Review Problems for Test 3

These problems are provided to help you study. The presence of a problem on this handout does not imply that there will be a similar problem on the test. And the absence of a topic does not imply that it won't appear on the test.

1. (a) Compute the exact value of \( \int_0^2 \int_0^{1+\sqrt{2x-x^2}} dy \, dx \).

(b) Compute the exact value of \( \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{x^2+y^2}^{4} dz \, dy \, dx \).

2. Find the volume of the solid lying below the paraboloid \( z = x^2 + y^2 \) and above the region in the \( x-y \) plane bounded by \( y = x^2 \) and \( y = x + 2 \).

3. (a) Compute \( \int_0^1 \int_{\sqrt{y}}^{\sqrt{x^4+4}} dx \, dy \).

(b) Compute \( \int_0^2 \int_{y/2}^{1} 3y^2 \cos(x^4) \, dx \, dy + \int_{-2}^0 \int_{-y/2}^{1} 3y^2 \cos(x^4) \, dx \, dy \).

4. Compute \( \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx \).

5. Find the volume of the region which lies above the cone \( z = \frac{1}{\sqrt{3}} \sqrt{x^2 + y^2} \) and below the hemisphere \( z = \sqrt{1 - x^2 - y^2} \).

6. Compute \( \int_0^1 \int_{0}^{\sqrt[3]{x^2+y^2}} \int_{0}^{x+y} \sqrt{x^2+y^2} \, dz \, dy \, dx \).

7. Find the area of the surface

\[
x = 2u \cos v, \quad y = u^2, \quad z = 2u \sin v, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi.
\]

8. Find the center of mass of the region in the first octant cut off by the plane \( 2x + 2y + z = 4 \), if the density is \( \rho(x, y, z) = 2z + 1 \).

9. Compute \( \int_R (x - 2y) \, dx \, dy \), where \( R \) is the parallelogram bounded by \( y = x + 1 \), \( y = x - 2 \), \( y = -\frac{1}{2}x + 1 \), and \( y = -\frac{1}{2}x + 4 \).

10. (a) Compute \( \int_{\vec{s}} (x + y) \, dx - (x - y) \, dy \), where \( \sigma \) is the path consisting of the segment from \((0, 1)\) to \((-1, 0)\), the segment from \((-1, 0)\) to \((0, -1)\), the segment from \((0, -1)\) to \((1, 0)\), and the segment from \((1, 0)\) to \((0, 1)\).

(b) Compute \( \int_{\vec{s}} \vec{F} \cdot d\vec{s} \), where \( \vec{s} \) is the curve of intersection of \( x^2 + y^2 = 1 \) and the plane \( z = 2 + 2x + 3y \), traversed counterclockwise as viewed from above, and \( \vec{F} = (-2y, 2x, 2) \).

11. Let \( \vec{F}(x, y, z) = (x^2y + z, xz, x + 3yz) \). Compute \( \text{curl} \, \vec{F} \) and \( \text{div} \, \vec{F} \).
12. Compute
\[ \int_{\vec{\sigma}} (y^2 + z^3)\,dx + (2xy - 2y)\,dy + (3xz^2 + 4)\,dz, \]
where \( \vec{\sigma}(t) \) is the path which consists of the curve \( \left( \frac{3t}{2t+1}, t e^{2(t-1)}, \frac{1}{8} t^2 (t^2 + 1)^3 \right) \) for \( 0 \leq t \leq 1 \), followed by the segment from \((1, 1, 1)\) to \((1, 2, -1)\).

13. Compute \( \int_{\vec{\sigma}} (x^2 y - xy^2)\,dx + \left( 2x^2 y + \frac{1}{3} x^3 \right)\,dy \), where \( \vec{\sigma} \) is the boundary of the square \( 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \), traversed in the counterclockwise direction.

14. Find the area of the region enclosed by the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} = 1 \).

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**Solutions to the Review Problems for Test 3**

1. (a) Compute the exact value of \( \int_0^2 \int_0^{1 + \sqrt{2x - x^2}} dy\,dx \).

