - 150}{(x - 6)^2(x + 2)} = \frac{a}{x - 6} + \frac{b}{(x - 6)^2} + \frac{c}{x + 2}$$

$$-3x^2 + 47x - 150 = a(x - 6)(x + 2) + b(x + 2) + c(x - 6)^2$$

Set $x = 6$. This gives

$$24 = b \cdot 8, \quad \text{so } b = 3.$$

Set $x = -2$. This gives

$$-256 = c \cdot 64, \quad \text{so } c = -4.$$

Plugging the values for b and c back in yields

$$-3x^2 + 47x - 150 = a(x - 6)(x + 2) + 3(x + 2) - 4(x - 6)^2.$$

Set $x = 0$. This gives

$$-150 = a \cdot (-12) + 6 - 144, \quad \text{so } a = 1.$$

Thus,

$$\frac{-3x^2 + 47x - 150}{(x - 6)^2(x + 2)} = \frac{1}{x - 6} + \frac{3}{(x - 6)^2} + \frac{-4}{x + 2}.$$

Hence,

$$\begin{aligned} \int \frac{-3x^2 + 47x - 150}{(x - 6)^2(x + 2)} dx &= \int \left(\frac{1}{x - 6} + \frac{3}{(x - 6)^2} + \frac{-4}{x + 2} \right) dx = \\ &= \ln|x - 6| - \frac{3}{x - 6} - 4 \ln|x + 2| + C. \quad \square \end{aligned}$$

19. Compute $\int \frac{3 + 4x + 5x^2 + 3x^3}{x^2(x + 3)} dx$.

The top and the bottom both have degree 3, so I must divide the top by the bottom:

$$\frac{3 + 4x + 5x^2 + 3x^3}{x^2(x + 3)} = 3 + \frac{3 + 4x - 4x^2}{x^2(x + 3)}.$$

I'll put the 3 aside for now, and work on the fraction:

$$\frac{3 + 4x - 4x^2}{x^2(x + 3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 3}.$$

Clear denominators:

$$3 + 4x - 4x^2 = Ax(x + 3) + B(x + 3) + Cx^2.$$

Set $x = 0$: I get $3 = 3B$, so $B = 1$.

Set $x = -3$: I get $-45 = 9C$, so $C = -5$.

Plug B and C back in:

$$3 + 4x - 4x^2 = Ax(x + 3) + (x + 3) - 5x^2.$$

Differentiate:

$$4 - 8x = A(x + 3) + Ax + 1 - 10x.$$

Set $x = 0$: I $4 = 3A + 1$, so $A = 1$.

Therefore,

$$\frac{3 + 4x - 4x^2}{x^2(x + 3)} = \frac{1}{x} + \frac{1}{x^2} - \frac{5}{x + 3}.$$

Hence,

$$\frac{3 + 4x + 5x^2 + 3x^3}{x^2(x + 3)} = 3 + \frac{1}{x} + \frac{1}{x^2} - \frac{5}{x + 3}.$$

Finally,

$$\int \frac{3 + 4x + 5x^2 + 3x^3}{x^2(x+3)} dx = \int \left(3 + \frac{1}{x} + \frac{1}{x^2} - \frac{5}{x+3} \right) dx = 3x + \ln|x| - \frac{1}{x} - 5 \ln|x+3| + C. \quad \square$$

20. Compute $\int \frac{1}{x^{1/2}(x^{1/3} + x^{1/4})} dx$.

Since the least common multiple of 2, 3, and 4 is 12, I'll let $x = u^{12}$:

$$\begin{aligned} \int \frac{1}{x^{1/2}(x^{1/3} + x^{1/4})} dx &= \int \frac{12u^{11} du}{u^6(u^4 + u^3)} = 12 \int \frac{u^2}{u+1} du = 12 \int \left(u - 1 + \frac{1}{u+1} \right) du = \\ & \quad [x = u^{12}, \quad dx = 12u^{11} du] \\ &= 12 \left(\frac{1}{2}u^2 - u + \ln|u+1| \right) + C = 12 \left(\frac{1}{2}x^{1/6} - x^{1/12} + \ln|x^{1/12} + 1| \right) + C. \quad \square \end{aligned}$$

