Math 322
9-29-2020

Review Problems for Test 2

These problems are intended to help you study. The presence of a problem on this sheet does not mean that a similar problem will occur on the test. And the absence of a problem from this sheet does not mean that a similar problem is not on the test.

1. Determine whether the set is a subspace. Check each axiom for a subspace. If the axiom holds, prove it. If the axiom doesn't hold, give a specific counterexample.

(a) The subset of the real vector space $\mathbb{R}^3$ given by

$$A = \left\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\right\}.$$

(b) The subset of the real vector space $\mathbb{R}^3$ given by

$$B = \left\{(x, y, z) \in \mathbb{R}^3 \mid xy = z\right\}.$$

(c) The subset of the real vector space $\mathbb{R}^3$ given by

$$C = \{(x, y, z) \mid x + 2y = 3z\}.$$

(d) For a fixed $n \times n$ real matrix A, the subset of the real vector space $M(n, \mathbb{R})$ given by

$$D = \{X \in M(n, \mathbb{R}) \mid AXA + 5X = 0\}.$$

2. Let $\mathbb{R}[x]$ denote the vector space of polynomials with real coefficients, regarded as a vector space over $\mathbb{R}$.

(a) Prove or disprove:

$$V = \{f(x) \in \mathbb{R}[x] \mid f(2) = 0\} \quad \text{is a subspace.}$$

(b) Prove or disprove:

$$W = \{f(x) \in \mathbb{R}[x] \mid f(2) = 3\} \quad \text{is a subspace.}$$

3. Let U and V be subspaces of a vector space W. Show that the intersection $U \cap V$ is a subspace of W.

4. (a) Complete the definition of **independent set**: A set S of vectors in a vector space V over a field F is **independent** if ....

(b) Complete the definition of **spanning set**: A set S of vectors in a vector space V over a field F **spans** V if ....

5. Bonzo McTavish says: "A set of vectors in $\mathbb{R}^n$ is independent if when you make a matrix with the vectors, the determinant of the matrix is nonzero." What is wrong with this?

6. Silas Hogwinder says: "A set of vectors in $\mathbb{R}^n$ is independent if when you make a matrix with the vectors, it row reduces to the identity." What is wrong with this?

7. (a) Explain why the following set of vectors in $\mathbb{R}^3$ is not independent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(b) Explain why the following set of vectors in $\mathbb{R}^3$ is not independent.

$$\left\{\begin{bmatrix}1\\2\\10\end{bmatrix}, \begin{bmatrix}-4\\7\\5\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$$

(c) Explain why the following set of vectors in $\mathbb{R}^3$ does not span $\mathbb{R}^3$.

$$\left\{\begin{bmatrix}6\\0\\-6\end{bmatrix}, \begin{bmatrix}3\\1\\13\end{bmatrix}\right\}.$$

8. Determine whether the following vectors are independent in $\mathbb{R}^4$. If they are dependent, find a nontrivial linear combination of the vectors which equals $\vec{0}$.

$$(1,-1,4,2), \quad (-2,1,-1,-1), \quad (-4,1,5,1)$$

9. Let V be a vector space, and suppose $\{u,v,w\}$ is a dependent set of vectors in V. Let

$$p = a_1u + b_1v + c_1w, \quad q = a_2u + b_2v + c_2w, \quad r = a_3u + b_3v + c_3w.$$

(The a's, b's, and c's are scalars.) Prove that $\{p,q,r\}$ is dependent.

10. Write the vector $(1,0,2)$ in $\mathbb{Z}_3^3$ as a linear combination of the vectors

$$(2,2,1), \quad (2,1,2), \quad (1,1,1).$$

11. Determine whether the following vectors span $\mathbb{R}^3$. If they don't, find the dimension of the subspace they span.

$$(1,-4,7), \quad (-2,2,-3), \quad (-4,-2,5)$$

12. Consider the following subspace of the real vector space $M(2,\mathbb{R})$:

$$W = \left\{\begin{bmatrix}a & a+c\\ b+c & b\end{bmatrix}\right\}.$$

Find a basis for W. Prove that your set is a basis by showing that it spans W and is independent.

13. Let

$$V = \left\{\begin{bmatrix}a & b\\ a+b & a-b\end{bmatrix} \,\middle|\, a,b \in \mathbb{R}\right\}.$$

(a) Show that V is a subspace of $M(2,\mathbb{R})$.