Rewrite \( y = 1 + \sqrt{2x - x^2} \):
\[
y - 1 = \sqrt{2x - x^2}, \quad (y - 1)^2 = 2x - x^2, \quad x^2 - 2x + (y - 1)^2 = 0, \quad x^2 - 2x + 1 + (y - 1)^2 = 1, \quad (x - 1)^2 + (y - 1)^2 = 1.
\]

Thus, \( y = 1 + \sqrt{2x - x^2} \) is the top half of the circle of radius 1 centered at \((1, 1)\). The region is bounded above by this semicircle, below by the \( x \)-axis, and on the sides by \( x = 0 \) and \( x = 2 \):

\[
\int_0^2 \int_0^{1 + \sqrt{2x - x^2}} dy\,dx = \frac{1}{2} \pi \cdot 1^2 + 2 \cdot 1 = 2 + \frac{\pi}{2}.
\]

(b) Compute the exact value of \( \int_{-4}^1 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^4 dz\,dy\,dx \).

The projection of the region into the \( x-y \) plane is
\[
-4 \leq x \leq 4, \quad -\sqrt{16 - x^2} \leq y \leq \sqrt{16 - x^2}.
\]
This is the circle of radius 4 centered at the origin. The bottom of the region is the cone \( z = \sqrt{x^2 + y^2} \) and the top is the plane \( z = 4 \).

Thus, the region is a cone with height \( h = 4 \) and radius \( r = 4 \). Since the integrand is 1, the integral represent the volume of the region. A cone of height \( h \) and radius \( r \) has volume \( \frac{1}{3} \pi r^2 h \). Therefore,

\[
\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^{4} dz \, dy \, dx = \frac{1}{3} \pi \cdot 4^2 \cdot 4 = \frac{64\pi}{3}. \]

2. Find the volume of the solid lying below the paraboloid \( z = x^2 + y^2 \) and above the region in the \( x-y \) plane bounded by \( y = x^2 \) and \( y = x + 2 \).

The projection into the \( x-y \) plane is shown in the first picture:

Since \( x^2 \) and \( x + 2 \) intersect at \( x = -1 \) and at \( x = 2 \), the region is described by the following inequalities:

\[-1 \leq x \leq 2, \quad x^2 \leq y \leq x + 2.\]

The top of the solid is \( z = x^2 + y^2 \). The bottom is the \( x-y \) plane \( z = 0 \). The second picture shows the top and the bottom; the solid is the region between them.

The volume is

\[
\int_{-1}^{2} \int_{-1}^{x+2} (x^2 + y^2) \, dy \, dx = \int_{-1}^{2} \left[ x^2 y + \frac{1}{3} y^3 \right]_{x^2}^{x^2 + 2} \, dx = \int_{-1}^{2} \left( x^3 + 2x^2 + \frac{1}{3}(x + 2)^3 - x^5 - \frac{1}{3}x^6 \right) \, dx = \left[ \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{12}(x + 2)^4 - \frac{1}{6}x^6 - \frac{1}{21}x^7 \right]_{-1}^{2} = \frac{639}{35} \approx 18.25714. \]
3. (a) Compute \( \int_0^1 \int_{\sqrt{y}}^1 \frac{1}{\sqrt{x^3 + 4}} \, dx \, dy \).

Interchange the order of integration:

\[
\begin{cases}
0 \leq y \leq 1 \\
\sqrt{y} \leq x \leq 1
\end{cases} \rightarrow \begin{cases}
0 \leq x \leq 1 \\
0 \leq y \leq x^2
\end{cases}
\]

Thus,

\[
\int_0^1 \int_{\sqrt{y}}^1 \frac{1}{\sqrt{x^3 + 4}} \, dx \, dy = \int_0^1 \int_0^{x^2} \frac{1}{\sqrt{x^3 + 4}} \, dy \, dx = \int_0^1 \frac{1}{\sqrt{x^3 + 4}} [y]_0^{x^2} \, dx = \int_0^1 \frac{x^2}{\sqrt{x^3 + 4}} \, dx = \left[ \frac{2}{3} (x^3 + 4)^{1/2} \right]_0^1 = \frac{2}{3} (\sqrt{5} - 2) \approx 0.15738.
\]