21. Compute $\int \frac{1}{x^{7/8} + x^{5/8}} dx$.

$$\begin{aligned} \int \frac{1}{x^{7/8} + x^{5/8}} dx &= \int \frac{1}{u^7 + u^5} \cdot 8u^7 du = 8 \int \frac{u^2}{u^2 + 1} du = 8 \int \frac{(u^2 + 1) - 1}{u^2 + 1} du = \\ & \quad [x = u^8, \quad dx = 8u^7 du; \quad u = x^{1/8}] \\ &= 8 \int \left(\frac{u^2 + 1}{u^2 + 1} - \frac{1}{u^2 + 1} \right) du = 8 \int \left(1 - \frac{1}{u^2 + 1} \right) du = 8(u - \tan^{-1} u) + c = 8(x^{1/8} - \tan^{-1} x^{1/8}) + c. \quad \square \end{aligned}$$

22. Compute $\int \frac{1}{\sqrt{7+6x-x^2}} dx$.

Complete the square:

$$\begin{aligned} 7 + 6x - x^2 &= -[x^2 - 6x - 7] \\ &= -[x^2 - 6x + 9 - 16] \\ &= -[(x-3)^2 - 16] \\ &= 16 - (x-3)^2 \end{aligned}$$

So

$$\begin{aligned} \int \frac{1}{\sqrt{7+6x-x^2}} dx &= \int \frac{1}{\sqrt{16-(x-3)^2}} dx = \int \frac{1}{\sqrt{16-u^2}} du = \sin^{-1} \frac{u}{4} + c = \sin^{-1} \frac{x-3}{4} + c. \\ & \quad [u = x-3, \quad du = dx] \quad \square \end{aligned}$$

23. Compute $\int \frac{1}{x^2 + 10x + 74} dx$.

Complete the square:

$$\begin{aligned} x^2 + 10x + 74 &= x^2 + 10x + 25 + 49 \\ &= (x+5)^2 + 49 \end{aligned}$$

So

$$\int \frac{1}{x^2 + 10x + 74} dx = \int \frac{1}{(x+5)^2 + 49} dx = \int \frac{1}{u^2 + 49} du = \frac{1}{7} \tan^{-1} \frac{u}{7} + c = \frac{1}{7} \tan^{-1} \frac{x+5}{7} + c.$$

$$[u = x + 5, \quad du = dx] \quad \square$$

24. Compute $\int \frac{x-2}{x^2-8x+25} dx$.

Since $\frac{1}{2} \cdot (-8) = -4$ and $(-4)^2 = 16$, I have

$$x^2 - 8x + 25 = x^2 - 8x + 16 + 9 = (x-4)^2 + 9.$$

Therefore,

$$\int \frac{x-2}{x^2-8x+25} dx = \int \frac{x-2}{(x-4)^2+9} dx = \int \frac{(u+4)-2}{u^2+9} du = \int \frac{u+2}{u^2+9} du =$$

$$[u = x - 4, \quad du = dx; \quad x = u + 4]$$

$$\int \frac{u}{u^2+9} du + \int \frac{2}{u^2+9} du = \frac{1}{2} \ln |u^2+9| + \frac{2}{3} \tan^{-1} \frac{u}{3} + C = \frac{1}{2} \ln |(x-4)^2+9| + \frac{2}{3} \tan^{-1} \frac{x-4}{3} + C.$$

I did the first part of the u -integral using the substitution $w = u^2 + 9$. \square

25. Compute $\int \frac{x+3}{\sqrt{-x^2-6x-8}} dx$.

First,

$$-x^2 - 6x - 8 = -(x^2 + 6x + 8) = -[(x^2 + 6x + 9) - 1] = -[(x+3)^2 - 1] = 1 - (x+3)^2.$$

I note that $\frac{6}{2} = 3$ and $3^2 = 9$, so I needed 9 to complete the square.

Thus,

$$\int \frac{x+3}{\sqrt{-x^2-6x-8}} dx = \int \frac{x+3}{\sqrt{1-(x+3)^2}} dx = \int \frac{u}{\sqrt{1-u^2}} du = \int \frac{u}{\sqrt{w}} \cdot \frac{dw}{-2u} =$$

$$\left[u = x + 3, \quad du = dx; w = 1 - u^2, \quad dw = -2u du, \quad du = \frac{dw}{-2u} \right]$$

$$-\frac{1}{2} \int \frac{1}{\sqrt{w}} dw = -\frac{1}{2} \cdot 2\sqrt{w} + C = -\sqrt{1-u^2} + C = -\sqrt{1-(x+3)^2} + C. \quad \square$$