(b) If the following set of matrices in V spans V, prove it. If it doesn't span V, find a specific element of V which is not in the span of the set

$$S = \left\{\begin{bmatrix}1 & 1\\ 2 & 0\end{bmatrix}, \begin{bmatrix}1 & -1\\ 0 & 2\end{bmatrix}\right\}.$$

14. Find bases for the row space, the column space, and the null space for the following real matrix:

$$\begin{bmatrix}1 & -3 & -1 & -3 & 2\\ 2 & -6 & 1 & -3 & 1\\ 0 & 0 & 1 & 1 & 1\end{bmatrix}.$$

15. Find bases for the row space, the column space, and the null space for the following matrix over $\mathbb{Z}_5$:

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & 1 & 3 \\ 4 & 1 & 2 & 0 \end{bmatrix}$$

16. Find bases for the row space, the column space, and the null space for the following real matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 5 \\ 2 & 4 & 2 & 3 & 3 \\ 1 & 2 & 1 & 2 & 1 \end{bmatrix}.$$

17. Find bases for the row space, the column space, and the null space for the following real matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

18. Show that the following set of matrices is an independent subset of $M(3, \mathbb{R})$.

$$\left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

19. Show that the set $\{x, x^2, e^x\}$ is an independent set in the real vector space of differentiable functions on $\mathbb{R}$.

20. Find a standard basis vector $u \in \mathbb{R}^5$ such that the following set forms a basis for $\mathbb{R}^5$:

$$\{(1, 2, 1, 1, 0), (2, -1, -1, 0, 3), (1, 4, -1, -1, 2), (3, 2, 2, 1, -6), u\}.$$

21. Find standard basis vectors $u, v \in \mathbb{Z}_3^4$ such that the following set forms a basis for $\mathbb{Z}_3^4$.

$$\{(1, 2, 2, 1), (2, 1, 1, 1), u, v\}$$

22. (a) Find a basis for the subspace of $\mathbb{Z}_2^4$ spanned by the set

$$\{(1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 0)\}.$$

(b) Find a subset of the following set S which forms a basis for the subspace of $\mathbb{Z}_2^4$ spanned by S.

$$S = \{(1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 0)\}.$$

23. Consider the following real matrices:

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 2 & 1 & -1 & 3 \\ 4 & 12 & 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 2 \end{bmatrix}.$$

Determine whether the row space of A is contained in the row space of B. If it is, prove it. If it isn't, find a vector in the row space of A which is not in the row space of B.

24. Consider the following real matrix, where $x \in \mathbb{R}$:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ x & 2 & 10 \end{bmatrix}.$$

(a) Find a value of x for which $\operatorname{rank} A = 1$.

(b) Find a value of x for which $\operatorname{rank} A = 2$.

25. In a 5×19 real matrix A, the 5 rows are independent. What is nullity A?

26. (a) Let $A, B \in M(n, F)$, where F is a field. Prove that

$$\text{null space}(B) \subset \text{null space}(AB).$$

(b) Let $A, B \in M(n, F)$, where F is a field. Suppose that A is invertible. Prove that

$$\text{null space}(B) = \text{null space}(AB).$$

(In (a) and (b), begin by writing down what it means for a vector x to be in the null space of B or the null space of AB.)

(c) Find a matrix $A \in M(2, \mathbb{R})$ such that

$$\text{null space}(A) \subsetneq \text{null space}(A^2).$$

(In words, you want to find a matrix A so that A^2 multiplies more vectors to $\vec{0}$ than A does.)

(d) Find a matrix $A \in M(2, \mathbb{R})$ such that

$$\text{row space}(A^2) \subsetneq \text{row space}(A).$$

(How can you choose A so that you can "build less stuff" with the rows of A^2 than with the rows of A?)

27. For the given function, check each axiom for a linear transformation. If the axiom holds, prove it. If the axiom does not hold, give a specific counterexample.

(a) The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x + \sin x.$$

(b) The function $g : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$g(x, y) = (x - y, x + y, 2x + 3y).$$

(c) The function $h : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$h(x, y) = (x + y, xy).$$

(d) The function $p : \mathbb{R}^2 \to M(2, \mathbb{R})$ given by

$$p(x, y) = \begin{bmatrix} 2x & 3y \\ 0 & y^2 \end{bmatrix}.$$

28. Let A be a *fixed* matrix in $M(2, \mathbb{R})$. Define $T : M(2, \mathbb{R}) \to M(2, \mathbb{R})$ by

$$T(X) = AX - XA \quad \text{for} \quad X \in M(2, \mathbb{R}).$$

Prove that T is a linear transformation.

29. Define $\operatorname{tr} : M(2, \mathbb{R}) \to \mathbb{R}$ by

$$\operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

tr is called the **trace function**.