Here’s the work for the integral:

\[
\int \frac{x^2}{\sqrt{x^3 + 4}} \, dx = \int \frac{x^2}{\sqrt{u}} \cdot \frac{du}{3x^2} = \frac{1}{3} \int \frac{du}{\sqrt{u}} = \frac{2}{3} \sqrt{u} + c = \frac{2}{3} \sqrt{x^3 + 4} + c.
\]

(b) Compute

\[
\int_0^2 \int_{y/2}^1 3y^2 \cos(x^4) \, dx \, dy + \int_{-2}^0 \int_{y/2}^1 3y^2 \cos(x^4) \, dx \, dy.
\]

Interchange the order of integration:

\[
\begin{cases}
0 \leq y \leq 2 \\
\frac{y}{2} \leq x \leq 1
\end{cases} \rightarrow \begin{cases}
0 \leq x \leq 1 \\
-2x \leq y \leq 2x
\end{cases}
\]

Thus,

\[
\int_0^2 \int_{y/2}^1 3y^2 \cos(x^4) \, dx \, dy + \int_{-2}^0 \int_{y/2}^1 3y^2 \cos(x^4) \, dx \, dy = \int_0^1 \int_{-2x}^{2x} 3y^2 \cos(x^4) \, dy \, dx = \ldots
\]
\[ \int_{0}^{1} \cos(x^4) \left[ y^3 \right]^{2x}_{-2x} dx = 16 \int_{0}^{1} x^3 \cos(x^4) dx = 4 \left[ \sin(x^4) \right]_{0}^{1} = 4 \sin 1 \approx 3.36588. \]

Here’s the work for the integral:

\[ \int x^3 \cos(x^4) \, dx = \int x^3 \cos u \cdot \frac{du}{4x^3} = \frac{1}{4} \int \cos u \, du = \frac{1}{4} \sin u + c = \frac{1}{4} \sin(x^4) + c. \]

\[ u = x^4, \quad du = 4x^3 \, dx, \quad dx = \frac{du}{4x^3}. \]

4. Compute \[ \int_{0}^{2} \int_{0}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx. \]

Note that \( y = \pm \sqrt{2x-x^2} \) may be rewritten as follows:

\[ y^2 = 2x - x^2, \quad x^2 - 2x + y^2 = 0, \quad x^2 - 2x + 1 + y^2 = 1, \quad (x - 1)^2 + y^2 = 1. \]

This is a circle of radius 1 centered at (1, 0).

I’ll convert to polar:

\[
\begin{aligned}
\left\{ \begin{array}{l}
0 \leq x \leq 2 \\
-\sqrt{2x-x^2} \leq y \leq \sqrt{2x-x^2}
\end{array} \right. & \quad \rightarrow \\
\left\{ \begin{array}{l}
-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\
0 \leq r \leq 2 \cos \theta
\end{array} \right.
\end{aligned}
\]

To get the polar equation for the circle, start with \( x^2 - 2x + y^2 = 0 \). Then

\[ x^2 + y^2 = 2x, \quad r^2 = 2r \cos \theta, \quad r = 2 \cos \theta. \]

Note that the whole circle is traced out once as \( \theta \) goes from \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\) (not, for example, from 0 to 2\(\pi\)).

The integrand is \( \sqrt{x^2+y^2} = \sqrt{r^2} = r \). So

\[
\int_{0}^{2} \int_{0}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\sqrt{2 \cos \theta}} r^2 \, dr \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ \frac{1}{3} r^3 \right]_{0}^{2 \cos \theta} \, d\theta = \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos \theta)^3 \, d\theta = \frac{8}{3} \left[ \sin \theta - \frac{1}{3} (\sin \theta)^3 \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{32}{9}.
\]