26. Compute $\int \frac{1}{2x^2+8x+10} dx$.

$$\int \frac{1}{2x^2+8x+10} dx = \frac{1}{2} \int \frac{1}{x^2+4x+5} dx = \frac{1}{2} \int \frac{1}{(x^2+4x+4)+1} dx = \frac{1}{2} \int \frac{1}{(x+2)^2+1} dx =$$

$$\frac{1}{2} \tan^{-1}(x+2) + C.$$

I completed the square by noting that $\frac{4}{2} = 2$ and $2^2 = 4$. You can do the integral using $u = x + 2$. \square

27. Compute $\int \frac{6x^3 - 24x^2 + 16x + 4}{x^4 - 4x^3 + 4x^2} dx$.

First, $x^4 - 4x^3 + 4x^2 = x^2(x-2)^2$.

$$\frac{6x^3 - 24x^2 + 16x + 4}{x^2(x-2)^2} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-2} + \frac{d}{(x-2)^2}$$

$$6x^3 - 24x^2 + 16x + 4 = ax(x-2)^2 + b(x-2)^2 + cx^2(x-2) + dx^2$$

Set $x = 0$. I get $4 = 4b$, so $b = 1$.

Set $x = 2$. I get $-12 = 4d$, so $d = -3$.

Then

$$6x^3 - 24x^2 + 16x + 4 = ax(x-2)^2 + (x-2)^2 + cx^2(x-2) - 3x^2.$$

At this point, you can plug other numbers in for x , or differentiate the equation and then plug numbers in. The idea is to get equations for a and c which you can solve.

For example, set $x = 1$. I get

$$2 = a + 1 - c - 3, \quad \text{or} \quad 4 = a - c.$$

Set $x = -1$. I get

$$-42 = -9a + 9 - 3c - 3, \quad \text{or} \quad 16 = 3a + c.$$

I have to solve $4 = a - c$ and $16 = 3a + c$. You can do this in various ways.

For instance, if I add the equations $4 = a - c$ and $16 = 3a + c$, I get $20 = 4a$, so $a = 5$. Then plugging $a = 5$ into $4 = a - c$, I get $4 = 5 - c$, so $c = 1$.

Thus,

$$\int \frac{6x^3 - 24x^2 + 16x + 4}{x^4 - 4x^3 + 4x^2} dx = \int \left(\frac{5}{x} + \frac{1}{x^2} + \frac{1}{x-2} - \frac{3}{(x-2)^2} \right) dx = 5 \ln|x| - \frac{1}{x} + \ln|x-2| + \frac{3}{x-2} + C. \quad \square$$

28. Compute $\int \frac{4x^3 + 2x^2 + 16x + 11}{(x^2 + 1)(x^2 + 4)} dx$.

$$\frac{4x^3 + 2x^2 + 16x + 11}{(x^2 + 1)(x^2 + 4)} = \frac{ax + b}{x^2 + 1} + \frac{cx + d}{x^2 + 4}$$

$$4x^3 + 2x^2 + 16x + 11 = (ax + b)(x^2 + 4) + (cx + d)(x^2 + 1)$$

Set $x = 0$: This gives

$$11 = 4b + d. \tag{1}$$

Differentiate the last x -equation (using the Product Rule on the two terms on the right):

$$12x^2 + 4x + 16 = (ax + b)(2x) + a(x^2 + 4) + (cx + d)(2x) + c(x^2 + 1).$$

Set $x = 0$:

$$16 = 4a + c. \tag{2}$$

Differentiate the last x -equation:

$$24x + 4 = (ax + b)(2) + (a)(2x) + (a)(2x) + (cx + d)(2) + (c)(2x) + (c)(2x).$$

Set $x = 0$:

$$4 = 2b + 2d, \quad \text{so} \quad 2 = b + d. \quad (3)$$

Differentiate the last x -equation:

$$24 = 2a + 2a + 2a + 2c + 2c + 2c, \quad \text{so} \quad 4 = a + c. \quad (4)$$

Solving (1) ($11 = 4b + d$) together with (3) ($2 = b + d$) gives $b = 3$ and $d = -1$.

Solving (2) ($16 = 4a + c$) together with (4) ($4 = a + c$) gives $a = 4$ and $c = 0$.