(a) Show that tr is a linear transformation.

(b) Find a basis for the null space of tr.

30. Define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(u, v) = (u, (u^2 + 1)\cos v, (u^2 + 1)\sin v).$$

(a) Compute the derivative $Df(u, v)$.

(b) Show that there no values (u, v) for which $Df(u, v)$ is the zero matrix.

31. Find a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that takes the parallelogram determined by the vectors $(2, 3)$ and $(2, 4)$ to the parallelogram determined by the vectors $(-2, 1)$ and $(-1, -3)$.

32. Construct a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects points in $\mathbb{R}^2$ across the line $y = 9x$.

33. Find an affine transformation which takes the unit square $\{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ to the parallelogram with vertices $A(2, 5)$, $B(3, 7)$, $C(2, 6)$, $D(1, 4)$ in such a way that the origin $(0, 0)$ goes to A, the vector $(1, 0)$ goes to $\overrightarrow{AB}$, and the vector $(0, 1)$ goes to $\overrightarrow{AD}$.

34. Consider the following bases for the real vector space $\mathbb{R}^2$:

$$\mathcal{B} = \left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\} \quad \text{and} \quad \mathcal{C} = \left\{\begin{bmatrix} 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}.$$

(a) Construct the translation matrices

$$[\mathcal{B} \to \text{std}], \quad [\text{std} \to \mathcal{B}], \quad [\mathcal{C} \to \text{std}], \quad [\text{std} \to \mathcal{C}], \quad [\mathcal{B} \to \mathcal{C}], \quad \text{and} \quad [\mathcal{C} \to \mathcal{B}].$$

(b) Write the vector $(-5, 4)$ in terms of $\mathcal{B}$.

(c) Write the vector $(3, 2)_{\mathcal{C}}$ in terms of the standard basis.

(d) Write the vector $(-1, 4)_{\mathcal{B}}$ in terms of $\mathcal{C}$.

(e) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = (2x - y, 3x + 8y).$$

Find $[f]_{\mathcal{B},\mathcal{C}}$.

35. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y + 3z \\ x + y - z \end{bmatrix}.$$

Consider the following bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\} \quad \text{and} \quad \mathcal{C} = \left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

(a) Find $[\mathcal{B} \to \text{std}]$, $[\mathcal{C} \to \text{std}]$, and $[\text{std} \to \mathcal{C}]$.

(b) Express $(3,4)$ relative to the basis $\mathcal{C}$.

(c) Express $(3,1,0)_{\mathcal{B}}$ relative to the standard basis.

(d) Find $[T]_{\mathcal{B},\mathcal{C}}$.

36. What is i^{101}?

37. Simplify $\dfrac{1+2i}{5-3i}$.

38. Compute $(1+i\sqrt{3})^{10}$.

- Hint: Express $\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}$ as $\cos\theta + i\sin\theta$ for a certain value of θ, then apply DeMoivre's formula.

39. Find the inverse of the following matrix. (Multiply any constants outside the matrix into the matrix and simplify.)

$$\begin{bmatrix} 2-3i & 4 \\ 1 & 2 \end{bmatrix}.$$

Solutions to the Review Problems for Test 2

1. Determine whether the set is a subspace. Check each axiom for a subspace. If the axiom holds, prove it. If the axiom doesn't hold, give a specific counterexample.

(a) The subset of the real vector space $\mathbb{R}^3$ given by

$$A = \left\{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=1\right\}.$$

(b) The subset of the real vector space $\mathbb{R}^3$ given by

$$B = \left\{(x,y,z) \in \mathbb{R}^3 \mid xy = z\right\}.$$

(c) The subset of the real vector space $\mathbb{R}^3$ given by

$$C = \{(x,y,z) \mid x+2y=3z\}.$$

(d) For a fixed $n \times n$ real matrix A, the subset of the real vector space $M(n,\mathbb{R})$ given by

$$D = \{X \in M(n,\mathbb{R}) \mid AXA + 5X = 0\}.$$

(a) The set is not a subspace, since the zero vector $(0,0,0)$ is not contained in the set. I'll check the axioms, anyway.

