Math 322
9-29-2020

Review Sheet for Test 1

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. Compute:

(a) $5 + 3^4$ in $\mathbb{Z}_{11}$.

(b) 7^{-1} in $\mathbb{Z}_{13}$.

(c) 3^{101} in $\mathbb{Z}_5$.

(d) -16 in $\mathbb{Z}_{19}$.

(e) $7 \cdot (3 + 13)$ in $\mathbb{Z}_{11}$.

2. (a) Find 8^{-1} in $\mathbb{Z}_{15}$.

(b) Prove that 10 does not have a multiplicative inverse in $\mathbb{Z}_{15}$.

3. Find a quadratic polynomial $x^2 + bx + c$ over $\mathbb{Z}_6$ which has 4 different roots in $\mathbb{Z}_6$.

4. (a) Complete the definition: "A field is a commutative ring with identity ...".

(b) Is $\mathbb{Z}$ a field? Why or why not?

5. Find two nonzero elements of $\mathbb{Z}_{14}$ whose product is 0.

6. Calvin Butterball is trying to solve $x^2 = 1$ in $\mathbb{Z}_8$. He takes the square root of both sides to get $x = 1$ and $x = -1 = 7$. Are those the only solutions?

7. List all the 2×2 row-reduced echelon matrices over $\mathbb{Z}_3$.

8. Consider the following matrix over $\mathbb{Z}_5$:

$$\begin{bmatrix} 1 & 2 & b & c \\ 0 & a & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Given that the matrix is in row-reduced echelon form, determine the values of a, b, and c. If you cannot determine the value of a, b, or c, explain why you can't.

9. Consider the following real matrices:

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & -2 \\ -1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 1 & 1 \\ 7 & 0 & -1 \end{bmatrix}.$$

Compute:

(a) $5A + 3D$.

(b) $(BA)^T + C$.

(c) $\operatorname{tr}(CA)$. (tr is the **trace** of the matrix, which is the sum of the entries on the main diagonal.)

(d) B^{-1}.

(e) $AC - 3I$.

10. Suppose A and B are invertible matrices. Solve for X:

$$A^2BXB^{-1} = A^3BAB.$$

11. A and B are invertible matrices. Simplify $(AB)^2B^{-1}A^{-2}A^3B^2$.

12. Prove that if A and B are matrices of the same dimension and k is a number, then

$$k(A + B) = kA + kB.$$

(You should use only the definitions of matrix equality, multiplication by scalars, and matrix addition.)

13. If A and B are $n \times n$ matrices, is it necessarily true that $(AB)^2 = A^2B^2$? If it is, prove it. If it isn't, give a specific counterexample.

14. (a) Suppose A and B are $n \times n$ matrices over $\mathbb{R}$ and B is not invertible. Prove that AB is not invertible.

(b) Suppose A and B are $n \times n$ matrices over $\mathbb{R}$, and A and B are invertible. Is $A + B$ necessarily invertible? Is $A + B$ necessarily not invertible?

15. Prove that if A is an $n \times n$ matrix, then $(AA^T)^2$ is symmetric.

(Remember that a matrix X is symmetric if $X = X^T$. What do you need to show in this problem?)

16. If A and B are symmetric matrices, is AB symmetric? If it is, prove it; if it isn't, give a specific counterexample.

(Try it out with some 2×2 symmetric matrices.)

17. Prove that if P is symmetric, then so is $P^2 + 5P + I$.

(Remember that a matrix X is symmetric if $X = X^T$. What do you need to show in this problem?)

18. Suppose the following matrix is skew-symmetric. Find a, b, and c.

$$\begin{bmatrix} a & b-3 & 9 \\ 5 & 0 & -6 \\ -9 & 2c & 0 \end{bmatrix}.$$

(Remember that a matrix X is skew-symmetric if $X = -X^T$. What does this imply about the main diagonal entries, and entries on opposite sides of the main diagonal?)

19. Using associativity of matrix addition and the rule $(M + N)^T = M^T + N^T$, prove that

$$(A + B + C)^T = A^T + B^T + C^T.$$

(Note: If A, B, and C are matrices, $A+B+C$ can be interpreted as either $(A+B)+C$ or as $A+(B+C)$.)

20. Solve the following system of linear equations over $\mathbb{R}$:

$$\begin{array}{ccccccccc} w & - & 2x & - & y & + & 5z & = & 1 \\ w & - & 2x & + & 3y & + & z & = & 9 \\ w & - & 2x & + & 2y & + & 2z & = & 7 \end{array}$$

21. Solve the following system of linear equations over $\mathbb{Z}_3$:

$$\begin{array}{rcrcl} 2x & + & y & = & 1 \\ 2x & + & 2y & = & 2 \end{array}$$

22. Solve the following system of linear equations over $\mathbb{Z}_5$:

$$\begin{array}{rcrcrcrcl} 2w & + & 2x & + & y & + & z & = & 4 \\ w & + & x & + & 3y & & & = & 1 \\ & & x & + & y & & & = & 1 \end{array}$$

23. Row reduce the following matrix over $\mathbb{R}$ to row-reduced echelon form:

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 2 & 4 & 1 & 12 \\ -1 & -2 & 4 & 3 \end{bmatrix}.$$

24. Row reduce the following matrix over $\mathbb{Z}_2$ to row-reduced echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

25. Find the inverse of the real matrix

$$\begin{bmatrix} 2 & 4 & -5 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}.$$

26. Find the inverse of the following matrix over $\mathbb{Z}_3$:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

27. Find the inverse of the following matrix over $\mathbb{R}$, where a and b are real numbers.

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}.$$

28. Write down the real 3×3 elementary matrix corresponding to each row operation.

(a) $r_2 \to 7r_2$.

(b) $r_3 \to r_3 + 4r_1$.

(c) $r_2 \leftrightarrow r_1$.

(d) $r_1 \to r_1 - 5r_2$.

29. What elementary row operations on real matrices are performed by left multiplying by the following elementary matrices?

(a) $\begin{bmatrix} 1 & -8 \\ 0 & 1 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 10 & 0 & 1 \end{bmatrix}$.

30. Of the following, which are **not** valid elementary row operations on real matrices?

$$r_1 \to r_1 + 3r_2, \quad r_3 \to r_2 - r_3, \quad r_4 \to 5r_4 + 7r_2, \quad r_2 \to -r_2, \quad r_2 \to r_3 + 7.$$

31. Express the real matrix $\begin{bmatrix} 2 & 6 \\ -3 & 7 \end{bmatrix}$ as a product of elementary matrices.

32. Express the real matrix $\begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ as a product of elementary matrices.

33. Express the matrix $\begin{bmatrix} 3 & 1 \\ 4 & 4 \end{bmatrix}$ over $\mathbb{Z}_5$ as a product of elementary matrices.

34. Calvin Butterball says: "Let A be an $n \times n$ matrix. A is invertible is equivalent to $Ax = \vec{0}$." What is wrong with this? What is the correct statement?

