Math 311
9-29-2020

Review Sheet for Test 2

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. Find the domain of the function $f(x,y) = \dfrac{x^2+y^2}{(x-1)(y-3)}$.

2. Find the domain and range of $f(x,y,z) = \dfrac{z^2+1}{\sqrt{1-x^2-y^2}}$.

3. Compute $\displaystyle\lim_{(x,y)\to(2,1)} \frac{3x+2y+51}{x^2+3y^2}$.

4. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^4+5y^4}{x^4+3x^2y^2+y^4}$ is undefined.

5. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^{3/2}}{x^2+y^2+1}$ by converting to polar coordinates.

6. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{x^4+3x^2y^2+y^4}$ is defined and find its value.

7. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{3x+y}{5y-6} & \text{if } (x,y) \neq (1,4) \\ \dfrac{2}{3} & \text{if } (x,y) = (1,4) \end{cases}$$

Determine whether f is continuous at $(1,4)$.

8. Compute the following partial derivatives:

(a) $\dfrac{\partial}{\partial x} x^2\sin(x^3+5y)$ and $\dfrac{\partial}{\partial y} x^2\sin(x^3+5y)$.

(b) $\dfrac{\partial}{\partial s}\dfrac{s^2}{s^3+t^3}$ and $\dfrac{\partial}{\partial t}\dfrac{s^2}{s^3+t^3}$.

(c) $\dfrac{\partial^3 f}{\partial x^2 \partial y}$, if

$$f(x,y) = e^{3x} + 4x^2y - \ln y.$$

(d) $\dfrac{\partial^3 f}{\partial x \partial y \partial z}$, if

$$f(x,y,z) = 3x + 8y - 2z + x^2y^3z^4.$$

9. Let

$$f(x,y) = x^3 + 5xy^2 - y^4.$$

Construct the Taylor series for f at the point $(2,1)$, writing terms through the 2^{nd} order.

10. For a differentiable function $f(x,y)$,

$$f(-2,4) = 6, \quad f_x(-2,4) = 3, \quad f_y(-2,4) = 1.$$

Use a 1$^{\text{st}}$-degree Taylor approximation at $(-2,4)$ to approximate $f(-2.1,4.1)$.

11. Find the tangent plane and the normal line to the surface

$$z = x(2x+y)^3 \quad \text{at} \quad (x,y) = (2,-3).$$

12. Find the tangent plane to the surface

$$x = u^2 - 3v^2, \quad y = \frac{4u}{v}, \quad z = 2u^2v^3 \quad \text{at} \quad (u,v) = (1,1).$$

13. Use a linear approximation to $z = f(x,y) = x^2 - y^2$ at the point $(2,1)$ to approximate $f(1.9,1.1)$.

14. Let $f(x,y) = \dfrac{(x+4)^2}{y}$.

(a) Find a unit vector at $(-3,1)$ which points in the direction of most rapid increase.

(b) Find the rate of most rapid increase at $(-3,1)$.

15. Find the gradient of $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2+1}}$ and show that it always points toward the origin.

16. Let $f(x,y) = \sqrt{x^2+2y+3}$. Find the directional derivative of f at the point $(3,2)$ in the direction of the vector $(-4,3)$.

17. Find the rate of change of $f(x,y,z) = xy - yz + xz$ at the point $(1,-2,-2)$ in the direction toward the origin. Is f increasing or decreasing in this direction?

18. The rate of change of $f(x,y)$ at $(1,-1)$ is 2 in the direction *toward* $(5,-1)$ and is $\dfrac{6}{5}$ in the direction of the vector $(-3,-4)$. Find $\nabla f(1,-1)$.

19. Calvin Butterball sits in his go-cart on the surface

$$z = x^3 - 2x^2y + x^2 + xy^2 - 2y^3 + y^2 \quad \text{at the point} \quad (1,1,0).$$

If his go-cart is pointed in the direction of the vector $\vec{v} = (15,-8)$, at what rate will it roll downhill?

20. Find the tangent plane to $x^2 - y^2 + 2yz + z^5 = 6$ at the point $(2,1,1)$.

21. Suppose that $z = f(x,y)$ and $(x,y) = g(u,v)$ are given by

$$z = x^4 + 3xy^2 - y^2, \quad (x,y) = (\sin 5u + \cos v, \cos 3u + \sin 2v).$$

Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

22. Let r and θ be the standard polar coordinates variables. Use the Chain Rule to find $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$, for $f(x,y) = xe^x + e^y$.

23. Suppose $u = f(x,y,z)$ and $x = \phi(s,t)$, $y = \psi(s,t)$, $z = \mu(s,t)$. Use the Chain Rule to write down an expression for $\dfrac{\partial u}{\partial t}$.

24. Suppose that $w = f(x,y)$, $x = g(r,s,t)$, and $y = h(r,t,s)$. Use the Chain Rule to find an expression for $\dfrac{\partial^2 f}{\partial t^2}$.

25. Locate and classify the critical points of

$$z = x^2y - 4xy + \frac{1}{3}y^3 - \frac{3}{2}y^2.$$

26. Locate and classify the critical points of

$$f(x, y) = 6xy^2 - 2x^3y + y^2.$$

27. Find the critical points of

$$z = (x^2 + y^2)e^{-x^2-4y^2}.$$

You do not need to classify them.