Here’s the work for the integral:

\[
\int (\cos \theta)^3 \, d\theta = \int (\cos \theta)^2 (\cos \theta) \, d\theta = \int (1 - (\sin \theta)^2) (\cos \theta) \, d\theta = \int (1 - u^2) \cos \theta \cdot \frac{du}{\cos \theta} = \int (1 - u^2) \, du = \left[ u = \sin \theta, \quad du = \cos \theta \, d\theta, \quad d\theta = \frac{du}{\cos \theta} \right]
\]
\[
u - \frac{1}{3}u^3 + c = \sin \theta - \frac{1}{3}(\sin \theta)^3 + c. \]

5. Find the volume of the region which lies above the cone \( z = \frac{1}{\sqrt{3}} \sqrt{x^2 + y^2} \) and below the hemisphere \( z = \sqrt{1 - x^2 - y^2} \).

I’ll do the integral in spherical coordinates. It’s pretty clear that the ranges for \( \theta \) and \( \rho \) are \( 0 \leq \theta \leq 2\pi \) and \( 0 \leq \rho \leq 1 \). What is the range for \( \phi \)? I need to figure out the angle between the side of the cone and the \( z \)-axis.

To do this, take a random point on the cone: For instance, if \( x = 1 \) and \( y = 0 \), then \( z = \frac{1}{\sqrt{3}} \). Here’s the picture:

\[
\begin{array}{c}
1/\sqrt{3} \\
2/\sqrt{3} \\
1 \rightarrow \pi/3 \\
\sqrt{3} \\
\end{array}
\]

I drew a triangle with horizontal side 1 (since \( r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 0^2} = 1 \)) and vertical side \( \frac{1}{\sqrt{3}} \) (the value of \( z \)). I found the hypotenuse using Pythagoras. Then I scaled the triangle up by multiplying all the sides by \( \sqrt{3} \) so I could see the ratios better. In the second triangle, I can clearly see that the cone angle is \( \frac{\pi}{3} \).

Therefore, the range on \( \phi \) is \( 0 \leq \phi \leq \frac{\pi}{3} \).

The volume is

\[
\begin{align*}
\iiint_R dx \, dy \, dz &= \int_{0}^{2\pi} \int_{0}^{\pi/3} \int_{0}^{\rho^2} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi \int_{0}^{\pi/3} \sin \phi \left[ \frac{1}{3} \rho^3 \right]_0^1 \, d\phi = \frac{2\pi}{3} \int_{0}^{\pi/3} \sin \phi \, d\phi = \\
&\frac{2\pi}{3} [\cos \phi]_{\pi/3}^0 = \frac{\pi}{3} \approx 1.04720.
\end{align*}
\]

6. Compute \( \int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{0}^{x+y} \sqrt{x^2 + y^2} \, dz \, dy \, dx \).

I’ll convert to cylindrical coordinates. The ranges \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq \sqrt{1-x^2} \) describe the interior
of the circle of radius 1 centered at the origin which lies in the first quadrant:

In polar coordinates, it is

\[ 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 1. \]

The limits on \( z \) become \( 0 \leq z \leq r \cos \theta + r \sin \theta \).

The integrand is \( \sqrt{x^2 + y^2} = \sqrt{r^2} = r \).

Therefore,

\[
\int_0^1 \int_0^1 \int_0^{r \cos \theta + r \sin \theta} r^2 \, dz \, dr \, d\theta = \int_0^\pi/2 \int_0^1 \int_0^{r \cos \theta + r \sin \theta} r^2 \, dz \, dr \, d\theta = 1.\]

7. Find the area of the surface

\[ x = 2u \cos v, \quad y = u^2, \quad z = 2u \sin v, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi. \]

\[ \vec{T}_u = (2 \cos v, 2u, 2 \sin v), \quad \vec{T}_v = (-2u \sin v, 0, 2u \cos v), \]

\[ \vec{T}_u \times \vec{T}_v = \begin{vmatrix} i & j & k \\ 2 \cos v & 2u & 2 \sin v \\ -2u \sin v & 0 & 2u \cos v \end{vmatrix} = \langle 4u^2 \cos v, -4u(\sin v)^2 - 4u(\cos v)^2, 4u^2 \sin v \rangle = \langle 4u^2 \cos v, -4u, 4u^2 \sin v \rangle, \]

\[ \|\vec{T}_u \times \vec{T}_v\| = \sqrt{16u^4(\cos v)^2 + 16u^2 + 16u^4(\sin v)^2} = \sqrt{16u^4 + 16u^2} = 4u \sqrt{u^2 + 1}. \]