35. In this problem, the systems are over $\mathbb{R}$.

(a) Use row reduction to solve the system

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(b) Use matrix inversion to solve the system

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

36. If A, B, and C are $n \times n$ matrices, $A \neq 0$, and $AB = AC$, does it follow that $B = C$? If it does, prove it; if it doesn't, give a counterexample.

37. Prove that if A is an $n \times n$ matrix and A is not invertible, then there is a nonzero $n \times n$ matrix B such that $AB = 0$ (where 0 denotes the $n \times n$ zero matrix).

38. The determinant of the following real matrix is given:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 12.$$

Compute the following determinants.

(a) $\det \begin{bmatrix} d & e & 3f \\ a & b & 3c \\ g & h & 3i \end{bmatrix}$.

(b) $\det \begin{bmatrix} g & i & h \\ d & f & e \\ a & c & b \end{bmatrix}$.

(c) $\det \begin{bmatrix} a & 2b & c \\ g + 5a & 2h + 10b & i + 5c \\ d & 2e & f \end{bmatrix}$.

39. Let R be a commutative ring with identity. Define a function $D : M(2, R) \to R$ by

$$D \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad.$$

Prove that D is linear in each row. Show by specific counterexample that D is not alternating.

40. Prove that

$$\begin{vmatrix} a & a+b & c \\ d+2a & d+2a+e+2b & f+2c \\ g & g+h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

41. Let x be a nonzero real number. Compute

$$\det \begin{bmatrix} 1 & x & x^2 \\ x & x^2 & x^3 \\ x^2 & x^3 & x^4 \end{bmatrix}.$$

42. Let M be the matrix obtained from the 5×5 identity matrix I by swapping rows 1 and 5, then swapping columns 2 and 4, then swapping rows 2 and 3, and finally swapping columns 1 and 3. What is the determinant of M?

43. Compute the determinant of the following real matrix by row reducing it to the identity.

$$\begin{bmatrix} 2 & -4 & 2 \\ 1 & -1 & 0 \\ 0 & 5 & 15 \end{bmatrix}.$$

44. Compute the determinant of the following matrix over $\mathbb{Z}_5$ by row reducing it to the identity.

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

45. Compute $\begin{vmatrix} 1 & 3 & -4 \\ 1 & 2 & 5 \\ -2 & -2 & 3 \end{vmatrix}$ over $\mathbb{R}$.

46. Suppose A is a 3×3 real matrix and

$$A^4 = \begin{bmatrix} 1 & 80 & 95 \\ 0 & 81 & 350 \\ 0 & 0 & 256 \end{bmatrix}.$$

Prove that A is invertible.

47. Suppose that A, B, and C are 4×4 matrices over $\mathbb{R}$, and

$$|A| = 6, \quad |B| = -3, \quad |C| = 5.$$

(a) Compute $|(ABC)^T|$.

(b) Compute $|3C|$.

(c) Compute $|A^{-1}B^2|$.

48. Let $a, b, c \in \mathbb{R}$. Compute the 3×3 Vandermonde determinant

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

49. Show that if A is an invertible $n \times n$ matrix, then

$$|\operatorname{adj} A| = |A|^{n-1}.$$

50. Use the adjoint formula to find the inverse of the following real matrix:

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 3 & 2 \\ -1 & -2 & -1 \end{bmatrix}.$$

51. Use the adjoint formula to find the inverse of the real matrix

$$A = \begin{bmatrix} 1 & a & -a \\ a & 1 & -1 \\ a & 1 & a \end{bmatrix}.$$

(Assume that $a \neq \pm 1$.)

52. Use the adjoint formula to find the inverse of the following matrix over $\mathbb{Z}_3$:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}.$$

53. Use Cramer's Rule to solve the following system over $\mathbb{R}$:

$$\begin{aligned} 3x - 7y &= -5 \\ x + 10y &= 4 \end{aligned}$$

54. Use Cramer's rule to solve the following system over $\mathbb{Z}_3$:

$$\begin{array}{ccccccc} 2x & + & y & + & z & = & 0 \\ x & + & y & + & z & = & 1 \\ x & + & 2y & + & z & = & 1 \end{array}$$

Solutions to the Review Sheet for Test 1

1. Compute:

(a) $5 + 3^4$ in $\mathbb{Z}_{11}$.

(b) 7^{-1} in $\mathbb{Z}_{13}$.

(c) 3^{101} in $\mathbb{Z}_5$.

(d) -16 in $\mathbb{Z}_{19}$.

(e) $7 \cdot (3 + 13)$ in $\mathbb{Z}_{11}$.

(a) $5 + 3^4 = 86 = 77 + 9 = 7 \cdot 11 + 9 = 9$ in $\mathbb{Z}_{11}$. □

(b) Since $7 \cdot 2 = 1$ in $\mathbb{Z}_{13}$, it follows that $7^{-1} = 2$. □

(c) First, $3^4 = 81 = 1$ in $\mathbb{Z}_5$. So

$$3^{101} = 3^{100} \cdot 3 = (3^4)^{25} \cdot 3 = 1^{25} \cdot 3 = 1 \cdot 3 = 3 \quad \text{in} \quad \mathbb{Z}_5. \quad \square$$

(d)

$$-16 = -16 + 19 = 3 \quad \text{in} \quad \mathbb{Z}_{19}. \quad \square$$

(e)

$$7 \cdot (3 + 13) = 7 \cdot 16 = 7 \cdot 5 = 35 = 2 \quad \text{in} \quad \mathbb{Z}_{11}. \quad \square$$

---

2. (a) Find 8^{-1} in $\mathbb{Z}_{15}$.

(b) Prove that 10 does not have a multiplicative inverse in $\mathbb{Z}_{15}$.

(a) Since $8 \cdot 2 = 16 = 1$ in $\mathbb{Z}_{15}$, it follows that $8^{-1} = 2$. $\square$

(b) Suppose that $10x = 1$ in $\mathbb{Z}_{15}$. Then

$$\begin{aligned} 3 \cdot 10 \cdot x &= 3 \cdot 1 \\ 30x &= 3 \\ 0 &= 3 \end{aligned}$$

This contradiction shows that there is no x such that $10x = 1$ in $\mathbb{Z}_{15}$. $\square$

---

3. Find a quadratic polynomial $x^2 + bx + c$ over $\mathbb{Z}_6$ which has 4 different roots in $\mathbb{Z}_6$.

There are many examples, which you can find by trial and error. For instance:

x	0	1	2	3	4	5
$x^2 + 3x + 2$	2	0	0	2	0	0

$x^3 + 3x + 2$ has roots 1, 2, 4, and 5. $\square$

---

4. (a) Complete the definition: "A field is a commutative ring with identity ...".

(b) Is $\mathbb{Z}$ a field? Why or why not?

(a) A field is a commutative ring with identity in which every nonzero element has a multiplicative inverse. $\square$

(b) It is not a field. For example, 2 is a nonzero integer, but it doesn't have a multiplicative inverse (note that $\dfrac{1}{2}$ is not an integer). $\square$

---

5. Find two nonzero elements of $\mathbb{Z}_{14}$ whose product is 0.

For example, $2 \cdot 7 = 0$ in $\mathbb{Z}_{14}$. $\square$

---

6. Calvin Butterball is trying to solve $x^2 = 1$ in $\mathbb{Z}_8$. He takes the square root of both sides to get $x = 1$ and $x = -1 = 7$. Are those the only solutions?