$(1,0,0) \in A$ since $1+0+0=1$. $(0,0,1) \in A$ since $0+0+1=1$. But

$$(1,0,0)+(0,0,1) = (1,0,1) \notin A, \quad \text{since} \quad 1+0+1=2 \neq 1.$$

$(1,0,0) \in A$ since $1+0+0=1$. But

$$2 \cdot (1,0,0) = (2,0,0) \notin A, \quad \text{since} \quad 2+0+0=2 \neq 1. \quad \Box$$

(b) $(2,3,6)$ is contained in B, since $2 \cdot 3 = 6$, and $(3,4,12)$ is contained in B, since $3 \cdot 4 = 12$. However,

$$(2,3,6) + (3,4,12) = (5,7,72) \notin B, \quad \text{since} \quad 5 \cdot 7 = 35 \neq 72.$$

$(2,3,6)$ is contained in B, since $2 \cdot 3 = 6$. But

$$2 \cdot (2,3,6) = (4,6,12) \notin B, \quad \text{since} \quad 4 \cdot 6 = 24 \neq 12. \quad \square$$

(c) Let $(a,b,c), (d,e,f) \in C$. By definition, this means that

$$a + 2b = 3z \quad \text{and} \quad d + 2e = 3f.$$

Consider the sum of the two vectors:

$$(a,b,c) + (d,e,f) = (a+d, b+e, c+f).$$

Then

$$(a+d) + 2(b+e) = (a+2b) + (d+2e) = 3c + 3f = 3(c+f).$$

The sum vector satisfies the defining condition, so the sum is contained in C.
Let $(x,y,z) \in C$ and let k be a number. By definition, $x + 2y = 3z$.
Consider the product

$$k \cdot (x,y,z) = (kx, ky, kz).$$

Then

$$kx + 2(ky) = k(x + 2y) = k(3z) = 3(kz).$$

The product vector satisfies the defining condition, so the product is contained in C.
Therefore, C is a subspace. $\square$

(d) Let $X, Y \in D$. Then

$$\begin{aligned} AXA + 5X &= 0 \\ AYA + 5Y &= 0 \\ \hline AXA + AYA &= 5X + 5Y \\ A(X+Y)A + 5(X+Y) &= 0 \end{aligned}$$

Therefore, $X + Y \in D$.
(Warning: Don't write things like "$A(X+Y)A + 5(X+Y) = 0 \in D$". Do you understand why this is wrong?)
Let $X \in D$. Then

$$\begin{aligned} AXA + 5X &= 0 \\ k(AXA + 5X) &= k \cdot 0 \\ kAXA + k(5X) &= 0 \\ A(kX)A + 5(kX) &= 0 \end{aligned}$$

Therefore, $kX \in D$.
Hence, D is a subspace. $\square$

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2. Let $\mathbb{R}[x]$ denote the vector space of polynomials with real coefficients, regarded as a vector space over $\mathbb{R}$.

(a) Prove or disprove:

$$V = \{f(x) \in \mathbb{R}[x] \mid f(2) = 0\} \quad \text{is a subspace.}$$

(b) Prove or disprove:

$$W = \{f(x) \in \mathbb{R}[x] \mid f(2) = 3\} \quad \text{is a subspace.}$$

(a) By the Root Theorem (from precalculus), to say that $f(2) = 0$ is the same as saying that $f(x)$ is divisible by $x - 2$; that is, $f(x) = (x - 2)p(x)$, for some $p(x) \in \mathbb{R}[x]$.

Let $f(x), g(x) \in V$. Then $f(x) = (x - 2)p(x)$ and $g(x) = (x - 2)q(x)$ for some $p(x), q(x) \in \mathbb{R}[x]$. Hence,

$$f(x) + g(x) = (x - 2)p(x) + (x - 2)q(x) = (x - 2)(p(x) + q(x)).$$

Since the last expression is divisible by $x - 2$, it follows that $f(x) + g(x) \in V$.

Let $f(x) \in V$, so $f(x) = (x - 2)p(x)$ for some $p(x) \in \mathbb{R}[x]$. Let $k \in \mathbb{R}$. Then

$$k \cdot f(x) = k \cdot (x - 2)p(x) = (x - 2)\left[k \cdot p(x)\right].$$

The last expression is divisible by $x - 2$, so $k \cdot f(x) \in V$.

Since V is closed under addition and scalar multiplication, V is a subspace. $\square$

(b) The zero polynomial 0 is not in W, since the zero polynomial does not map 2 to 3. But every subspace contains the zero vector, so W is not a subspace. $\square$

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3. Let U and V be subspaces of a vector space W. Show that the intersection $U \cap V$ is a subspace of W.