The area is

\[
\int_0^{2\pi} \int_0^1 4u \sqrt{u^2 + 1} \, du \, dv = 8\pi \int_0^1 u \sqrt{u^2 + 1} \, du = 8\pi \left[ \frac{1}{3} (u^2 + 1)^{3/2} \right]_0^1 = \frac{8\pi}{3} (2^{3/2} - 1) \approx 15.3178. \]

Here’s the work for the integral:

\[
\int u \sqrt{u^2 + 1} \, du = \int u \sqrt{w} \frac{dw}{2u} = \frac{1}{2} \int \sqrt{w} \, dw = \frac{1}{3} w^{3/2} + c = \frac{1}{3} (u^2 + 1)^{3/2} + c. \]
\[
\begin{align*}
\left[ w = u^2 + 1, \quad dw = 2u \, du, \quad du = \frac{dw}{2u} \right]
\end{align*}
\]

8. Find the center of mass of the region in the first octant cut off by the plane \(2x + 2y + z = 4\), if the density is \(\rho(x, y, z) = 2z + 1\).

The region is shown in the first picture. The top is the plane \(z = 4 - 2x - 2y\), and the bottom is \(z = 0\). Thus, the range for \(z\) is \(0 \leq z \leq 4 - 2x - 2y\).

The projection into the \(x-y\) plane is shown in the second picture. It is \(0 \leq x \leq 2\) and \(0 \leq y \leq 2-x\).

The mass is

\[
\begin{align*}
&\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (2z + 1) \, dz \, dy \, dx = \int_0^2 \int_0^{2-x} \left[ z^2 + z \right]_0^{4-2x-2y} \, dy \, dx = \\
&\int_0^2 \int_0^{2-x} \left[ (4-2x-2y)^2 + (4-2x-2y) \right] \, dy \, dx = \int_0^2 \left[ -\frac{1}{6} (4-2x-2y)^3 - \frac{1}{4} (4-2x-2y)^2 \right]_0^{2-x} \, dx = \\
&\int_0^2 \left( \frac{1}{6} (4-2x)^3 + \frac{1}{4} (4-2x)^2 \right) \, dx = \left[ -\frac{1}{48} (4-2x)^4 - \frac{1}{24} (4-2x)^3 \right]_0^2 = 8.
\end{align*}
\]

The region is symmetric and \(x\) and \(y\), and so is the density function. Therefore, \(\overline{x} = \overline{y}\).

I’m going to omit the ugly details of the integrations for the moments.

The \(x\)-moment is

\[
\begin{align*}
&\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} x(2z + 1) \, dz \, dy \, dx = \frac{52}{15}.
\end{align*}
\]

Hence, \(\overline{x} = \frac{13}{30}\) Likewise, \(\overline{y} = \frac{13}{30}\).

The \(z\)-moment is

\[
\begin{align*}
&\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} z(2z + 1) \, dz \, dy \, dx = \frac{56}{5}.
\end{align*}
\]

Hence, \(\overline{z} = \frac{7}{5}\). \(\square\)
9. Compute \( \int \int_R (x - 2y) \, dx \, dy \), where \( R \) is the parallelogram bounded by \( y = x + 1 \), \( y = x - 2 \), \( y = -\frac{1}{2}x + 1 \), and \( y = -\frac{1}{2}x + 4 \).

![Diagram of parallelogram]

I graphed the lines and found the intersection points. Next, I’ll construct a transformation from the square \( 0 \leq u \leq 1, \, 0 \leq v \leq 1 \), onto the parallelogram. I’ll use \((0, 1)\) as my reference point.