No. In fact, $3^2 = 1$ and $5^2 = 1$ as well. You could find all the solutions by trying all the elements in $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. $\square$

---

7. List all the 2×2 row-reduced echelon matrices over $\mathbb{Z}_3$.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}. \quad \square$$

---

8. Consider the following matrix over $\mathbb{Z}_5$:

$$\begin{bmatrix} 1 & 2 & b & c \\ 0 & a & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Given that the matrix is in row-reduced echelon form, determine the values of a, b, and c. If you cannot determine the value of a, b, or c, explain why you can't.

First, if $a \neq 0$, then being the first nonzero element in the second row, it must be a 1, and it's a leading coefficient. But there's a nonzero element in the $(1, 2)^{\text{th}}$ position (the "2") contradicting the fact that a leading coefficient must be the only nonzero element in its column. Hence, $a = 0$.

This makes the "1" in the $(2, 3)^{\text{th}}$ position a leading coefficient. Therefore, $b = 0$.

Since c is not in the same column as a leading coefficient, there is no restriction on its value — i.e. c could be any one of 0, 1, 2, 3, or 4. $\square$

---

9. Consider the following real matrices:

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & -2 \\ -1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 1 & 1 \\ 7 & 0 & -1 \end{bmatrix}.$$

Compute:

(a) $5A + 3D$.

(b) $(BA)^T + C$.

(c) $\operatorname{tr}(CA)$.

(d) B^{-1}.

(e) $AC - 3I$.

(a)

$$5A + 3D = \begin{bmatrix} 5 & 15 & -5 \\ 10 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 3 & 3 \\ 21 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 11 & 18 & -2 \\ 31 & 5 & -3 \end{bmatrix}. \quad \square$$

(b)

$$(BA)^T + C = \begin{bmatrix} 12 & 11 & -2 \\ 9 & 7 & -1 \end{bmatrix}^T + \begin{bmatrix} 1 & 0 \\ 2 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 9 \\ 11 & 7 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 9 \\ 13 & 5 \\ -3 & 2 \end{bmatrix}. \quad \square$$

(c)

$$\operatorname{tr}(CA) = \operatorname{tr} \begin{bmatrix} 1 & 3 & -1 \\ -2 & 4 & -2 \\ 5 & 0 & 1 \end{bmatrix} = 1 + 4 + 1 = 6. \quad \square$$

(d)

$$B^{-1} = \frac{1}{2 \cdot 4 - 1 \cdot 5} \begin{bmatrix} 4 & -5 \\ -1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4 & -5 \\ -1 & 2 \end{bmatrix}. \quad \square$$

(e)

$$AC - 3I = \begin{bmatrix} 8 & -9 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -9 \\ 4 & -5 \end{bmatrix}. \quad \square$$

---

10. Suppose A and B are invertible matrices. Solve for X:

$$A^2 B X B^{-1} = A^3 BAB.$$

$$\begin{array}{ccc} A^2 B X B^{-1} & = & A^3 BAB \\ A^{-2} A^2 B X B^{-1} & = & A^{-2} A^3 BAB \\ B X B^{-1} & = & ABAB \\ B^{-1} B X B^{-1} & = & B^{-1} ABAB \\ X B^{-1} & = & B^{-1} ABAB \\ X B^{-1} B & = & B^{-1} ABABB \\ X & = & B^{-1} ABAB^2 \quad \square \end{array}$$

---

11. A and B are invertible matrices. Simplify $(AB)^2 B^{-1} A^{-2} A^3 B^2$.

$$(AB)^2 B^{-1} A^{-2} A^3 B^2 = ABABB^{-1} A^{-2} A^3 B^2 = ABAA^{-2} A^3 B^2 = ABA^2 B^2. \quad \square$$

---

12. Prove that if A and B are matrices of the same dimension and k is a number, then

$$k(A + B) = kA + kB.$$

You can prove properties of matrices by showing that corresponding elements are equal. In this case, I consider the $(i, j)^{\text{th}}$ elements of the matrices on the left and right and show that those elements are equal.

$$\begin{array}{cccc} [k(A+B)]_{ij} & = & k \cdot (A+B)_{ij} & \text{(Definition of scalar multiplication)} \\ & = & k \cdot (A_{ij} + B_{ij}) & \text{(Definition of matrix addition)} \\ & = & k \cdot A_{ij} + k \cdot B_{ij} & \text{(Distributive Law)} \\ & = & (kA)_{ij} + (kB)_{ij} & \text{(Definition of scalar multiplication)} \\ & = & (kA + kB)_{ij} & \text{(Definition of matrix addition)} \end{array}$$

Since the $(i, j)^{\text{th}}$ elements of $k(A + B)$ and $kA + kB$ are equal, it follows that $k(A + B) = kA + kB$ by definition of matrix equality. $\square$

---

13. If A and B are $n \times n$ matrices, is it necessarily true that $(AB)^2 = A^2 B^2$? If it is, prove it. If it isn't, give a specific counterexample.

By definition, U^2 means $U \cdot U$, so $(AB)^2 = (AB) \cdot (AB) = ABAB$. But this isn't necessarily the same as $A^2 B^2$, because matrix multiplication isn't necessarily commutative.

For example, consider the real matrices

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}.$$

Then

$$(AB)^2 = \begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix}, \quad \text{but} \quad A^2B^2 = \begin{bmatrix} -23 & 29 \\ -8 & -24 \end{bmatrix}.$$

In this case, $(AB)^2 \neq A^2B^2$. □

---

14. (a) Suppose A and B are $n \times n$ matrices over $\mathbb{R}$ and B is not invertible. Prove that AB is not invertible.

(b) Suppose A and B are $n \times n$ matrices over $\mathbb{R}$, and A and B are invertible. Is $A + B$ necessarily invertible? Is $A + B$ necessarily not invertible?