Thus, I have

$$\int \frac{4x^3 + 2x^2 + 16x + 11}{(x^2 + 1)(x^2 + 4)} dx = \int \left(\frac{4x}{x^2 + 1} + \frac{3}{x^2 + 1} - \frac{1}{x^2 + 4} \right) dx = 2 \ln(x^2 + 1) + 3 \tan^{-1} x - \frac{1}{2} \tan^{-1} \frac{x}{2} + C.$$

The first integral is computed using $u = x^2 + 1$; the second and third use the inverse tangent formula:

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C. \quad \square$$

29. Compute $\int \frac{2x^2 - 9x + 11}{(x - 1)^3} dx$.

$$\frac{2x^2 - 9x + 11}{(x - 1)^3} = \frac{a}{x - 1} + \frac{b}{(x - 1)^2} + \frac{c}{(x - 1)^3}$$

$$2x^2 - 9x + 11 = a(x - 1)^2 + b(x - 1) + c$$

Set $x = 1$: This gives $4 = c$. Plug this into the last equation:

$$2x^2 - 9x + 11 = a(x - 1)^2 + b(x - 1) + 4.$$

There are a number of ways to proceed. For instance, you can plug in two other values for x (say $x = 0$ and $x = 2$) to obtain two equations involving a and b , which you can solve simultaneously.

Alternatively, differentiate the last equation:

$$4x - 9 = 2a(x - 1) + b.$$

Set $x = 1$: This gives $-5 = b$. Plug this into the last equation:

$$4x - 9 = 2a(x - 1) - 5.$$

Set $x = 0$: This gives

$$-9 = -2a - 5$$

$$-4 = -2a$$

$$2 = a$$

Plugging $a = 2$, $b = -5$, and $c = 4$ into the original decomposition, I have

$$\int \frac{2x^2 - 9x + 11}{(x - 1)^3} dx = \int \left(\frac{2}{x - 1} - \frac{5}{(x - 1)^2} + \frac{4}{(x - 1)^3} \right) dx = 2 \ln |x - 1| + \frac{5}{x - 1} - \frac{2}{(x - 1)^2} + C. \quad \square$$

30. How would you try to decompose $\frac{2(x-2)^2}{x^4(x^2+4)^3}$ using partial fractions?

$$\frac{2(x-2)^2}{x^4(x^2+4)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{Ex+F}{x^2+4} + \frac{Gx+H}{(x^2+4)^2} + \frac{Ix+J}{(x^2+4)^3}. \quad \square$$

31. What is wrong with the following “partial fractions decomposition”?

$$\frac{5x}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x+1}?$$

Partial fractions is the opposite of combining fractions over a common denominator. In this case, the question is: “What fractions would add up to $\frac{5x}{(x-1)^2(x+1)}$?” The decompositions above *could* occur, since it has $(x-1)^2(x+1)$ as the common denominator.

However, since you don’t know beforehand what the fractions are, you must assume the “worst case” — namely, that there might be an $\frac{A}{x-1}$ term. And in fact, there is — if you work out the decomposition, it comes out to

$$\frac{5x}{(x-1)^2(x+1)} = -\frac{5}{4} \frac{1}{x+1} + \frac{5}{4} \frac{1}{x-1} + \frac{5}{2} \frac{1}{(x-1)^2}.$$

Notice the term $\frac{5}{4} \frac{1}{x-1}$. \square

32. What is wrong with the following “partial fractions decomposition”?

$$\frac{7}{x(x-1)} = \frac{A}{x} + \frac{B}{x} + \frac{C}{x-1}.$$

The first two terms could be combined into a single term $\frac{D}{x}$, so they’re redundant. There is no reason to list the same denominator twice. \square

33. Find the partial fractions decomposition of

$$\frac{-3x^4 + x^3 - 6x^2 - 3}{x(x^2+1)^2}.$$

Try the decomposition

$$\frac{-3x^4 + x^3 - 6x^2 - 3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Clear denominators:

$$-3x^4 + x^3 - 6x^2 - 3 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x).$$

Set $x = 0$: I get $A = -3$. Plug it back in:

$$-3x^4 + x^3 - 6x^2 - 3 = -3(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x).$$

Differentiate:

$$12x^3 + 3x^2 - 12x = -12x(x^2 + 1) + B(x^3 + x) + (Bx + C)(3x^2 + 1) + 2Dx + E.$$

Set $x = 0$: I get $C + E = 0$.