Let $x, y \in U \cap V$. I must show that $x + y \in U \cap V$.
Since $x, y \in U \cap V$, $x, y \in U$. Since U is a subspace, $x + y \in U$.
Since $x, y \in U \cap V$, $x, y \in V$. Since V is a subspace, $x + y \in V$.
Since $x + y \in U$ and $x + y \in V$, it follows that $x + y \in U \cap V$.
Next, let $x \in U \cap V$ and let k be a scalar. I must show that $kx \in U \cap V$.
Since $x \in U \cap V$, $x \in U$. Since U is a subspace, $kx \in U$.
Since $x \in U \cap V$, $x \in V$. Since V is a subspace, $kx \in V$.
Since $kx \in U$ and $kx \in V$, it follows that $kx \in U \cap V$.
Since $U \cap V$ is closed under addition and scalar multiplication, $U \cap V$ is a subspace. $\square$

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4. (a) Complete the definition of **independent set**: A set S of vectors in a vector space V over a field F is **independent** if ....

(b) Complete the definition of **spanning set**: A set S of vectors in a vector space V over a field F **spans** V if ....

(a) A set S of vectors in a vector space V over a field F is **independent** if whenever $a_1, \ldots, a_n \in F$ and $v_1, \ldots, v_n \in S$,

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = \vec{0} \quad \text{implies} \quad a_1 = a_2 = \cdots = a_n = 0.$$

If you said something about making a matrix with the vectors, you didn't get the definition right. How you *check* whether *certain kinds of vectors* are independent is *not* the *definition* of independence. $\square$

(b) A set S of vectors in a vector space V over a field F **spans** V if for every $v \in V$, there are numbers $a_1, \ldots, a_n \in F$ and vectors $v_1, \ldots, v_n \in S$ such that

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

If you said something about making a matrix with the vectors, you didn't get the definition right.) How you *check* whether *certain kinds of vectors* form a spanning set is *not* the *definition* of spanning. $\square$

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5. Bonzo McTavish says: "A set of vectors in $\mathbb{R}^n$ is independent if when you make a matrix with the vectors, the determinant of the matrix is nonzero." What is wrong with this?

Consider the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{in} \mathbb{R}^3.$$

The set is independent, but the matrix formed by the vectors (either as the rows or the columns) isn't square, so you can't take the determinant.

Don't confuse the *procedure* you use for checking independence in a special case (e.g. when you have n vectors in $real^n$) with what the concept of *independence* means. □

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6. Silas Hogwinder says: "A set of vectors in $\mathbb{R}^n$ is independent if when you make a matrix with the vectors, it row reduces to the identity." What is wrong with this?

Consider the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{in} \mathbb{R}^3.$$

The set is independent, but the matrix formed by the vectors (either as the rows or the columns) isn't square, so it doesn't row reduce to the identity.

Don't confuse the *procedure* you use for checking independence in a special case (e.g. when you have n vectors in $real^n$) with what the concept of *independence* means. □

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7. (a) Explain why the following set of vectors in $\mathbb{R}^3$ is not independent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(b) Explain why the following set of vectors in $\mathbb{R}^3$ is not independent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}, \begin{bmatrix} -4 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(c) Explain why the following set of vectors in $\mathbb{R}^3$ does not span $\mathbb{R}^3$.

$$\left\{ \begin{bmatrix} 6 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 13 \end{bmatrix} \right\}.$$

(a) A set containing the zero vector is dependent. (Can you prove it?) □

(b) $\mathbb{R}^3$ has dimension 3 (since, for instance, the standard basis for $\mathbb{R}^3$ has 3 elements). Therefore, any set of vectors in $\mathbb{R}^3$ with more than 3 vectors must be dependent. □

(c) $\mathbb{R}^3$ has dimension 3 (since, for instance, the standard basis for $\mathbb{R}^3$ has 3 elements). Therefore, any set of vectors in $\mathbb{R}^3$ with fewer than 3 vectors cannot span $\mathbb{R}^3$. □

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8. Determine whether the following vectors are independent in $\mathbb{R}^4$. If they are dependent, find a nontrivial linear combination of the vectors which equals $\vec{0}$.

$$(1, -1, 4, 2), \quad (-2, 1, -1, -1), \quad (-4, 1, 5, 1)$$

Let

$$a \cdot (1,-1,4,2) + b \cdot (-2,1,-1,-1) + c \cdot (-4,1,5,1) = (0,0,0,0).$$

The set is independent if this implies that $a = b = c = 0$.
The equation above is equivalent to the matrix equation

$$\begin{bmatrix} 1 & -2 & -4 \\ -1 & 1 & 1 \\ 4 & -1 & 5 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -2 & -4 & 0 \\ -1 & 1 & 1 & 0 \\ 4 & -1 & 5 & 0 \\ 2 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding equations are $a + 2c = 0$ and $b + 3c = 0$. Setting $c = 1$ gives $a = -2$ and $b = -3$. Therefore,

$$(-2) \cdot (1,-1,4,2) + (-3) \cdot (-2,1,-1,-1) + 1 \cdot (-4,1,5,1) = (0,0,0,0).$$