(a) It is easy to do this using determinants, but I'll give a proof which uses facts about solutions to systems. Since B is not invertible, there is a nonzero vector x such that $Bx = 0$. Then

$$\begin{aligned} Bx &= 0 \\ A(Bx) &= A \cdot 0 \\ (AB)x &= 0 \end{aligned}$$

Since there is a nonzero vector (namely x) such that $(AB)x = 0$, it follows that AB is not invertible. □

(b) Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then A and B are invertible. But

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, $A + B$ is not invertible.
On the other hand, let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then A and B are invertible. Moreover,

$$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So in this case, $A + B$ is invertible.
Thus, if A and B is invertible, $A + B$ might be invertible or not invertible. □

---

15. Prove that if A is an $n \times n$ matrix, then $(AA^T)^2$ is symmetric.

$$\begin{aligned} [(AA^T)^2]^T &= [AA^TAA^T]^T && \text{(Since } (MN)^2 = MNMN) \\ &= (A^T)^TA^T(A^T)^TA^T && \text{(Since } (MN)^T = N^TM^T) \\ &= AA^TAA^T && \text{(Since } (M^T)^T = M) \\ &= (AA^T)^2 && \text{(Since } (MN)^2 = MNMN) \end{aligned}$$

Hence, $(AA^T)^2$ is symmetric. □

---

16. If A and B are symmetric matrices, is AB symmetric? If it is, prove it; if it isn't, give a specific counterexample.

The following real matrices are symmetric:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}.$$

But their product is not symmetric:

$$AB = \begin{bmatrix} -1 & 4 \\ 13 & -8 \end{bmatrix}. \quad \square$$

---

17. Prove that if P is symmetric, then so is $P^2 + 5P + I$.

Suppose P is symmetric, so $P = P^T$. Then

$$\begin{aligned} (P^2 + 5P + I)^T &= (P^2)^T + (5P)^T + I^T && \text{(Using } (M+N)^T = M^T + N^T\text{, or the last problem)} \\ &= (P^T)^2 + (5P)^T + I^T && \text{(Since } (MN)^T = N^T M^T\text{)} \\ &= (P^T)^2 + 5 \cdot P^T + I^T && \text{(Since } (kM)^T = k \cdot M^T\text{)} \\ &= (P^T)^2 + 5 \cdot P^T + I && \text{(Since } I^T = I\text{)} \\ &= P^2 + 5P + I && \text{(Since } P^T = P\text{)} \end{aligned}$$

It follows that $P^2 + 5P + I$ is symmetric. $\square$

---

18. Suppose the following matrix is skew-symmetric. Find a, b, and c.

$$\begin{bmatrix} a & b-3 & 9 \\ 5 & 0 & -6 \\ -9 & 2c & 0 \end{bmatrix}.$$

Since the matrix is skew-symmetric, it's equal to the negative of its transpose:

$$\begin{bmatrix} a & b-3 & 9 \\ 5 & 0 & -6 \\ -9 & 2c & 0 \end{bmatrix} = \begin{bmatrix} -a & -5 & 9 \\ -b+3 & 0 & -2c \\ -9 & 6 & 0 \end{bmatrix}.$$

Equate corresponding entries. Since $a = -a$, I have $a = 0$. Since $b - 3 = -5$, I have $b = -2$. Since $2c = 6$, I have $c = 3$. With these values inserted, the matrix is

$$\begin{bmatrix} 0 & -5 & 9 \\ 5 & 0 & -6 \\ -9 & 6 & 0 \end{bmatrix}. \quad \square$$

---

19. Using associativity of matrix addition and the rule $(M + N)^T = M^T + N^T$, prove that

$$(A + B + C)^T = A^T + B^T + C^T.$$

Note that $A + B + C$ can be interpreted as meaning either $(A + B) + C$ or $A + (B + C)$, by associativity of matrix addition. I'll use $(A + B) + C$. I have

$$\begin{aligned} (A + B + C)^T &= [(A + B) + C]^T \\ &= (A + B)^T + C^T \\ &= (A^T + B^T) + C^T \\ &= A^T + B^T + C^T \end{aligned}$$

I used the rule $(M + N)^T = M^T + N^T$ for the second and third equalities. $\square$

Note: To prove this result for a summ with an arbitrary number of terms, you'd use **mathematical induction**.

---

20. Solve the following system of linear equations over $\mathbb{R}$:

$$\begin{array}{rcrcrcrcr} w & - & 2x & - & y & + & 5z & = & 1 \\ w & - & 2x & + & 3y & + & z & = & 9 \\ w & - & 2x & + & 2y & + & 2z & = & 7 \end{array}$$

$$\begin{bmatrix} 1 & -2 & -1 & 5 & 1 \\ 1 & -2 & 3 & 1 & 9 \\ 1 & -2 & 2 & 2 & 7 \end{bmatrix} \underset{r_3 \to r_3 - r_1}{\to} \begin{bmatrix} 1 & -2 & -1 & 5 & 1 \\ 1 & -2 & 3 & 1 & 9 \\ 0 & 0 & 3 & -3 & 6 \end{bmatrix} \underset{r_2 \to r_2 - r_1}{\to}$$

$$\begin{bmatrix} 1 & -2 & -1 & 5 & 1 \\ 0 & 0 & 4 & -4 & 8 \\ 0 & 0 & 3 & -3 & 6 \end{bmatrix} \underset{r_2 \to r_2/4}{\to} \begin{bmatrix} 1 & -2 & -1 & 5 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 3 & -3 & 6 \end{bmatrix} \underset{r_3 \to r_3 - 3r_2}{\to}$$

$$\begin{bmatrix} 1 & -2 & -1 & 5 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{r_1 \to r_1 + r_2}{\to} \begin{bmatrix} 1 & -2 & 0 & 4 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding equations are

$$w - 2x + 4z = 3 \quad \text{and} \quad y - z = 2.$$

Set $x = s$ and $z = t$ and plug in:

$$w - 2s + 4t = 3, \quad \text{so} \quad w = 2s - 4t + 3,$$

$$y - t = 2, \quad \text{so} \quad y = t + 2.$$

The solution is

$$w = 2s - 4t + 3, \quad x = s, \quad y = t + 2, \quad z = t. \quad \square$$

---

21. Solve the following system of linear equations over $\mathbb{Z}_3$:

$$\begin{array}{rcrcr} 2x & + & y & = & 1 \\ 2x & + & 2y & = & 2 \end{array}$$

The system can be written as

$$\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2^{-1} \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \cdot \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The solution is $x = 0$ and $y = 1$. (You could also do this problem by row-reduction.) $\square$

---

22. Solve the following system of linear equations over $\mathbb{Z}_5$:

$$\begin{array}{ccccccccc} 2w & + & 2x & + & y & + & z & = & 4 \\ w & + & x & + & 3y & & & = & 1 \\ & & x & + & y & & & = & 1 \end{array}$$

$$\begin{bmatrix} 2 & 2 & 1 & 1 & 4 \\ 1 & 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \underset{r_2 \leftrightarrow r_1}{\rightarrow} \begin{bmatrix} 1 & 1 & 3 & 0 & 1 \\ 2 & 2 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \underset{r_2 \to r_2 + 3r_1}{\rightarrow} \begin{bmatrix} 1 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \underset{r_2 \leftrightarrow r_3}{\rightarrow}$$

$$\begin{bmatrix} 1 & 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{r_1 \to r_1 - r_2}{\rightarrow} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The equations are

$$w + 2y = 0, \quad x + y = 1, \quad z = 2.$$

Set $y = s$. The solution is

$$w = -2s = 3s, \quad x = -s + 1 = 4s + 1, y = s, \quad z = 2. \quad \square$$

---

23. Row reduce the following matrix over $\mathbb{R}$ to row-reduced echelon form:

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 2 & 4 & 1 & 12 \\ -1 & -2 & 4 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 2 & 4 & 1 & 12 \\ -1 & -2 & 4 & 3 \end{bmatrix} \underset{r_2 \to r_2 - 2r_1}{\rightarrow} \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 5 & 10 \\ -1 & -2 & 4 & 3 \end{bmatrix} \underset{r_3 \to r_3 + r_1}{\rightarrow} \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 2 & 4 \end{bmatrix} \underset{r_2 \to r_2/5}{\rightarrow}$$

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \underset{r_3 \to r_3 - 2r_2}{\rightarrow} \begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{r_1 \to r_1 + 2r_2}{\rightarrow} \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \square$$

---

24. Row reduce the following matrix over $\mathbb{Z}_2$ to row-reduced echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Remember that $1 + 1 = 0$ in $\mathbb{Z}_2$!