Differentiate again:

$$-36x^2 + 6x - 12 = -36x^2 - 12 + B(3x^2 + 1) + B(3x^2 + 1) + (Bx + C)(6x) + 2D.$$

Set $x = 0$: I get $B + D = 0$.

Cancel the $-36x^2$ and -12 terms in the previous equation, then differentiate:

$$6x = B(3x^2 + 1) + B(3x^2 + 1) + (Bx + C)(6x) + 2D,$$

$$6 = 6Bx + 6Bx + 6Bx + (Bx + C)(6).$$

Set $x = 0$: I get $C = 1$. Since $C + E = 0$, it follows that $E = -1$.

Plug $C = 1$ back in, then simplify the equation:

$$6 = 24Bx + 6, \quad \text{or} \quad 0 = 24Bx.$$

Set $x = 1$: I get $B = 0$. But $B + D = 0$, so $D = 0$.

Hence,

$$\frac{-3x^4 + x^3 - 6x^2 - 3}{x(x^2 + 1)^2} = -\frac{3}{x} + \frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)^2}. \quad \square$$

34. (a) Compute $\int \frac{-x^2 + 8x - 4}{x(x^2 + x - 2)} dx$.

(b) Calvin Butterball tries to use the antiderivative from (a) to compute

$$\int_{-1}^{1/2} \frac{-x^2 + 8x - 4}{x(x^2 + x - 2)} dx.$$

He gets

$$[2 \ln |x| + \ln |x - 1| - 4 \ln |x + 2|]_{-1}^{1/2} = \left(2 \ln \frac{1}{2} + \ln \frac{1}{2} - 4 \ln 3 \right) - (2 \ln 1 + \ln 2 - 4 \ln 1) \approx -4.39445.$$

Does this computation make sense? Why or why not?

(a) $x(x^2 + x - 2) = x(x - 1)(x + 2)$, so I try

$$\frac{-x^2 + 8x - 4}{x(x^2 + x - 2)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 2}.$$

Clear denominators:

$$-x^2 + 8x - 4 = A(x - 1)(x + 2) + Bx(x + 2) + Cx(x - 1).$$

Set $x = 0$: I get $-4 = -2A$, or $A = 2$.

Set $x = 1$: I get $3 = 3B$, or $B = 1$.

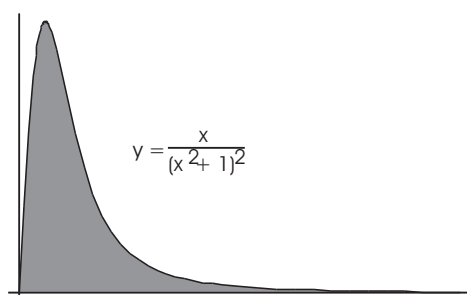
Set $x = -2$: I get $-24 = 6C$, or $C = -4$.

Therefore,

$$\int \frac{-x^2 + 8x - 4}{x(x^2 + x - 2)} dx = \int \left(\frac{2}{x} + \frac{1}{x-1} - \frac{4}{x+2} \right) dx = 2 \ln |x| + \ln |x-1| - 4 \ln |x+2| + C. \quad \square$$

(b) The computation is incorrect, because the antiderivative is valid only within intervals which don't contain the singularities at $x = -2$, $x = 0$, and $x = 1$. The interval $-1 \leq x \leq \frac{1}{2}$ includes the singularity at $x = 0$. It is not legal to simply "plug in the endpoints" — to do this definite integral correctly, you should set it up as two **improper integrals**. \square

35. Find the area of the region under $y = \frac{x}{(x^2 + 1)^2}$ from $x = 0$ to ∞ .



The area is

$$A = \int_0^{\infty} \frac{x}{(x^2 + 1)^2} dx = \lim_{c \rightarrow \infty} \int_0^c \frac{x}{(x^2 + 1)^2} dx = \lim_{c \rightarrow \infty} \left[-\frac{1}{2(x^2 + 1)} \right]_0^c = -\frac{1}{2} \lim_{c \rightarrow \infty} \left(\frac{1}{c^2 + 1} - 1 \right) = \frac{1}{2}.$$