28. Find the points on the sphere $x^2 + y^2 + z^2 = 36$ which are closest to and farthest from the point $(4, -3, 12)$.

29. A rectangular box (with a bottom and a top) is to have a total surface area of $6c^2$, where $c > 0$. Show that the box of largest volume satisfying this condition is a cube with sides of length c.

30. (a) Find the critical points of

$$w = 4xyz \quad \text{subject to the constraint} \quad x + y + z = 3.$$

(b) Express w as a function of x and y by eliminating z, then consider the behavior of w for $x = y$. Explain why the critical points in (a) can't give absolute maxes or mins.

31. Find the largest and smallest values of $f(x, y) = 4x^2y$ subject to the constraint $x^2 + y^2 = 36$.

Solutions to the Review Sheet for Test 2

1. Find the domain of the function $f(x, y) = \dfrac{x^2 + y^2}{(x - 1)(y - 3)}$.

Since the denominator of the fraction can't be 0, the domain is

$$\{(x, y) \mid x \neq 1 \quad \text{and} \quad y \neq 3\}.$$

It consists of all points except those lying on the lines $x = 1$ or $y = 3$. $\square$

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2. Find the domain and range of $f(x, y, z) = \dfrac{z^2 + 1}{\sqrt{1 - x^2 - y^2}}$.

Since the expression inside the square root must be positive, the function is defined for $1 - x^2 - y^2 > 0$. Therefore, the domain is the set of points (x, y, z) such that $x^2 + y^2 < 1$ — that is, the interior of the cylinder $x^2 + y^2 = 1$ of radius 1 whose axis is the z-axis. (There are no restrictions on z.)

To find the range, note that $z^2 + 1 \geq 1$. Also,

$$1 - x^2 - y^2 \leq 1, \quad \text{and} \quad \sqrt{1 - x^2 - y^2} \leq 1, \quad \text{so} \quad \frac{1}{\sqrt{1 - x^2 - y^2}} \geq 1.$$

Hence,

$$f(x,y,z) = \frac{z^2+1}{\sqrt{1-x^2-y^2}} \ge 1 \cdot 1 = 1.$$

This shows that every output of f is greater than or equal to 1.
On the other hand, suppose $k \ge 1$. Then

$$f(0,0,\sqrt{k-1}) = \frac{(\sqrt{k-1})^2+1}{\sqrt{1-0-0}} = k.$$

This shows that every number greater than or equal to 1 is an output of f.
Hence, the range of f is the set of numbers w such that $w \ge 1$. $\square$

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3. Compute $\displaystyle\lim_{(x,y)\to(2,1)} \frac{3x+2y+51}{x^2+3y^2}$.

$$\lim_{(x,y)\to(2,1)} \frac{3x+2y+5}{x^2+3y^2} = \frac{6+2+5}{4+3} = \frac{13}{7}. \quad \square$$

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4. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^4+5y^4}{x^4+3x^2y^2+y^4}$ is undefined.

If you approach $(0,0)$ along the x-axis ($y = 0$), you get

$$\lim_{(x,y)\to(0,0)} \frac{3x^4+5y^4}{x^4+3x^2y^2+y^4} = \lim_{(x,y)\to(0,0)} \frac{3x^4}{x^4} = \lim_{(x,y)\to(0,0)} 3 = 3.$$

If you approach $(0,0)$ along the line $y = x$, you get

$$\lim_{(x,y)\to(0,0)} \frac{3x^4+5y^4}{x^4+3x^2y^2+y^4} = \lim_{(x,y)\to(0,0)} \frac{3x^4+5x^4}{x^4+3x^4+x^4} = \lim_{(x,y)\to(0,0)} \frac{8x^4}{5x^4} = \lim_{(x,y)\to(0,0)} \frac{8}{5} = \frac{8}{5}.$$

Since the function approaches different values as you approach $(0,0)$ in different ways, the limit is undefined. $\square$

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5. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^{3/2}}{x^2+y^2+1}$ by converting to polar coordinates.

Set $r^2 = x^2+y^2$. As $(x,y) \to (0,0)$, I have $r \to 0$. So

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^{3/2}}{x^2+y^2+1} = \lim_{r\to 0} \frac{(r^2)^{3/2}}{r^2+1} = \lim_{r\to 0} \frac{r^3}{r^2+1} = \frac{0}{0+1} = 0. \quad \square$$

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6. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{x^4+3x^2y^2+y^4}$ is defined and find its value.

$$\left|\frac{x^4y^4}{x^4+3x^2y^2+y^4}\right| \le \left|\frac{x^4y^4}{x^4}\right| = |y^4| \to 0 \quad \text{as} \quad (x,y) \to (0,0).$$

Therefore,

$$\lim_{(x,y)\to(0,0)} \left|\frac{x^4y^4}{x^4+3x^2y^2+y^4}\right| = 0.$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{x^4+3x^2y^2+y^4} = 0. \quad \square$$

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7. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{3x+y}{5y-6} & \text{if } (x,y) \neq (1,4) \\ \dfrac{2}{3} & \text{if } (x,y) = (1,4) \end{cases}$$

Determine whether f is continuous at $(1,4)$.

$$\lim_{(x,y)\to(1,4)} f(x,y) = \lim_{(x,y)\to(1,4)} \frac{3x+y}{5y-6} = \frac{3+4}{20-6} = \frac{1}{2}.$$

Since $f(1,4) = \dfrac{1}{2}$,

$$\lim_{(x,y)\to(1,4)} f(x,y) \neq f(1,4).$$

Therefore, f is not continuous at $(1,4)$. □

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8. Compute the following partial derivatives:

(a) $\dfrac{\partial}{\partial x} x^2 \sin(x^3+5y)$ and $\dfrac{\partial}{\partial y} x^2 \sin(x^3+5y)$.