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \underset{r_2 \to r_2 + r_1}{\rightarrow} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \underset{r_2 \leftrightarrow r_3}{\rightarrow} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \underset{r_1 \to r_1 + r_2}{\rightarrow}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \underset{r_1 \to r_1 + r_3}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \underset{r_2 \to r_2 + r_3}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \square$$

25. Find the inverse of the real matrix
$$\begin{bmatrix} 2 & 4 & -5 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 4 & -5 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \underset{r_1 \leftrightarrow r_3}{\rightarrow} \begin{bmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & 4 & -5 & 1 & 0 & 0 \end{bmatrix} \underset{r_2 \rightarrow r_2 - 2r_1}{\rightarrow}$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & -4 & 0 & 1 & -2 \\ 2 & 4 & -5 & 1 & 0 & 0 \end{bmatrix} \underset{r_3 \rightarrow r_3 - 2r_1}{\rightarrow} \begin{bmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & -4 & 0 & 1 & -2 \\ 0 & 6 & -9 & 1 & 0 & -2 \end{bmatrix} \underset{r_2 \rightarrow r_2/3}{\rightarrow}$$

$$\begin{bmatrix} 1 & -1 & 2 & 0 & 0 & 1 \\ 0 & 1 & -\frac{4}{3} & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 6 & -9 & 1 & 0 & -2 \end{bmatrix} \underset{r_1 \rightarrow r_1 + r_2}{\rightarrow} \begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{4}{3} & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 6 & -9 & 1 & 0 & -2 \end{bmatrix} \underset{r_3 \rightarrow r_3 - 6r_2}{\rightarrow}$$

$$\begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{4}{3} & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & -1 & 1 & -2 & 2 \end{bmatrix} \underset{r_3 \rightarrow -r_3}{\rightarrow} \begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{4}{3} & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -1 & 2 & -2 \end{bmatrix} \underset{r_1 \rightarrow r_1 - (2/3)r_3}{\rightarrow}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} & -1 & \frac{5}{3} \\ 0 & 1 & -\frac{4}{3} & 0 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -1 & 2 & -2 \end{bmatrix} \underset{r_2 \rightarrow r_2 + (4/3)r_3}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} & -1 & \frac{5}{3} \\ 0 & 1 & 0 & -\frac{4}{3} & 3 & -\frac{10}{3} \\ 0 & 0 & 1 & -1 & 2 & -2 \end{bmatrix}$$

Hence,
$$\begin{bmatrix} 2 & 4 & -5 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{3} & -1 & \frac{5}{3} \\ -\frac{4}{3} & 3 & -\frac{10}{3} \\ -1 & 2 & -2 \end{bmatrix}. \quad \square$$

26. Find the inverse of the following matrix over $\mathbb{Z}_3$:
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \underset{r_2 \rightarrow r_2 + r_1}{\rightarrow} \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \underset{r_3 \rightarrow r_3 - r_1}{\rightarrow} \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 \end{bmatrix} \underset{r_2 \leftrightarrow r_3}{\rightarrow}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \underset{r_1 \rightarrow r_1 - r_2}{\rightarrow} \begin{bmatrix} 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \underset{r_2 \rightarrow 2r_2}{\rightarrow} \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Therefore,
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}. \quad \square$$

27. Find the inverse of the following matrix over $\mathbb{R}$, where a and b are real numbers.

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & a & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ b & 0 & 1 & 0 & 0 & 1 \end{array}\right] r_3 \to r_3 - br_1 \left[\begin{array}{ccc|ccc} 1 & 0 & a & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1-ab & -b & 0 & 1 \end{array}\right] r_3 \to r_3/(1-ab)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & a & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\dfrac{b}{1-ab} & 0 & \dfrac{1}{1-ab} \end{array}\right] r_1 \to r_1 - ar_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1+\dfrac{ab}{1-ab} & 0 & -\dfrac{a}{1-ab} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\dfrac{b}{1-ab} & 0 & \dfrac{1}{1-ab} \end{array}\right]$$

Hence,

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1+\dfrac{ab}{1-ab} & 0 & -\dfrac{a}{1-ab} \\ 0 & 1 & 0 \\ -\dfrac{b}{1-ab} & 0 & \dfrac{1}{1-ab} \end{bmatrix}. \quad \square$$

28. Write down the real 3×3 elementary matrix corresponding to each row operation.

In each case, you can find the elementary matrix by applying the row operation to the identity matrix.

(a) $r_2 \to 7r_2$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

(b) $r_3 \to r_3 + 4r_1$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}. \quad \square$$

(c) $r_2 \leftrightarrow r_1$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

(d) $r_1 \to r_1 - 5r_2$.

$$\begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

29. What elementary row operations on real matrices are performed by left multiplying by the following elementary matrices?

(a) $\begin{bmatrix} 1 & -8 \\ 0 & 1 \end{bmatrix}$.

The row operation is $r_1 \to r_1 - 8r_2$. $\square$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

The row operation is $r_2 \leftrightarrow r_3$. $\square$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 10 & 0 & 1 \end{bmatrix}$.

The row operation is $r_3 \to r_3 + 10r_1$. $\square$

---

30. Of the following, which are **not** valid elementary row operations on real matrices?

$$r_1 \to r_1 + 3r_2, \quad r_3 \to r_2 - r_3, \quad r_4 \to 5r_4 + 7r_2, \quad r_2 \to -r_2, \quad r_2 \to r_3 + 7.$$

$r_1 \to r_1 + 3r_2$ and $r_2 \to -r_2$ are valid elementary row operations.

$r_3 \to r_2 - r_3$, $r_4 \to 5r_4 + 7r_2$, and $r_2 \to r_3 + 7$ are not valid elementary row operations. $\square$

---

31. Express the real matrix $\begin{bmatrix} 2 & 6 \\ -3 & 7 \end{bmatrix}$ as a product of elementary matrices.

First, I row reduce the matrix to the identity:

$$\begin{bmatrix} 2 & 6 \\ -3 & 7 \end{bmatrix} \underset{r_1 \to r_1/2}{\to} \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix} \underset{r_2 \to r_2 + 3r_1}{\to} \begin{bmatrix} 1 & 3 \\ 0 & 16 \end{bmatrix} \underset{r_2 \to r_2/16}{\to} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \underset{r_1 \to r_1 - 3r_2}{\to} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I express the row operations in terms of elementary matrices:

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Next, solve for the original matrix and express the inverses as elementary matrices:

$$\begin{bmatrix} 2 & 6 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & \frac{1}{16} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}. \quad \square$$

---

32. Express the real matrix $\begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ as a product of elementary matrices.