Here's the work for the antiderivative:

$$\int \frac{x}{(x^2 + 1)^2} dx = \int \frac{x}{u^2} \cdot \frac{du}{2x} = \frac{1}{2} \int \frac{du}{u^2} = -\frac{1}{2u} + C = -\frac{1}{2(x^2 + 1)} + C.$$

$$\left[u = x^2 + 1, \quad du = 2x dx, \quad dx = \frac{du}{2x} \right] \quad \square$$

36. Compute $\int_4^6 \frac{1}{\sqrt{x-4}} dx$.

Since $\frac{1}{\sqrt{x-4}}$ is undefined at $x = 4$ and $x = 4$ is in the interval of integration (it's one of the endpoints), the integral is improper. I replace the "4" with a parameter a , then take the limit as a approaches 4 from the right.

$$\int_4^6 \frac{1}{\sqrt{x-4}} dx = \lim_{a \rightarrow 4^+} \int_a^6 \frac{1}{\sqrt{x-4}} dx = \lim_{a \rightarrow 4^+} [2\sqrt{x-4}]_a^6 = 2 \lim_{a \rightarrow 4^+} (\sqrt{2} - \sqrt{a-4}) = 2\sqrt{2}. \quad \square$$

37. Compute $\int_{-\infty}^0 xe^{x^2} dx$.

$$\int_{-\infty}^0 xe^{x^2} dx = \lim_{b \rightarrow -\infty} \int_b^0 xe^{x^2} dx = \lim_{b \rightarrow -\infty} \int_{b^2}^0 xe^u \cdot \frac{du}{2x} = \frac{1}{2} \lim_{b \rightarrow -\infty} \int_{b^2}^0 e^u du =$$

$$\left[u = x^2, \quad du = 2x dx, \quad dx = \frac{du}{2x}; \quad x = b, \quad u = b^2; \quad x = 0, \quad u = 0 \right]$$

$$\frac{1}{2} \lim_{b \rightarrow -\infty} [e^u]_{b^2}^0 = \frac{1}{2} \lim_{b \rightarrow -\infty} (1 - e^{b^2}).$$

As $b \rightarrow -\infty$, I have $b^2 \rightarrow \infty$, and $e^{b^2} \rightarrow \infty$. Therefore, the integral diverges (to $-\infty$, since the e^{b^2} term was negated). \square

38. Compute $\int_0^\infty xe^{-3x} dx$.

$$\int_0^\infty xe^{-3x} dx = \lim_{b \rightarrow +\infty} \int_0^b xe^{-3x} dx = \lim_{b \rightarrow +\infty} \left[-\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} \right]_0^b =$$

$$\lim_{b \rightarrow +\infty} \left(-\frac{1}{3}be^{-3b} - \frac{1}{9}e^{-3b} + \frac{1}{9} \right) = -0 - 0 + \frac{1}{9} = \frac{1}{9}.$$

Here's the work for the two limits:

$$\lim_{b \rightarrow +\infty} e^{-3b} = \lim_{b \rightarrow +\infty} \frac{1}{e^{3b}} = 0,$$

$$\lim_{b \rightarrow +\infty} be^{-3b} = \lim_{b \rightarrow +\infty} \frac{b}{e^{3b}} = \lim_{b \rightarrow +\infty} \frac{1}{3e^{3b}} = 0.$$

I used L'Hôpital's Rule to compute the second limit.

Here's the work for the antiderivative:

$$\begin{array}{rcl} \frac{d}{dx} & & \int dx \\ + & x & e^{-3x} \\ & \searrow & \\ - & 1 & -\frac{1}{3}e^{-3x} \\ & \searrow & \\ + & 0 & \frac{1}{9}e^{-3x} \end{array}$$

$$\int xe^{-3x} dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C. \quad \square$$

39. Compute $\int_0^\infty \cos 3x dx$.

$$\int_0^\infty \cos 3x dx = \lim_{k \rightarrow \infty} \int_0^k \cos 3x dx = \lim_{k \rightarrow \infty} \left[\frac{1}{3} \sin 3x \right]_0^k = \lim_{k \rightarrow \infty} \frac{1}{3} \sin 3k.$$

$\lim_{k \rightarrow \infty} \sin 3k$ is undefined. Hence, the integral diverges. (It doesn't diverge to ∞ or $-\infty$; the limit is simply undefined.) \square