The vector from \((0, 1)\) to \((2, 0)\) is \(\langle 2, -1 \rangle\). The vector from \((0, 1)\) to \((2, 3)\) is \(\langle 2, 2 \rangle\). The transformation is

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u \\
v
\end{bmatrix} + \begin{bmatrix} 0 \\
1
\end{bmatrix}.
\]

If I multiply out and combine terms on the right, then equate corresponding components, I get

\[x = 2u + 2v, \quad y = -u + 2v + 1, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 1.\]

The Jacobian is

\[
\left| \det \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \right| = 6.
\]

The integrand is

\[x - 2y = (2u + 2v) - 2(-u + 2v + 1) = 4u - 2v - 2.\]

Hence,

\[
\int \int_R (x - 2y) \, dx \, dy = \int_0^1 \int_0^1 (4u - 2v - 2)(6) \, du \, dv = 6 \int_0^1 [2u^2 - 2uv - 2u]_0^1 \, dv = 6 \int_0^1 (-2v) \, dv = -12 \left[ \frac{1}{2}v^2 \right]_0^1 = -6.
\]

10. (a) Compute \( \int_\sigma (x + y) \, dx - (x - y) \, dy \), where \( \sigma \) is the path consisting of the segment from \((0, 1)\) to \((-1, 0)\), the segment from \((-1, 0)\) to \((0, -1)\), the segment from \((0, -1)\) to \((1, 0)\), and the segment from \((1, 0)\)
to $(0, 1)$.

I’ll break the integral up into four segments, as shown in the picture.
Segment $A$ is $y = x + 1$. $x + y = 2x + 1$, $x - y = -1$, $dy = dx$, and $x$ goes from $0$ to $-1$. The integral is

$$\int_{0}^{-1} ((2x + 1) \, dx - (-1) \, dx) = \int_{0}^{-1} (2x + 2) \, dx = [x^2 + 2x]^{-1}_0 = -1.$$  

Segment $B$ is $y = -x - 1$. $x + y = -1$, $x - y = 2x + 1$, $dy = -dx$, and $x$ goes from $-1$ to $0$. The integral is

$$\int_{-1}^{0} (-dx - (2x + 1) \, dx) = \int_{-1}^{0} (-2x - 2) \, dx = [-x^2 - 2x]^0_{-1} = -1.$$  

Segment $C$ is $y = x - 1$. $x + y = 2x - 1$, $x - y = 1$, $dy = dx$, and $x$ goes from $0$ to $1$. The integral is

$$\int_{0}^{1} ((2x - 1) \, dx - dx) = \int_{0}^{1} (2x - 2) \, dx = [x^2 - 2x]^1_0 = -1.$$  

Segment $D$ is $y = 1 - x$. $x + y = 1$, $x - y = 2x - 1$, $dy = -dx$, and $x$ goes from $1$ to $0$. The integral is

$$\int_{1}^{0} (dx - (2x - 1) \, dx) = \int_{1}^{0} (2 - 2x) \, dx = [2x - x^2]^0_{1} = -1.$$  

Therefore,

$$\int_{\sigma} (x + y) \, dx - (x - y) \, dy = -1 + (-1) + (-1) + (-1) = -4.$$  

(b) Compute $\int_{\sigma} \vec{F} \cdot d\vec{s}$, where $\sigma$ is the curve of intersection of $x^2 + y^2 = 1$ and the plane $z = 2 + 2x + 3y$, traversed counterclockwise as viewed from above, and $\vec{F} = (-2y, 2x, 2)$. 

\[ \]
The projection of the curve into the \(x,y\) plane is the circle \(x^2 + y^2 = 1\), which may be parametrized by \(x = \cos t, y = \sin t, 0 \leq t \leq 2\pi\). Note that this parameter range traverses the circle counterclockwise as viewed from above.