First, I row reduce the matrix to the identity:

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} \underset{r_2 \to r_2/2}{\to} \begin{bmatrix} 1 & 4 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix} \underset{r_1 \to r_1 - 4r_2}{\to} \begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix} \underset{r_3 \to r_3/3}{\to}$$

$$\begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \underset{r_1 \to r_1 + 13r_3}{\to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \underset{r_2 \to r_2 - 3r_3}{\to} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I express the row operations in terms of elementary matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next, solve for the original matrix and express the inverses as elementary matrices:

$$\begin{bmatrix} 1 & 4 & -1 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{3} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

Note: Your answer may be different if you did different row operations.

---

33. Express the matrix $\begin{bmatrix} 3 & 1 \\ 4 & 4 \end{bmatrix}$ over $\mathbb{Z}_5$ as a product of elementary matrices.

First, I row reduce the matrix to the identity:

$$\begin{bmatrix} 3 & 1 \\ 4 & 4 \end{bmatrix} \underset{r_1 \to 2r_1}{\to} \begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix} \underset{r_2 \to r_2 + r_1}{\to} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \underset{r_1 \to r_1 + 3r_2}{\to} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I express the row operations in terms of elementary matrices:

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Next, solve for the original matrix and express the inverses as elementary matrices:

$$\begin{bmatrix} 3 & 1 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}. \quad \square$$

---

34. Calvin Butterball says: "Let A be an $n \times n$ matrix. A is invertible is equivalent to $Ax = \vec{0}$." What is wrong with this? What is the correct statement?

To say "$Ax = \vec{0}$" *alone* just means that there is some vector x such that $Ax = \vec{0}$. It does not mean that $x = \vec{0}$, nor does it mean that $x = \vec{0}$ is the *only* vector which satisfies the equation. Mathematics is not just equations and symbols; *words are necessary.*

A correct statement is: "Let A be an $n \times n$ matrix. A is invertible is equivalent to $Ax = \vec{0}$ having only the trivial solution." $\square$

---

35. In this problem, the systems are over $\mathbb{R}$.

(a) Use row reduction to solve the system

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(b) Use matrix inversion to solve the system

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(a)

$$\begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & -1 \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{\rightarrow} \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 1 \end{bmatrix} \underset{r_2 \rightarrow r_2 - 2r_1}{\rightarrow} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -2 & 3 \end{bmatrix} \underset{r_2 \rightarrow -r_2/2}{\rightarrow}$$

$$\begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -\frac{3}{2} \end{bmatrix} \underset{r_1 \rightarrow r_1 - 3r_2}{\rightarrow} \begin{bmatrix} 1 & 0 & \frac{7}{2} \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

Therefore, $x = \frac{7}{2}$ and $y = -\frac{3}{2}$. □

(b)

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} \\ -\frac{3}{2} \end{bmatrix}. \quad \square$$

---

36. If A, B, and C are $n \times n$ matrices, $A \neq 0$, and $AB = AC$, does it follow that $B = C$? If it does, prove it; if it doesn't, give a counterexample.

The statement is false. Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}.$$

Then $A \neq 0$,

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = AC, \quad \text{but} \quad B \neq C. \quad \square$$

---

37. Prove that if A is an $n \times n$ matrix and A is not invertible, then there is a nonzero $n \times n$ matrix B such that $AB = 0$ (where 0 denotes the $n \times n$ zero matrix).

Since A is not invertible, the system $Ax = 0$ does not have only the trivial solution. This means that there must be a *nonzero* vector x such that $Ax = 0$. Let B be the $n \times n$ matrix all of whose columns are equal to x. Then B is a nonzero matrix, and

$$AB = A \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ x & x & \cdots & x \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ Ax & Ax & \cdots & Ax \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ 0 & 0 & \cdots & 0 \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}. \quad \square$$

Here's a specific example. The following matrix is not invertible:

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}.$$

I can find a nonzero vector x so that $Ax = 0$. For example,

$$\begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

I make a matrix B using the vector $[2 \ \ 1]^T$ in each column:

$$B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

You can check that $AB = 0$. $\square$

---

38. The determinant of the following real matrix is given:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 12.$$

Compute the following determinants.

(a) $\det \begin{bmatrix} d & e & 3f \\ a & b & 3c \\ g & h & 3i \end{bmatrix}$.

(b) $\det \begin{bmatrix} g & i & h \\ d & f & e \\ a & c & b \end{bmatrix}$.

(c) $\det \begin{bmatrix} a & 2b & c \\ g+5a & 2h+10b & i+5c \\ d & 2e & f \end{bmatrix}$.

(a)

$$\det \begin{bmatrix} d & e & 3f \\ a & b & 3c \\ g & h & 3i \end{bmatrix} \underset{c_3 \to c_3/3}{=} 3\det \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{=} -3\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (-3)(12) = -36. \quad \square$$

(b)

$$\det \begin{bmatrix} g & i & h \\ d & f & e \\ a & c & b \end{bmatrix} \underset{c_2 \leftrightarrow c_3}{=} -\det \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix} \underset{r_1 \leftrightarrow r_3}{=} \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 12. \quad \square$$

(c)

$$\det \begin{bmatrix} a & 2b & c \\ g+5a & 2h+10b & i+5c \\ d & 2e & f \end{bmatrix} \underset{c_2 \to c_2/2}{=} 2\det \begin{bmatrix} a & b & c \\ g+5a & h+5b & i+5c \\ d & e & f \end{bmatrix} \underset{r_2 \to r_2 - 5r_1}{=}$$

$$2\det \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} \underset{r_2 \leftrightarrow r_3}{=} -2\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (-2)(12) = -24. \quad \square$$

---

39. Let R be a commutative ring with identity. Define a function $D : M(2, R) \to R$ by

$$D \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad.$$

Prove that D is linear in each row. Show by specific counterexample that D is not alternating.