Hence, the vectors are dependent. □

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9. Let V be a vector space, and suppose $\{u, v, w\}$ is a dependent set of vectors in V. Let

$$p = a_1 u + b_1 v + c_1 w, \quad q = a_2 u + b_2 v + c_2 w, \quad r = a_3 u + b_3 v + c_3 w.$$

(The a's, b's, and c's are scalars.) Prove that $\{p, q, r\}$ is dependent.

Let $W = \langle u, v, w \rangle$ be the span of $\{u, v, w\}$ in V. Since $\{u, v, w\}$ is dependent, I can express one of the vectors in terms of the others. Without loss of generality, suppose w can be expressed as a linear combination of u and v. Then

$$\langle u, v, w \rangle = \langle u, v \rangle.$$

If $\{u, v\}$ is independent, then it's a basis for W. If it's dependent, then I can express one of u, v in terms of the other. Again without loss of generality, suppose v is a multiple of u. Then

$$\langle u, v, w \rangle = \langle u, v \rangle = \langle u \rangle.$$

If u is nonzero, then $\{u\}$ is independent, and it's a basis for W. The only other possibility is that $u = 0$. In that case, $v = w = 0$, so $W = \{0\}$.

Considering all of these cases, I've shown that the dimension of W is 2, 1, or 0. Since $\{p, q, r\}$ is a set of 3 vectors in W, it follows that $\{p, q, r\}$ must be dependent. □

---

10. Write the vector $(1, 0, 2)$ in $\mathbb{Z}_3^3$ as a linear combination of the vectors

$$(2,2,1), \quad (2,1,2), \quad (1,1,1).$$

I want to find a, b, c such that

$$a \cdot (2,2,1) + b \cdot (2,1,2) + c \cdot (1,1,1) = (1,0,2).$$

This gives the matrix equation

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Form the augmented matrix and row reduce:

$$\begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The last matrix says $a = 2$, $b = 1$, and $c = 1$. Thus,

$$2 \cdot (2, 2, 1) + 1 \cdot (2, 1, 2) + 1 \cdot (1, 1, 1) = (1, 0, 2). \quad \square$$

---

11. Determine whether the following vectors span $\mathbb{R}^3$. If they don't, find the dimension of the subspace they span.

$$(1, -4, 7), \quad (-2, 2, -3), \quad (-4, -2, 5)$$

Construct a matrix with the vectors as rows and row reduce:

$$\begin{bmatrix} 1 & -4 & 7 \\ -2 & 2 & -3 \\ -4 & -2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\dfrac{1}{3} \\ 0 & 1 & -\dfrac{11}{6} \\ 0 & 0 & 0 \end{bmatrix}$$

The row reduced echelon matrix has the same row space as the original matrix and the nonzero rows of the row reduced echelon matrix form a basis for the row space. Since there are two nonzero rows, the given vectors do not span $\mathbb{R}^3$; they span a subspace of dimension 2. $\square$

---

12. Consider the following subspace of the real vector space $M(2, \mathbb{R})$:

$$W = \left\{ \begin{bmatrix} a & a + c \\ b + c & b \end{bmatrix} \right\}.$$

Find a basis for W. Prove that your set is a basis by showing that it spans W and is independent.

Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

$\mathcal{B}$ spans W:

$$\begin{bmatrix} a & a + c \\ b + c & b \end{bmatrix} = \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b & b \end{bmatrix} + \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} =$$

$$a \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

To show $\mathcal{B}$ is independent, let

$$a \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} a & a+c \\ b+c & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Equating $(1,1)^{\text{th}}$ entries gives $a = 0$. Equating $(2,2)^{\text{th}}$ entries gives $b = 0$. Equating $(2,1)^{\text{th}}$ entries gives $b + c = 0$; since $b = 0$, I get $c = 0$. This proves that the set is independent.

Hence, $\mathcal{B}$ is a basis for W. □

---

13. Let

$$V = \left\{ \begin{bmatrix} a & b \\ a+b & a-b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

(a) Show that V is a subspace of $M(2, \mathbb{R})$.