Plugging these expressions into \(z = 2 + 2x + 3y\), I get \(z = 2 + 2\cos t + 3\sin t\). Hence, the curve of intersection is

\[
\vec{\sigma}(t) = (\cos t, \sin t, 2 + 2\cos t + 3\sin t).
\]

Therefore,

\[
\vec{\sigma}'(t) = (-\sin t, \cos t, -2\sin t + 3\cos t).
\]

The integrand is

\[
\vec{F}(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t) = (-2\sin t, 2\cos t, 2) \cdot (-\sin t, \cos t, -2\sin t + 3\cos t) = 2(\sin t)^2 + 2(\cos t)^2 - 4\sin t + 6\cos t = 2 - 4\sin t + 6\cos t.
\]

The integral is

\[
\int_{\vec{\sigma}} \vec{F} \cdot d\vec{s} = \int_{0}^{2\pi} (2 - 4\sin t + 6\cos t) \, dt = [2t + 4\cos t + 6\sin t]_{0}^{2\pi} = 4\pi \approx 12.56637.
\]

11. Let \(\vec{F}(x, y, z) = (x^2y + z, xz, x + 3yz)\). Compute \(\text{curl} \vec{F}\) and \(\text{div} \vec{F}\).

\[
\text{curl} \vec{F} = \begin{vmatrix}
  \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
  i & j & k \\
  x^2y + z & xz & x + 3yz
\end{vmatrix} = (3z - x, 1 - 1, z - x^2) = (3z - x, 0, z - x^2).
\]

\[
\text{div} \vec{F} = 2xy + 0 + 3y = 2xy + 3y.
\]

12. Compute

\[
\int_{\vec{\sigma}} (y^2 + z^3) \, dx + (2xy - 2y) \, dy + (3xz^2 + 4) \, dz,
\]

where \(\vec{\sigma}(t)\) is the path which consists of the curve \(\left\langle \frac{3t}{2t + 1}, te^{2(t-1)}, \frac{1}{8}t^2(t^2 + 1)^3 \right\rangle\) for \(0 \leq t \leq 1\), followed by the segment from \((1, 1, 1)\) to \((1, 2, -1)\).

It would be very tedious to compute the line integral directly, and it should lead you to ask yourself whether there might not be an easier way. Well,

\[
\text{curl}(y^2 + z^3, 2xy - 2y, 3xz^2 + 4) = \begin{vmatrix}
  \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
  i & j & k \\
  y^2 + z^3 & 2xy - 2y & 3xz^2 + 4
\end{vmatrix} = (0, 3z^2 - 3z^2, 2y - 2y) = (0, 0, 0).
\]

The field is conservative. I’ll find a potential function \(f\). I want

\[
\frac{\partial f}{\partial x} = y^2 + z^3, \quad \frac{\partial f}{\partial y} = 2xy - 2y, \quad \frac{\partial f}{\partial z} = 3xz^2 + 4.
\]

Integrate the first equation with respect to \(x\):

\[
f(x, y, z) = \int (y^2 + z^3) \, dx = xy^2 + xz^3 + C(y, z).
\]
\( C(y, z) \) is an arbitrary constant depending on \( y \) and \( z \). Differentiate with respect to \( y \) and set the result equal to \( \frac{\partial f}{\partial y} = 2xy - 2y \): 

\[
2xy + \frac{\partial C}{\partial y} = \frac{\partial f}{\partial y} = 2xy - 2y.
\]

Cancelling \( 2xy \)'s, I get \( \frac{\partial C}{\partial y} = -2y \), so 

\[
C = \int -2y \, dy = -y^2 + D(z).
\]

\( D(z) \) is an arbitrary constant depending on \( z \). Then 

\[
f(x, y, z) = xy^2 + xz^3 - y^2 + D(z).
\]

Differentiate with respect to \( z \) and set the result equal to \( \frac{\partial f}{\partial z} = 3xz^2 + 4 \): 

\[
3xz^2 + D'(z) = \frac{\partial f}{\partial z} = 3xz^2 + 4.
\]

Cancelling \( 3xz^2 \)'s, I get \( D'(z) = 4 \), so 

\[
D(z) = 4z + E.
\]

Now \( E \) is a numerical arbitrary constant, and since I need some potential function, I can take \( E = 0 \). Then 

\[
f(x, y, z) = xy^2 + xz^3 - y^2 + 4z.
\]

Soon you will have forgotten the world, and the world will have forgotten you. - Marcus Aurelius