To show that D is linear in the first row, start with the matrices

$$\begin{bmatrix} ka & kb \\ c & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a' & b' \\ c & d \end{bmatrix}.$$

The matrix obtained by adding the first rows, keeping the second row fixed, is

$$\begin{bmatrix} ka + a' & kb + b' \\ c & d \end{bmatrix}.$$

(This is not the sum of the two matrices above.) Then

$$D\left(\begin{bmatrix} ka + a' & kb + b' \\ c & d \end{bmatrix}\right) = (ka + a') \cdot d = kad + a'd.$$

On the other hand,

$$k \cdot D \begin{bmatrix} a & b \\ c & d \end{bmatrix} + D \begin{bmatrix} a' & b' \\ c & d \end{bmatrix} = k(ad) + a'd = kad + a'd.$$

For the second row, start with the matrices

$$\begin{bmatrix} a & b \\ kc & kd \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c' & d' \end{bmatrix}.$$

The matrix obtained by adding the second rows, keeping the first row fixed, is

$$\begin{bmatrix} a & b \\ kc + c' & kd + d' \end{bmatrix}.$$

Then

$$D\left(\begin{bmatrix} a & b \\ kc + c' & kd + d' \end{bmatrix}\right) = a \cdot (kd + d') = kad + ad'.$$

On the other hand,

$$k \cdot D \begin{bmatrix} a & b \\ c & d \end{bmatrix} + D \begin{bmatrix} a & b \\ c' & d' \end{bmatrix} = k(ad) + ad' = kad + ad'.$$

D is not alternating:

$$D \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 1 \cdot 1 = 1 \neq 0. \quad \square$$

---

40. Prove that

$$\begin{vmatrix} a & a + b & c \\ d + 2a & d + 2a + e + 2b & f + 2c \\ g & g + h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

$$\begin{vmatrix} a & a + b & c \\ d + 2a & d + 2a + e + 2b & f + 2c \\ g & g + h & i \end{vmatrix} \underset{c_2 \to c_2 - c_1}{=} \begin{vmatrix} a & b & c \\ d + 2a & e + 2b & f + 2c \\ g & h & i \end{vmatrix} \underset{r_2 \to r_2 - 2r_1}{=} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}. \quad \square$$

---

41. Let x be a nonzero real number. Compute

$$\det \begin{bmatrix} 1 & x & x^2 \\ x & x^2 & x^3 \\ x^2 & x^3 & x^4 \end{bmatrix}.$$

Factoring out an x from the second row, I have

$$\det \begin{bmatrix} 1 & x & x^2 \\ x & x^2 & x^3 \\ x^2 & x^3 & x^4 \end{bmatrix} = x \cdot \det \begin{bmatrix} 1 & x & x^2 \\ 1 & x & x^2 \\ x^2 & x^3 & x^4 \end{bmatrix} = 0.$$

The determinant of the second matrix is 0 because it has two equal rows. □

---

42. Let M be the matrix obtained from the 5×5 identity matrix I by swapping rows 1 and 5, then swapping columns 2 and 4, then swapping rows 2 and 3, and finally swapping columns 1 and 3. What is the determinant of M?

$\det I = 1$, and each row or column swap multiplies the determinant by -1. Since there were 4 swaps, the determinant is multiplied by $(-1)^4 = 1$. Hence, $\det M = 1$. □

---

43. Compute the determinant of the following real matrix by row reducing it to the identity.

$$\begin{bmatrix} 2 & -4 & 2 \\ 1 & -1 & 0 \\ 0 & 5 & 15 \end{bmatrix}.$$

$$\det \begin{bmatrix} 2 & -4 & 2 \\ 1 & -1 & 0 \\ 0 & 5 & 15 \end{bmatrix} \underset{r_2 \to r_2/2}{=} 2 \cdot \det \begin{bmatrix} 1 & -2 & 1 \\ 1 & -1 & 0 \\ 0 & 5 & 15 \end{bmatrix} \underset{r_2 \to r_2 - r_1}{=} 2 \cdot \det \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 5 & 15 \end{bmatrix} \underset{r_1 \to r_1 + 2r_2}{=}$$

$$2 \cdot \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 5 & 15 \end{bmatrix} \underset{r_3 \to r_3 - 5r_2}{=} 2 \cdot \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 20 \end{bmatrix} \underset{r_3 \to r_3/20}{=} (2)(20) \cdot \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \underset{r_1 \to r_1 + r_3}{=}$$

$$(2)(20) \cdot \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \underset{r_2 \to r_2 + r_3}{=} (2)(20) \cdot \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 40.$$

Notice that in the first step, the row operation "divide row 1 by 2" means, in terms of determinants, that you *factor* 2 out of row 1 — and similarly for the fifth row operation, where 20 is factored out of row 3. (You might find it easiest to think of *factoring numbers out of rows.* □

---

44. Compute the determinant of the following matrix over $\mathbb{Z}_5$ by row reducing it to the identity.

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 4 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix} \underset{r_1 \to r_1/2}{=} 2 \cdot \begin{vmatrix} 1 & 3 & 0 \\ 4 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix} \underset{r_2 \to r_1 + r_2}{=} 2 \cdot \begin{vmatrix} 1 & 3 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 1 \end{vmatrix} \underset{r_2 \to r_2/4}{=} (2)(4) \cdot \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \end{vmatrix} \underset{r_1 \to r_1 + 2r_2}{=}$$

$$(2)(4)\cdot\begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 2 & 1 \end{vmatrix} \underset{r_3\to r_3+3r_2}{=} (2)(4)\cdot\begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{vmatrix} \underset{r_3\to r_3/3}{=} (2)(4)(3)\cdot\begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} \underset{r_1\to r_1+2r_3}{=}$$

$$(2)(4)(3)\cdot\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} \underset{r_2\to r_2+r_3}{=} (2)(4)(3)\cdot\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 24 = 4.$$

Notice that in the first step, the row operation "divide row 1 by 2" means, in terms of determinants, that you *factor* 2 out of row 1. The "3" in the second column of the second determinant results from the fact that $1 = 2\cdot 3$ in $\mathbb{Z}_5$: The 2 gets factored out of 1, leaving 3 behind.

Likewise, in the third row operation, dividing row 2 by 4 means that you factor 4 out of row 2. So the "4" that appears in row 2, column 3 comes from $1 = 4\cdot 4$: The 4 gets factored out of 1, leaving 4 behind. □

---

45. Compute $\begin{vmatrix} 1 & 3 & -4 \\ 1 & 2 & 5 \\ -2 & -2 & 3 \end{vmatrix}$ over $\mathbb{R}$.

$$\begin{vmatrix} 1 & 3 & -4 \\ 1 & 2 & 5 \\ -2 & -2 & 3 \end{vmatrix} \underset{r_2\to r_2-r_1}{=} \begin{vmatrix} 1 & 3 & -4 \\ 0 & -1 & 9 \\ -2 & -2 & 3 \end{vmatrix} \underset{r_3\to r_3+3r_1}{=} \begin{vmatrix} 1 & 3 & -4 \\ 0 & -1 & 9 \\ 0 & 4 & -5 \end{vmatrix} =$$

$$1\cdot\begin{vmatrix} -1 & 9 \\ 4 & -5 \end{vmatrix} = 5 - 36 = -31. \quad \square$$

---

46. Suppose A is a 3×3 real matrix and

$$A^4 = \begin{bmatrix} 1 & 80 & 95 \\ 0 & 81 & 350 \\ 0 & 0 & 256 \end{bmatrix}.$$

Prove that A is invertible.

Taking the determinant of both sides, I get

$$\det A^4 = 1\cdot 81\cdot 256 = 20736.$$

So

$$(\det A)^4 = 20736, \quad \text{and} \quad \det A = 20736^{1/4} = 12 \neq 0.$$

Since $\det A \neq 0$, it follows that A is invertible. □

---

47. Suppose that A, B, and C are 4×4 matrices over $\mathbb{R}$, and

$$|A| = 6, \quad |B| = -3, \quad |C| = 5.$$

(a) Compute $|(ABC)^T|$.