(b) $\dfrac{\partial}{\partial s} \dfrac{s^2}{s^3+t^3}$ and $\dfrac{\partial}{\partial t} \dfrac{s^2}{s^3+t^3}$.

(c) $\dfrac{\partial^3 f}{\partial x^2 \partial y}$, if

$$f(x,y) = e^{3x} + 4x^2y - \ln y.$$

(d) $\dfrac{\partial^3 f}{\partial x \partial y \partial z}$, if

$$f(x,y,z) = 3x + 8y - 2z + x^2y^3z^4.$$

(a)

$$\frac{\partial}{\partial x} x^2 \sin(x^3+5y) = 3x^4 \cos(x^3+5y) + 2x \sin(x^3+5y).$$

$$\frac{\partial}{\partial y} x^2 \sin(x^3+5y) = 5x^2 \cos(x^3+5y). \quad \square$$

(b)

$$\frac{\partial}{\partial s} \frac{s^2}{s^3+t^3} = \frac{(s^3+t^3)(2s) - (s^2)(3s^2)}{(s^3+t^3)^2}.$$

$$\frac{\partial}{\partial t} \frac{s^2}{s^3+t^3} = -\frac{3s^2t^2}{(s^3+t^3)^2}. \quad \square$$

(c)

$$\frac{\partial f}{\partial y} = 4x^2 - \frac{1}{y}.$$

$$\frac{\partial^2 f}{\partial x \partial y} = 8x.$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = 8. \quad \square$$

(d)

$$\frac{\partial f}{\partial z} = -2 + 4x^2y^3z^3.$$

$$\frac{\partial^2 f}{\partial y \partial z} = 12x^2y^2z^3.$$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = 24xy^2z^3. \quad \square$$

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9. Let

$$f(x, y) = x^3 + 5xy^2 - y^4.$$

Construct the Taylor series for f at the point $(2, 1)$, writing terms through the 2^{nd} order.

$$\frac{\partial f}{\partial x} = 3x^2 + 5y^2, \quad \frac{\partial f}{\partial y} = 10xy - 4y^3.$$

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial x \partial y} = 10y, \quad \frac{\partial^2 f}{\partial y^2} = 10x - 12y^2.$$

At $(2, 1)$,

$$f(2, 1) = 17, \quad \frac{\partial f}{\partial x}(2, 1) = 17, \quad \frac{\partial f}{\partial y}(2, 1) = 16.$$

$$\frac{\partial^2 f}{\partial x^2}(2, 1) = 12, \quad \frac{\partial^2 f}{\partial x \partial y}(2, 1) = 10, \quad \frac{\partial^2 f}{\partial y^2}(2, 1) = 8.$$

The series is

$$f(x, y) = 17 + (17(x - 2) + 16(y - 1)) + \frac{1}{2!}\left(12(x - 2)^2 + 20(x - 2)(y - 1) + 8(y - 1)^2\right) + \cdots. \quad \square$$

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10. For a differentiable function $f(x, y)$,

$$f(-2, 4) = 6, \quad f_x(-2, 4) = 3, \quad f_y(-2, 4) = 1.$$

Use a 1^{st}-degree Taylor approximation at $(-2, 4)$ to approximate $f(-2.1, 4.1)$.

The 1^{st}-degree Taylor approximation is

$$f(x, y) \approx 6 + (3(x + 2) + (y - 4)).$$

Hence,

$$f(-2.1, 4.1) \approx 6 + 3(-0.1) + 0.1 = 5.8. \quad \square$$

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11. Find the tangent plane and the normal line to the surface

$$z = x(2x + y)^3 \quad \text{at} \quad (x, y) = (2, -3).$$

When $(x, y) = (2, -3)$,

$$z = 2 \cdot 1^3 = 2.$$

The point of tangency is $(2, -3, 2)$.

$$\frac{\partial f}{\partial x} = 6x(2x+y)^2 + (2x+y)^3, \quad \frac{\partial f}{\partial x}(2,-3) = 13.$$

$$\frac{\partial f}{\partial y} = 3x(2x+y)^2, \quad \frac{\partial f}{\partial y}(2,-3) = 6.$$

The normal vector is

$$\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) = (-13, -6, 1).$$

The normal line is

$$x - 2 = -13t, \quad y + 3 = -6t, \quad z - 2 = t.$$

The tangent plane is

$$-13(x-2) - 6(y+3) + (z-2) = 0, \quad \text{or} \quad -13x - 6y + z = -6. \quad \square$$

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12. Find the tangent plane to the surface

$$x = u^2 - 3v^2, \quad y = \frac{4u}{v}, \quad z = 2u^2v^3 \quad \text{at} \quad (u, v) = (1, 1).$$