(b) If the following set of matrices in V spans V, prove it. If it doesn't span V, find a specific element of V which is not in the span of the set

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \right\}.$$

(a) Let

$$\begin{bmatrix} a & b \\ a+b & a-b \end{bmatrix}, \begin{bmatrix} c & d \\ c+d & c-d \end{bmatrix} \in V.$$

Then

$$\begin{bmatrix} a & b \\ a+b & a-b \end{bmatrix} + \begin{bmatrix} c & d \\ c+d & c-d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ (a+c)+(b+d) & (a+c)-(b+d) \end{bmatrix} \in V.$$

Also, if $k \in \mathbb{R}$, then

$$k \cdot \begin{bmatrix} a & b \\ a+b & a-b \end{bmatrix} = \begin{bmatrix} ka & kb \\ ka+kb & ka-kb \end{bmatrix} \in V.$$

Therefore, V is a subspace of $M(2, \mathbb{R})$. □

(b) Take an arbitrary matrix in V and try to write it as a linear combination of the elements of S:

$$\begin{bmatrix} a & b \\ a+b & a-b \end{bmatrix} = x \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} + y \cdot \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ a+b & a-b \end{bmatrix} = \begin{bmatrix} x+y & x-y \\ 2x & 2y \end{bmatrix}.$$

Equating corresponding entries, I obtain four equations:

$$x + y = a, \quad x - y = b, \quad 2x = a + b, \quad 2y = a - b.$$

The last two equations give

$$x = \frac{1}{2}(a+b), \quad y = \frac{1}{2}(a-b).$$

If I plug these into the first two equations, they are satisfied. This means that I can find a linear combination of the elements of S which is equal to any element of V. Specifically,

$$\begin{bmatrix} a & b \\ a+b & a-b \end{bmatrix} = \left(\frac{1}{2}(a+b)\right) \cdot \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} + \left(\frac{1}{2}(a-b)\right) \cdot \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

Therefore, S spans V. □

14. Find bases for the row space, the column space, and the null space for the following real matrix:

$$\begin{bmatrix} 1 & -3 & -1 & -3 & 2 \\ 2 & -6 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Row reduce:

$$\begin{bmatrix} 1 & -3 & -1 & -3 & 2 \\ 2 & -6 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A basis for the row space is given by the nonzero rows of the row reduced echelon matrix:

$$\{[1 \quad -3 \quad 0 \quad -2 \quad 0], [0 \quad 0 \quad 1 \quad 1 \quad 0], [0 \quad 0 \quad 0 \quad 0 \quad 1]\}.$$

The leading coefficients occur in the first, third, and fifth columns. Therefore, the first, third, and fifth columns *of the original matrix* form a basis for the column space of the original matrix:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

To find a basis for the null space, use a, b, c, d, and e as variables, and regard the row-reduced echelon matrix as the coefficient matrix for a homogeneous system. The corresponding equations are

$$a - 3b - 2d = 0, \quad c + d = 0, \quad e = 0.$$

Solve for the leading coefficient variables:

$$a = 3b + 2d, \quad c = -d, \quad e = 0.$$

The parametric solution is

$$a = 3s + 2t, \quad b = s, \quad c = -t, \quad d = t, \quad e = 0.$$

Hence,

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = s \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore, a basis for the null space is

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}. \quad \square$$

---

15. Find bases for the row space, the column space, and the null space for the following matrix over $\mathbb{Z}_5$:

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & 1 & 3 \\ 4 & 1 & 2 & 0 \end{bmatrix}$$

Row reduce the matrix:

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & 1 & 3 \\ 4 & 1 & 2 & 0 \end{bmatrix} \underset{r_2 \to r_2 + 2r_1}{\to} \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 0 & 3 & 2 \\ 4 & 1 & 2 & 0 \end{bmatrix} \underset{r_3 \to r_3 + r_1}{\to} \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 3 & 2 \end{bmatrix} \underset{r_2 \to r_2 - r_3}{\to}$$

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{r_2 \to 2r_2}{\to} \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{r_1 \to r_1 - r_2}{\to} \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the row space is given by the nonzero rows of the row reduced echelon matrix:

$$\{[1 \quad 4 \quad 0 \quad 3], [0 \quad 0 \quad 1 \quad 4]\}.$$

The leading coefficients occur in the first and third columns. Therefore, the first and third columns *of the original matrix* form a basis for the column space of the original matrix:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

To find a basis for the null space, use a, b, c, and d as variables, and regard the row-reduced echelon matrix as the coefficient matrix for a homogeneous system. The corresponding equations are

$$a + 4b + 3d = 0, \quad c + 4d = 0.$$

Solve for the leading coefficient variables:

$$a = b + 2d, \quad c = d.$$

The parametric solution is

$$a = s + 2t, \quad b = s, \quad c = t, \quad d = t.$$

Therefore,

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Hence, a basis for the null space is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}. \quad \square$$