(b) Compute $|3C|$.

(c) Compute $|A^{-1}B^2|$.

(a)

$$|(ABC)^T| = |ABC| = |A||B||C| = (6)(-3)(5) = -90. \quad \square$$

(b) In computing $3C$, you multiply *each* row by 3. Since C is a 4×4 matrix,

$$|3C| = 3^4|C| = 81 \cdot 5 = 405. \quad \square$$

(c)

$$|A^{-1}B^2| = |A|^{-1}|B|^2 = \left(\frac{1}{6}\right)(-3)^2 = \frac{3}{2}. \quad \square$$

---

48. Let $a, b, c \in \mathbb{R}$. Compute the 3×3 Vandermonde determinant

$$\left|\begin{matrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{matrix}\right|.$$

Subtracting row 1 from rows 2 and 3 does not change the determinant, so I do this, then expand by cofactors of the first column:

$$\left|\begin{matrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{matrix}\right| = \left|\begin{matrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{matrix}\right| = \left|\begin{matrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{matrix}\right| = (b-a)(c^2-a^2) - (c-a)(b^2-a^2) =$$

$$(b-a)(c-a)(c+a) - (c-a)(b-a)(b+a) = (b-a)(c-a)[(c+a)-(b+a)] = (b-a)(c-a)(c-b). \quad \square$$

---

49. Show that if A is an invertible $n \times n$ matrix, then

$$|\operatorname{adj} A| = |A|^{n-1}.$$

The adjoint formula says

$$A \cdot \operatorname{adj} A = |A| \cdot I = \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |A| \end{bmatrix}.$$

Taking the determinant of both sides, I have

$$|A||\operatorname{adj} A| = \left|\begin{matrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |A| \end{matrix}\right| = |A|^n|I| = |A|^n.$$

(I got $|A|^n$ by factoring $|A|$ out of each of the n rows.) Dividing both sides by $|A|$ (which is nonzero, since A is invertible), I have

$$|\operatorname{adj} A| = |A|^{n-1}.$$

As an example, suppose A is a 3×3 matrix and $|\operatorname{adj} A| = 64$. Then $|A|^2 = 64$, so $|A| = 8$. $\square$

---

50. Use the adjoint formula to find the inverse of the following real matrix:

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 3 & 2 \\ -1 & -2 & -1 \end{bmatrix}.$$

The determinant is

$$\begin{vmatrix} 1 & 3 & 2 \\ 0 & 3 & 2 \\ -1 & -2 & -1 \end{vmatrix} = 1.$$

Compute the matrix of cofactors:

$$\begin{matrix} (+)\begin{vmatrix} 3 & 2 \\ -2 & -1 \end{vmatrix} = 1 & (-)\begin{vmatrix} 0 & 2 \\ -1 & -1 \end{vmatrix} = -2 & (+)\begin{vmatrix} 0 & 3 \\ -1 & -2 \end{vmatrix} = 3 \\ (-)\begin{vmatrix} 3 & 2 \\ -2 & -1 \end{vmatrix} = -1 & (+)\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} = 1 & (-)\begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} = -1 \\ (+)\begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} = 0 & (-)\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = -2 & (+)\begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} = 3 \end{matrix}$$

The inverse is:

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 1 & -2 \\ 3 & -1 & 3 \end{bmatrix}. \quad \square$$

---

51. Use the adjoint formula to find the inverse of the real matrix

$$A = \begin{bmatrix} 1 & a & -a \\ a & 1 & -1 \\ a & 1 & a \end{bmatrix}.$$

(Assume that $a \neq \pm 1$.)

First, I'll find the determinant:

$$|A| = \begin{vmatrix} 1 & a & -a \\ a & 1 & -1 \\ a & 1 & a \end{vmatrix} = \underset{r_2 \to r_2 - ar_1}{\rightarrow} \begin{vmatrix} 1 & a & -a \\ 0 & 1-a^2 & -1+a^2 \\ a & 1 & a \end{vmatrix} = \underset{r_3 \to r_3 - ar_1}{\rightarrow} \begin{vmatrix} 1 & a & -a \\ 0 & 1-a^2 & -1+a^2 \\ 0 & 1-a^2 & a+a^2 \end{vmatrix} =$$

$$(1)\begin{vmatrix} 1-a^2 & a^2-1 \\ 1-a^2 & a+a^2 \end{vmatrix} = (1-a^2)(a+a^2) - (a^2-1)(1-a^2) = 1 + a - a^2 - a^3.$$

Next, compute the transpose of the matrix of cofactors. For example, consider the element -1 which is in the $(2,3)^{\text{th}}$ position of the matrix. The cofactor is

$$-\begin{vmatrix} 1 & a & * \\ * & * & * \\ a & 1 & * \end{vmatrix} = a^2 - 1.$$

Therefore, I write $a^2 - 1$ in the $(3,2)^{\text{th}}$ position of the transposed cofactor matrix.
Continuing in this way, I find that the inverse is

$$\begin{bmatrix} 1 & a & -a \\ a & 1 & -1 \\ a & 1 & a \end{bmatrix}^{-1} = \frac{1}{1 + a - a^2 - a^3} \begin{bmatrix} a+1 & -a^2-a & 0 \\ -a^2-a & a^2+a & 1-a^2 \\ 0 & a^2-1 & 1-a^2 \end{bmatrix}. \quad \square$$

---

52. Use the adjoint formula to find the inverse of the following matrix over $\mathbb{Z}_3$:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}.$$

First, compute the determinant:

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} \underset{r_1 \to r_1 + r_2}{=} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = (1) \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 = 2.$$

Then $|A|^{-1} = 2^{-1} = 2$.
Next, compute the adjoint:

$$\operatorname{adj} A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Hence,

$$A^{-1} = |A|^{-1} \cdot \operatorname{adj} A = 2 \cdot \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}. \quad \square$$

---

53. Use Cramer's Rule to solve the following system over $\mathbb{R}$:

$$\begin{aligned} 3x - 7y &= -5 \\ x + 10y &= 4 \end{aligned}$$

$$x = \frac{\begin{vmatrix} -5 & -7 \\ 4 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & -7 \\ 1 & 10 \end{vmatrix}} = -\frac{22}{37}.$$

$$x = \frac{\begin{vmatrix} 3 & -5 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -7 \\ 1 & 10 \end{vmatrix}} = \frac{17}{37}.$$

---

54. Use Cramer's rule to solve the following system over $\mathbb{Z}_3$:

$$\begin{array}{ccccccc} 2x & + & y & + & z & = & 0 \\ x & + & y & + & z & = & 1 \\ x & + & 2y & + & z & = & 1 \end{array}$$

$$x = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}^{-1} = 1 \cdot 2^{-1} = 1 \cdot 2 = 2,$$

$$y = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}^{-1} = 0 \cdot 2^{-1} = 0 \cdot 2 = 0,$$

$$z = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}^{-1} = 1 \cdot 2^{-1} = 1 \cdot 2 = 2. \quad \square$$

When things go wrong, don't go with them. - Anonymous