$u = 1$ and $v = 1$ give the point of tangency: $(x, y, z) = (-2, 4, 2)$.
Next,

$$\vec{T}_u = \left(2u, \frac{4}{v}, 4uv^3\right) \quad \text{and} \quad \vec{T}_v = \left(-6v, -\frac{4u}{v^2}, 6u^2v^2\right).$$

Thus,

$$\vec{T}_u(1,1) = (2,4,4) \quad \text{and} \quad \vec{T}_v(1,1) = (-6,-4,6).$$

The normal vector is given by

$$\vec{T}_u(1,1) \times \vec{T}_v(1,1) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 4 & 4 \\ -6 & -4 & 6 \end{vmatrix} = (40, -36, 16).$$

The tangent plane is

$$40(x+2) - 36(y-4) + 16(z-2) = 0, \quad \text{or} \quad 10x - 9y + 4z = -48. \quad \square$$

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13. Use a linear approximation to $z = f(x, y) = x^2 - y^2$ at the point $(2, 1)$ to approximate $f(1.9, 1.1)$.

$f(2, 1) = 3$, so the point of tangency is $(2, 1, 3)$. A normal vector for a function $z = f(x, y)$ is given by

$$\vec{N} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right) = (2x, -2y, -1), \quad \vec{N}(2,1) = (4, -2, -1).$$

Hence, the tangent plane is

$$4(x-2) - 2(y-1) - (z-3) = 0, \quad \text{or} \quad z = 3 + 4(x-2) - 2(y-1).$$

Substitute $x = 1.9$ and $y = 1.1$:

$$z = 3 + 4(-0.1) - 2(0.1) = 2.4. \quad \square$$

14. Let $f(x, y) = \dfrac{(x+4)^2}{y}$.

(a) Find a unit vector at $(-3, 1)$ which points in the direction of most rapid increase.

(b) Find the rate of most rapid increase at $(-3, 1)$.

$$\nabla f(x, y) = \left(\frac{2(x+4)}{y}, -\frac{(x+4)^2}{y^2} \right).$$

$$\nabla f(-3, 1) = (2, -1), \quad \|\nabla f(-3, 1)\| = \sqrt{5}.$$

(a) Find a unit vector at $(-3, 1)$ which points in the direction of most rapid increase is $\dfrac{1}{\sqrt{5}}(2, -1)$. □

(b) Find the rate of most rapid increase at $(-3, 1)$ is $\sqrt{5}$. □

15. Find the gradient of $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2 + 1}}$ and show that it always points toward the origin.

$$\nabla f = \left(\frac{-x}{(x^2 + y^2 + z^2 + 1)^{3/2}}, \frac{-y}{(x^2 + y^2 + z^2 + 1)^{3/2}}, \frac{-z}{(x^2 + y^2 + z^2 + 1)^{3/2}} \right) =$$

$$\frac{-1}{(x^2 + y^2 + z^2 + 1)^{3/2}}(x, y, z).$$

(x, y, z) is the **radial vector** from the origin $(0, 0, 0)$ to the point (x, y, z). Since ∇f is a negative multiple of this vector ∇f always points *inward* toward the origin. □

16. Let $f(x, y) = \sqrt{x^2 + 2y + 3}$. Find the directional derivative of f at the point $(3, 2)$ in the direction of the vector $(-4, 3)$.

$$\nabla f(x, y) = \left(\frac{x}{\sqrt{x^2 + 2y + 3}}, \frac{1}{\sqrt{x^2 + 2y + 3}} \right).$$

$$\nabla f(3, 2) = \left(\frac{3}{4}, \frac{1}{4} \right).$$

Hence,

$$Df_{(-4,3)}(3, 2) = \left(\frac{3}{4}, \frac{1}{4} \right) \cdot \frac{(-4, 3)}{\|(-4, 3)\|} = \left(\frac{3}{4}, \frac{1}{4} \right) \cdot \frac{(-4, 3)}{5} = -\frac{9}{20}. \quad \square$$

17. Find the rate of change of $f(x, y, z) = xy - yz + xz$ at the point $(1, -2, -2)$ in the direction toward the origin. Is f increasing or decreasing in this direction?

First, compute the gradient at the point:

$$\nabla f = (y + z, x - z, -y + x)\,, \quad \nabla f(1, -2, -2) = (-4, 3, 3).$$

Next, determine the direction vector. The point is $P(1, -2, -2)$, so the direction toward the origin $Q(0, 0, 0)$ is

$$\overrightarrow{PQ} = (-1, 2, 2).$$

Make this into a unit vector by dividing by its length:

$$\frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{1}{3}(-1, 2, 2).$$

Finally, take the dot product of the unit vector with the gradient:

$$Df_{\vec{v}}(1, -2, -2) = \nabla f(1, -2, -2) \cdot \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = (-4, 3, 3) \cdot \frac{1}{3}(-1, 2, 2) = \frac{16}{3}.$$

f is increasing in this direction, since the directional derivative is positive. $\square$

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18. The rate of change of $f(x, y)$ at $(1, -1)$ is 2 in the direction *toward* $(5, -1)$ and is $\frac{6}{5}$ in the direction of the vector $(-3, -4)$. Find $\nabla f(1, -1)$.