40. Compute $\int_1^\infty \frac{x^2 + 4}{x^3 + 12x + 1} dx$.

Do the antiderivative:

$$\int \frac{x^2 + 4}{x^3 + 12x + 1} dx = \int \frac{x^2 + 4}{u} \cdot \frac{du}{3(x^2 + 4)} = \frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln |u| + c = \frac{1}{3} \ln |x^3 + 12x + 1| + c.$$

$$\left[u = x^3 + 12x + 1, \quad du = (3x^2 + 12) dx = 3(x^2 + 4), \quad dx = \frac{du}{3(x^2 + 4)} \right]$$

Do the integral:

$$\int_1^\infty \frac{x^2 + 4}{x^3 + 12x + 1} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{x^2 + 4}{x^3 + 12x + 1} dx = \lim_{b \rightarrow \infty} \left[\frac{1}{3} \ln |x^3 + 12x + 1| \right]_1^b =$$

$$\frac{1}{3} \lim_{b \rightarrow \infty} (\ln |b^3 + 12b + 1| - \ln 14) = +\infty.$$

(Note that $\lim_{b \rightarrow \infty} \ln |b^3 + 12b + 1| = +\infty$.)

The integral diverges to $+\infty$. \square

41. Compute $\int_1^\infty \frac{x^2 + 4}{(x^3 + 12x + 1)^2} dx$.

Do the antiderivative:

$$\int \frac{x^2 + 4}{(x^3 + 12x + 1)^2} dx = \int \frac{x^2 + 4}{u^2} \cdot \frac{du}{3(x^2 + 4)} = \frac{1}{3} \int \frac{1}{u^2} du = \frac{1}{3} \cdot \frac{-1}{u} + c = -\frac{1}{3} \frac{1}{x^3 + 12x + 1} + c.$$

$$\left[u = x^3 + 12x + 1, \quad du = (3x^2 + 12) dx = 3(x^2 + 4), \quad dx = \frac{du}{3(x^2 + 4)} \right]$$

Do the integral:

$$\int_1^\infty \frac{x^2 + 4}{(x^3 + 12x + 1)^2} dx = \lim_{b \rightarrow \infty} \int_1^b \frac{x^2 + 4}{(x^3 + 12x + 1)^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{3} \frac{1}{x^3 + 12x + 1} \right]_1^b =$$

$$-\frac{1}{3} \lim_{b \rightarrow \infty} \left(\frac{1}{b^3 + 12b + 1} - \frac{1}{14} \right) = \frac{1}{42}.$$

(Note that $\lim_{b \rightarrow \infty} \frac{1}{b^3 + 12b + 1} = 0$.)

The integral converges to $\frac{1}{42}$. \square

42. Compute $\int_2^{11} \frac{1}{\sqrt[3]{x-3}} dx$.

$$\int_2^{11} \frac{1}{\sqrt[3]{x-3}} dx = \int_2^3 \frac{1}{\sqrt[3]{x-3}} dx + \int_3^{11} \frac{1}{\sqrt[3]{x-3}} dx.$$

The first integral is

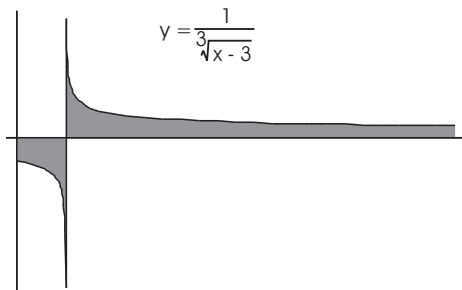
$$\int_2^3 \frac{1}{\sqrt[3]{x-3}} dx = \lim_{a \rightarrow 3^-} \int_2^a \frac{1}{\sqrt[3]{x-3}} dx = \lim_{a \rightarrow 3^-} \left[\frac{3}{2} (x-3)^{2/3} \right]_2^a = \frac{3}{2} \lim_{a \rightarrow 3^-} \left((a-3)^{2/3} - 1 \right) = -\frac{3}{2}.$$

The second integral is

$$\int_3^{11} \frac{1}{\sqrt[3]{x-3}} dx = \lim_{b \rightarrow 3^+} \int_b^{11} \frac{1}{\sqrt[3]{x-3}} dx = \lim_{b \rightarrow 3^+} \left[\frac{3}{2}(x-3)^{2/3} \right]_b^{11} = \frac{3}{2} \lim_{b \rightarrow 3^+} (4 - (b-3)^{2/3}) = 6.$$