The direction from $(1, -1)$ *toward* the point $(5, -1)$ is given by the vector $(4, 0)$. This vector has length 4, so

$$2 = \nabla f(1, -1) \cdot \frac{(4, 0)}{4} = (f_x, f_y) \cdot \frac{(4, 0)}{4} = f_x.$$

The vector $(-3, -4)$ has length 5, so

$$\frac{6}{5} = \nabla f(1, -1) \cdot \frac{(-3, -4)}{5} = (f_x, f_y) \cdot \frac{(-3, -4)}{5} = -\frac{3}{5} f_x - \frac{4}{5} f_y.$$

Thus, $6 = -3f_x - 4f_y$.

I have two equations involving f_x and f_y. Solving simultaneously, I obtain $f_x = 2$ and $f_y = -3$. Hence, $\nabla f(1, -1) = (2, -3)$. $\square$

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19. Calvin Butterball sits in his go-cart on the surface

$$z = x^3 - 2x^2 y + x^2 + xy^2 - 2y^3 + y^2 \quad \text{at the point} \quad (1, 1, 0).$$

If his go-cart is pointed in the direction of the vector $\vec{v} = (15, -8)$, at what rate will it roll downhill?

The rate at which he rolls is given by the directional derivative. The gradient is

$$\nabla f = (3x^2 - 4xy + 2x + y^2, -2x^2 + 2xy - 6y^2 + 2y), \quad \text{and} \quad \nabla f(1, 1) = (2, -4).$$

Since $\|(15, -8)\| = 17$,

$$Df_{\vec{v}}(1, 1) = (2, -4) \cdot \frac{(15, -8)}{17} = \frac{62}{17} = 3.64705\ldots. \quad \square$$

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20. Find the tangent plane to $x^2 - y^2 + 2yz + z^5 = 6$ at the point $(2, 1, 1)$.

Write $w = x^2 - y^2 + 2yz + z^5 - 6$. (Take the original surface and drag everything to one side of the equation.) The original surface is $w = 0$, so it's a level surface of w. Since the gradient ∇w is perpendicular to the level surfaces of w, it follows that ∇w must be perpendicular to the original surface.

The gradient is

$$\nabla w = (2x, -2y + 2z, 2y + 5z^4), \quad \nabla w(2, 1, 1) = (4, 0, 7).$$

The vector $(4, 0, 7)$ is perpendicular to the tangent plane. Hence, the plane is

$$4(x - 2) + 0 \cdot (y - 1) + 7(z - 1) = 0, \quad \text{or} \quad 4x + 7z = 15. \quad \square$$

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21. Suppose that $z = f(x, y)$ and $(x, y) = g(u, v)$ are given by

$$z = x^4 + 3xy^2 - y^2, \quad (x, y) = (\sin 5u + \cos v, \cos 3u + \sin 2v).$$

Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (4x^3 + 3y^2)(5\cos 5u) + (6xy - 2y)(-3\sin 3u).$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (4x^3 + 3y^2)(-\sin v) + (6xy - 2y)(2\cos 2v). \quad \square$$

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22. Let r and θ be the standard polar coordinates variables. Use the Chain Rule to find $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$, for $f(x, y) = xe^x + e^y$.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = (xe^x + e^x)(\cos\theta) + (e^y)(\sin\theta),$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = (xe^x + e^x)(-r\sin\theta) + (e^y)(r\cos\theta). \quad \square$$

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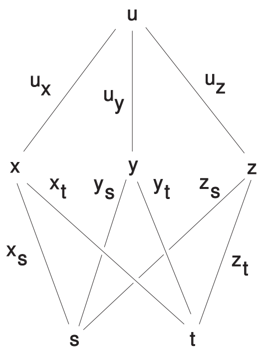
23. Suppose $u = f(x, y, z)$ and $x = \phi(s, t)$, $y = \psi(s, t)$, $z = \mu(s, t)$. Use the Chain Rule to write down an expression for $\dfrac{\partial u}{\partial t}$.

This diagram shows the dependence of the variables.



There are 3 paths from u to t, which give rise to the 3 terms in the following sum:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}. \quad \square$$

---

24. Suppose that $w = f(x, y)$, $x = g(r, s, t)$, and $y = h(r, t, s)$. Use the Chain Rule to find an expression for $\dfrac{\partial^2 f}{\partial t^2}$.

By the Chain Rule,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}.$$

Next, differentiate with respect to t, applying the Product Rule to the terms on the right:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right) + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right).$$

Since $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ are functions of x and y, I must apply the Chain Rule in computing their derivatives with respect to t. I get

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial t}\left(\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)\frac{\partial y}{\partial t}\right) + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t}\left(\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)\frac{\partial y}{\partial t}\right) =$$

$$\frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial x}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial y}{\partial t}\right) + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t}\left(\frac{\partial^2 w}{\partial x \partial y}\frac{\partial x}{\partial t} + \frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial t}\right). \quad \square$$

---

25. Locate and classify the critical points of

$$z = x^2 y - 4xy + \frac{1}{3}y^3 - \frac{3}{2}y^2.$$

$$\frac{\partial z}{\partial x} = 2xy - 4y, \quad \frac{\partial z}{\partial y} = x^2 - 4x + y^2 - 3y,$$

$$\frac{\partial^2 z}{\partial x^2} = 2y, \quad \frac{\partial^2 z}{\partial x \partial y} = 2x - 4, \quad \frac{\partial^2 z}{\partial y^2} = 2y - 3.$$

Set the first partials equal to 0:

$$2xy - 4y = 0, \quad (x-2)y = 0.$$

$$x^2 - 4x + y^2 - 3y = 0.$$

Solve simultaneously:

$$\begin{array}{cccccc}
 & & (x-2)y = 0 & & & \\
 & \swarrow & & \searrow & & \\
 & x = 2 & & & y = 0 & \\
 & x^2 - 4x + y^2 - 3y = 0 & & & x^2 - 4x + y^2 - 3y = 0 & \\
 & y^2 - 3y - 4 = 0 & & & x^2 - 4x = 0 & \\
 & (y-4)(y+1) = 0 & & & x(x-4) = 0 & \\
\swarrow & \downarrow & & & \downarrow & \searrow \\
y = 4 & y = -1 & & & x = 0 & x = 4 \\
(2, 4) & (2, -1) & & & (0, 0) & (4, 0)
\end{array}$$

Test the critical points:

point	z_{xx}	z_{yy}	z_{xy}	Δ	result
$(2, 4)$	8	5	0	40	min
$(2, -1)$	-2	-5	0	10	max
$(0, 0)$	0	-3	-4	-16	saddle
$(4, 0)$	0	-3	4	-16	saddle

$\square$

---

26. Locate and classify the critical points of

$$f(x, y) = 6xy^2 - 2x^3y + y^2.$$

$$f_x = 6y^2 - 6x^2y, \quad f_y = 12xy - 2x^3 + 2y,$$

$$f_{xx} = -12xy, \quad f_{xy} = 12y - 6x^2, \quad f_{yy} = 12x + 2.$$

Set the first partials equal to 0:

$$6y^2 - 6x^2y = 0, \quad y(y - x^2) = 0, \tag{1}$$

$$12xy - 2x^3 + 2y = 0, \quad 6xy - x^3 + y = 0. \tag{2}$$

Solve simultaneously:

$$(1) \quad y(y - x^2) = 0$$

$$\swarrow \qquad\qquad \searrow$$

$$y = 0$$
$$(2) \quad 6xy - x^3 + y = 0$$
$$x^3 = 0$$
$$x = 0$$
$$(0, 0)$$

$$y = x^2$$
$$(2) \quad 6xy - x^3 + y = 0$$
$$6x^3 - x^3 + x^2 = 0$$
$$5x^3 + x^2 = 0$$
$$x^2(5x + 1) = 0$$

$$\swarrow \qquad\qquad \searrow$$

$$x^2 = 0$$
$$x = 0$$
$$y = 0$$
$$(0, 0)$$

$$5x + 1 = 0$$
$$x = -\frac{1}{5}$$
$$y = \frac{1}{25}$$
$$\left(-\frac{1}{5}, \frac{1}{25}\right)$$

Test the critical points:

point	$f_{xx} = -12xy$	$f_{yy} = 12x + 2$	$f_{xy} = 12y - 6x^2$	Δ	result
$(0, 0)$	0	2	0	0	test fails
$\left(-\frac{1}{5}, \frac{1}{25}\right)$	$\frac{12}{125}$	$-\frac{2}{5}$	$\frac{6}{25}$	$-\frac{12}{125}$	saddle

$\square$

27. Find the critical points of

$$z = (x^2 + y^2)e^{-x^2-4y^2}.$$

You do not need to classify them.

$$z_x = -2x(x^2 + y^2)e^{-x^2-4y^2} + 2xe^{-x^2-4y^2} = -2x(x^2 + y^2 - 1)e^{-x^2-4y^2},$$

$$z_y = -8y(x^2 + y^2)e^{-x^2-4y^2} + 2ye^{-x^2-4y^2} = -2y(4x^2 + 4y^2 - 1)e^{-x^2-4y^2}.$$

Set the first partials equal to 0:

$$-2x(x^2 + y^2 - 1)e^{-x^2-4y^2} = 0, \quad x(x^2 + y^2 - 1) = 0.$$

$$-2y(4x^2 + 4y^2 - 1)e^{-x^2-4y^2} = 0, \quad y(4x^2 + 4y^2 - 1) = 0.$$

Solve simultaneously:

$$\begin{array}{ccc} & x(x^2 + y^2 - 1) = 0 & \\ \swarrow & & \searrow \\ x = 0 & & x^2 + y^2 = 1 \\ y(4x^2 + 4y^2 - 1) = 0 & & y(4x^2 + 4y^2 - 1) = 0 \\ y(4y^2 - 1) = 0 & & 3y = 0 \\ (A) & & (B) \end{array}$$

$$\begin{array}{ccccc} & (A) & & & \\ & y(4y^2 - 1) = 0 & & & \\ \swarrow & & \searrow & & \\ y = 0 & & & 4y^2 - 1 = 0 & \\ & & \swarrow & & \searrow \\ & & y = \dfrac{1}{2} & & y = -\dfrac{1}{2} \\ (0, 0) & & \left(0, \dfrac{1}{2}\right) & & \left(0, -\dfrac{1}{2}\right) \end{array}$$

$$\begin{array}{ccc} & (B) & \\ & 3y = 0 & \\ & y = 0 & \\ & x^2 = 1 & \\ \swarrow & & \searrow \\ x = 1 & & x = -1 \\ (1, 0) & & (-1, 0) \end{array}$$