Therefore,

$$\int_2^{11} \frac{1}{\sqrt[3]{x-3}} dx = -\frac{3}{2} + 6 = \frac{9}{2}.$$



The graph of $\frac{1}{\sqrt[3]{x-3}}$ has a vertical asymptote at $x = 3$, but the (signed) area on each side is finite. The negative area to the left of $x = 3$ partially cancels the positive area to the right of $x = 3$. Thus, the integral in this problem does not represent the actual area bounded by the graph, the x -axis, and the lines $x = 2$, $x = 3$, and $x = 11$. \square

In the next few problems, I'll use the following fact. If f and g are integrable functions on every finite interval $[a, b]$ and $f(x) \geq g(x) \geq 0$, then if $\int_a^\infty f(x) dx$ converges, then $\int_a^\infty g(x) dx$ converges.

Intuitively, if the bigger function's integral converges to a number, then the smaller function's integral must converge, because it's caught between that number and 0.

43. Prove that $\int_0^\infty e^{-x^4} dx$ converges.

The interval of integration is $x \geq 0$. On this interval, $x \leq x^4$, so $-x \geq -x^4$, and $e^{-x} \geq e^{-x^4}$. Therefore, if $\int_0^\infty e^{-x} dx$ converges, then $\int_0^\infty e^{-x^4} dx$ converges.

Now

$$\int_0^\infty e^{-x} dx = \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx = \lim_{b \rightarrow \infty} [-e^{-x}]_0^b = \lim_{b \rightarrow \infty} (-e^{-b} + 1) = -0 + 1 = 1.$$

Since $\int_0^\infty e^{-x} dx$ converges, it follows that $\int_0^\infty e^{-x^4} dx$ converges as well. \square

44. Prove that $\int_0^\infty \frac{(\sin x)^2}{x^2 + 1} dx$ converges.

I have

$$\begin{aligned} (\sin x)^2 &\leq 1 \\ \frac{(\sin x)^2}{x^2 + 1} &\leq \frac{1}{x^2 + 1} \end{aligned}$$

(I built up from a known fact about trig functions to get an inequality with the function I'm trying to integrate on the "small" side.)

Now

$$\int_0^{\infty} \frac{1}{x^2+1} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{1}{x^2+1} dx = \lim_{b \rightarrow \infty} [\tan^{-1} x]_0^b = \frac{\pi}{2}.$$

Since $\int_0^{\infty} \frac{1}{x^2+1} dx$ converges and $\frac{1}{x^2+1} \geq \frac{(\sin x)^2}{x^2+1} \geq 0$, it follows that $\int_0^{\infty} \frac{(\sin x)^2}{x^2+1} dx$ converges as well. \square

45. (a) Show that the following integrals both diverge:

$$\int_0^{\infty} x dx \quad \text{and} \quad \int_{-\infty}^0 x dx.$$

(It follows that $\int_{-\infty}^{\infty} x dx$ diverges as well.)

(b) Show that $\lim_{b \rightarrow \infty} \int_{-b}^b x dx$ converges. (This is called the **Cauchy principal value** of the integral; this problem shows that $\lim_{b \rightarrow \infty} \int_{-b}^b x dx$ is not the same as $\int_{-\infty}^{\infty} x dx$.)

(a)

$$\begin{aligned} \int_0^{\infty} x dx &= \lim_{b \rightarrow \infty} \int_0^b x dx = \lim_{b \rightarrow \infty} \left[\frac{1}{2} x^2 \right]_0^b = \lim_{b \rightarrow \infty} \frac{1}{2} b^2 = \infty. \\ \int_{-\infty}^0 x dx &= \lim_{b \rightarrow -\infty} \int_b^0 x dx = \lim_{b \rightarrow -\infty} \left[\frac{1}{2} x^2 \right]_b^0 = \lim_{b \rightarrow -\infty} \left(-\frac{1}{2} b^2 \right) = -\infty. \quad \square \end{aligned}$$

(b)

$$\lim_{b \rightarrow \infty} \int_{-b}^b x dx = \lim_{b \rightarrow \infty} \left[\frac{1}{2} x^2 \right]_{-b}^b = \lim_{b \rightarrow \infty} \left(\frac{1}{2} b^2 - \frac{1}{2} b^2 \right) = \lim_{b \rightarrow \infty} 0 = 0. \quad \square$$

To think is not enough; you must think of something. - JULES RENARD