□

---

28. Find the points on the sphere $x^2 + y^2 + z^2 = 36$ which are closest to and farthest from the point $(4, -3, 12)$.

The (square of the) distance from (x, y, z) to $(4, -3, 12)$ is

$$w = (x - 4)^2 + (y + 3)^2 + (z - 12)^2.$$

The constraint is $g(x, y, z) = x^2 + y^2 + z^2 - 36 = 0$.
The equations to be solved are

$$2(x - 4) = 2x\lambda, \quad x - 4 = x\lambda.$$

$$2(y + 3) = 2y\lambda, \quad y + 3 = y\lambda.$$

$$2(z-12) = 2z\lambda, \quad z - 12 = z\lambda.$$
$$x^2 + y^2 + z^2 = 36.$$

Note that if $x = 0$ in the first equation, the equation becomes $-4 = 0$, which is impossible. Therefore, $x \neq 0$, and I may divide by x.

Solve simultaneously:

$$\begin{gathered}
x - 4 = x\lambda \\
\lambda = \frac{x-4}{x} \\
y + 3 = y\lambda \\
y + 3 = \frac{y(x-4)}{x} \\
xy + 3x = yx - 4y \\
y = -\frac{3}{4}x \\
z - 12 = z\lambda \\
z - 12 = \frac{z(x-4)}{x} \\
xz - 12x = xz - 4z \\
z = 3x \\
x^2 + y^2 + z^2 = 36 \\
x^2 + \frac{9}{16}x^2 + 9x^2 = 36 \\
169x^2 = 576 \\
x^2 = \frac{576}{169}
\end{gathered}$$

$$\swarrow \qquad\qquad\qquad\qquad \searrow$$

$$\begin{array}{cc}
\begin{gathered}
x = \frac{24}{13} \\
y = -\frac{18}{13} \\
z = \frac{72}{13} \\
\left(\frac{24}{13}, -\frac{18}{13}, \frac{72}{13}\right)
\end{gathered}
&
\begin{gathered}
x = -\frac{24}{13} \\
y = \frac{18}{13} \\
z = -\frac{72}{13} \\
\left(-\frac{24}{13}, \frac{18}{13}, -\frac{72}{13}\right)
\end{gathered}
\end{array}$$

Test the points:

	$\left(\frac{24}{13}, -\frac{18}{13}, \frac{72}{13}\right)$	$\left(-\frac{24}{13}, \frac{18}{13}, -\frac{72}{13}\right)$
$w(x,y,z)$	49	361

$\left(\frac{24}{13}, -\frac{18}{13}, \frac{72}{13}\right)$ is closest to $(4, -3, 12)$ and $\left(-\frac{24}{13}, \frac{18}{13}, -\frac{72}{13}\right)$ is farthest from $(4, -3, 12)$. □

---

29. A rectangular box (with a bottom and a top) is to have a total surface area of $6c^2$, where $c > 0$. Show that the box of largest volume satisfying this condition is a cube with sides of length c.

Suppose the dimensions of the box are x, y, and z. Then the volume is

$$V = xyz.$$

The surface area is

$$6c^2 = 2xy + 2yz + 2xz, \quad \text{so} \quad 3c^2 = xy + yz + xz.$$

The constraint is

$$g(x,y,z) = xy + yz + xz - 3c^2 = 0.$$

Set up the multiplier equation:

$$\nabla V = \lambda \nabla g$$
$$(yz, xz, xy) = \lambda(y + z, x + z, x + y)$$

This gives the equations

$$yz = \lambda(y + z).$$

$$xz = \lambda(x + z).$$

$$xy = \lambda(x + y).$$

$$3c^2 = xy + yz + xz.$$

Note that $x = y = z = c$ satisfies the constraint and gives a volume of c^3. Thus, the solution to the problem certainly has $V > 0$. If any of x, y, or z is 0, the volume is 0, which is not a max. So I may assume $x, y, z > 0$.

Note that this also implies that $y + z > 0$, so I may divide by $y + z$.

Now solve the equations:

$$\begin{aligned}
yz &= \lambda(y + z) \\
\lambda &= \frac{yz}{y + z} \\
xz &= \lambda(x + z) \\
xz &= \frac{yz}{y + z}(x + z) \\
xz(y + z) &= yz(x + z) \\
xyz + xz^2 &= xyz + yz^2 \\
xz^2 &= yz^2 \\
x &= y
\end{aligned}$$

$$\begin{aligned}
xy &= \lambda(x + y) \\
xy &= \frac{yz}{y + z}(x + y) \\
xy(y + z) &= yz(x + y) \\
xy^2 + xyz &= xyz + y^2 z \\
xy^2 &= y^2 z \\
x &= z \\
3c^2 &= xy + yz + xz \\
3c^2 &= x^2 + x^2 + x^2 \\
x &= c \\
y &= c \\
z &= c
\end{aligned}$$

The critical point is (c, c, c), which is a cube with sides of length c. $\square$

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30. (a) Find the critical points of

$$w = 4xyz \quad \text{subject to the constraint} \quad x + y + z = 3.$$

(b) Express w as a function of x and y by eliminating z, then consider the behavior of w for $x = y$. Explain why the critical points in (a) can't give absolute maxes or mins.

The constraint is

$$g(x, y, z) = x + y + z - 3 = 0.$$

Set up the multiplier equation:

$$\nabla f = \lambda \nabla g$$
$$(4yz, 4xz, 4xy) = \lambda(1, 1, 1)$$

This gives the equations

$$4yz = \lambda.$$
$$4xz = \lambda.$$
$$4xy = \lambda.$$
$$x + y + z = 3.$$

Solve the equations:

$$\begin{gathered} 4yz = \lambda \\ 4xz = \lambda \\ 4yz = 4xz \\ yz - xz = 0 \\ (y - x)z = 0 \end{gathered}$$

$$\swarrow \qquad\qquad\qquad \searrow$$

$$\begin{gathered} y = x \\ 4xy = \lambda \\ 4xy = 4xz \\ xy - xz = 0 \\ x(y - z) = 0 \end{gathered} \qquad\qquad\qquad \begin{gathered} z = 0 \\ \lambda = 0 \\ 4xy = 0 \\ xy = 0 \end{gathered}$$

$$\swarrow \qquad \downarrow \qquad\qquad\qquad\qquad \downarrow \qquad \searrow$$

$$\begin{gathered} x = 0 \\ y = 0 \\ x + y + z = 3 \\ z = 3 \\ (0, 0, 3) \end{gathered} \qquad \begin{gathered} y = z \\ x + y + z = 3 \\ 3x = 3 \\ x = 1 \\ y = 1 \\ z = 1 \\ (1, 1, 1) \end{gathered} \qquad\qquad \begin{gathered} x = 0 \\ x + y + z = 3 \\ y = 3 \\ (0, 3, 0) \end{gathered} \qquad \begin{gathered} y = 0 \\ x + y + z = 3 \\ x = 3 \\ (3, 0, 0) \end{gathered}$$

Test the points:

point	$w = 4xyz$
$(3, 0, 0)$	0
$(0, 3, 0)$	0
$(0, 0, 3)$	0
$(1, 1, 1)$	1

$\square$

(b) Solving the constraint for z gives $z = 3 - x - y$. Then

$$w = 4xy(3 - x - y).$$

Consider the behavior of w along the line $x = y$:

$$w = 4x^2(3 - 2x).$$

The factor $4x^2$ is positive. As $x \to infty$, the term $3 - 2x$ becomes large and negative, so $w \to -\infty$. As $x \to -\infty$, the term $3 - 2x$ becomes large and positive, so $w \to \infty$.

This means that you can find values of x, y, and z satisfying the constraint for which w is arbitrarily big or small. Hence, the critical points found in (a) can't be absolute maxes or mins. □

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31. Find the largest and smallest values of $f(x, y) = 4x^2y$ subject to the constraint $x^2 + y^2 = 36$.

The constraint is $g(x, y) = x^2 + y^2 - 36 = 0$.
Set up the multiplier equation:

$$\nabla f = \lambda \nabla g$$

$$(8xy, 4x^2) = \lambda(2x, 2y)$$

This gives two equations:

$$8xy = 2x\lambda, \quad 4xy = x\lambda = 0, \quad x(4y - \lambda) = 0.$$

$$4x^2 = 2y\lambda.$$

Solve those equations simultaneously with the constraint:

$$\begin{tikzcd}
 & & x(4y-\lambda)=0 \arrow[dl] \arrow[dr] & & \\
 & \begin{array}{c} x=0 \\ x^2+y^2=36 \\ y^2=36 \end{array} \arrow[dl] \arrow[d] & & \begin{array}{c} \lambda=4y \\ 4x^2=2y\lambda \\ 4x^2=2y\cdot 4y \\ x^2=2y^2 \\ 2y^2+y^2=36 \\ 3y^2=36 \\ y^2=12 \end{array} \arrow[dl] \arrow[d] & \\
\begin{array}{c} y=6 \\ (0,6) \end{array} & \begin{array}{c} y=-6 \\ (0,-6) \end{array} & \begin{array}{c} y=2\sqrt{3} \\ x^2=24 \end{array} \arrow[dl] \arrow[d] & \begin{array}{c} y=-2\sqrt{3} \\ x^2=24 \end{array} \arrow[d] \arrow[dr] & \\
 & \begin{array}{c} x=2\sqrt{6} \\ \left(2\sqrt{6}, 2\sqrt{3}\right) \end{array} & \begin{array}{c} x=-2\sqrt{6} \\ \left(-2\sqrt{6}, 2\sqrt{3}\right) \end{array} & \begin{array}{c} x=2\sqrt{6} \\ \left(2\sqrt{6}, -2\sqrt{3}\right) \end{array} & \begin{array}{c} x=-2\sqrt{6} \\ \left(-2\sqrt{6}, -2\sqrt{3}\right) \end{array}
\end{tikzcd}$$

Test the points:

	$(0,6)$	$(0,-6)$	$\left(2\sqrt{6}, 2\sqrt{3}\right)$	$\left(-2\sqrt{6}, 2\sqrt{3}\right)$	$\left(2\sqrt{6}, -2\sqrt{3}\right)$	$\left(-2\sqrt{6}, -2\sqrt{3}\right)$
$f(x,y)$	0	0	$192\sqrt{3}$	$192\sqrt{3}$	$-192\sqrt{3}$	$-192\sqrt{3}$
			max	max	min	min

□

To be conscious that you are ignorant is a great step to knowledge. - BENJAMIN DISRAELI