---

16. Find bases for the row space, the column space, and the null space for the following real matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 5 \\ 2 & 4 & 2 & 3 & 3 \\ 1 & 2 & 1 & 2 & 1 \end{bmatrix}.$$

Row reduce the matrix:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 5 \\ 2 & 4 & 2 & 3 & 3 \\ 1 & 2 & 1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the row space is given by the nonzero rows of the row reduced echelon matrix:

$$\{[1 \quad 2 \quad 0 \quad 0 \quad -1], [0 \quad 0 \quad 1 \quad 0 \quad 4], [0 \quad 0 \quad 0 \quad 1 \quad -1]\}.$$

The leading coefficients occur in the first, third, and fourth columns. Hence, the first, third, and fourth columns *of the original matrix* form a basis for the column space:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

To find a basis for the null space, use a, b, c, d, and e as variables, and regard the row-reduced echelon matrix as the coefficient matrix for a homogeneous system. The corresponding equations are

$$a + 2b - e = 0, \quad c + 4e = 0, \quad d - e = 0.$$

Solve for the leading coefficient variables:

$$a = -2b + e, \quad c = -4e, \quad d = e.$$

The parametric solution is

$$a = -2s + t, \quad b = s, \quad c = -4t, \quad d = t.$$

Hence,

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = s \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ 0 \\ -4 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, a basis for the null space is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -4 \\ 1 \\ 1 \end{bmatrix} \right\}. \quad \square$$

---

17. Find bases for the row space, the column space, and the null space for the following real matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix is in row reduced echelon form.
A basis for the row space is given by the nonzero rows of the matrix:

$$\{[0 \quad 1 \quad 0 \quad 1], [0 \quad 0 \quad 1 \quad 0]\}.$$

The leading coefficients occur in the second and third columns. *Since the given matrix is in row reduced echelon form*, its second and third columns form a basis for the column space:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

To find a basis for the null space, use a, b, c, and d as variables, and regard the matrix as the coefficient matrix for a homogeneous system. The corresponding equations are

$$b + d = 0, \quad c = 0.$$

Solve for the leading coefficient variables:

$$b = -d, \quad c = 0.$$

The parametric solution is

$$a = s, \quad b = -t, \quad c = 0, \quad d = t.$$

Hence,

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, a basis for the null space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}. \quad \square$$

---

18. Show that the following set of matrices is an independent subset of $M(3, \mathbb{R})$.

$$\left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}.$$

Suppose that

$$a \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} + b \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + d \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

I must show that $a = b = c = d = 0$. I have

$$\begin{bmatrix} a+b & c & a+b \\ c+d & a & c+d \\ a+b & c & a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Equating entries gives

$$a + b = 0, \quad a = 0, \quad c + d = 0, \quad c = 0.$$

Plugging $a = 0$ into $a + b = 0$ gives $b = 0$. Plugging $c = 0$ into $c + d = 0$ gives $d = 0$. Hence, the set is independent. $\square$

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19. Show that the set $\{x, x^2, e^x\}$ is an independent set in the real vector space of differentiable functions on $\mathbb{R}$.

I'll use the Wronskian. I have

$$W(x,x^2,e^x) = \begin{vmatrix} x & x^2 & e^x \\ 1 & 2x & e^x \\ 0 & 2 & e^x \end{vmatrix} \underset{r_1 \to r_1 - xr_2}{=} \begin{vmatrix} 0 & -x^2 & e^x - xe^x \\ 1 & 2x & e^x \\ 0 & 2 & e^x \end{vmatrix} = (-1) \begin{vmatrix} -x^2 & e^x - xe^x \\ 2 & e^x \end{vmatrix} =$$

$$(-1)(-x^2e^x + 2xe^x - 2e^x) = x^2e^x - 2xe^x + 2e^x.$$

When $x = 0$, I have $W(x,x^2,e^x) = 2 \neq 0$. Therefore, the set is independent. □

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20. Find a standard basis vector $u \in \mathbb{R}^5$ such that the following set forms a basis for $\mathbb{R}^5$:

$$\{(1,2,1,1,0),(2,-1,-1,0,3),(1,4,-1,-1,2),(3,2,2,1,-6),u\}.$$

If I make a matrix with these vectors as rows, the row reduced echelon form will have the same row space.

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 2 & -1 & -1 & 0 & 3 \\ 1 & 4 & -1 & -1 & 2 \\ 3 & 2 & 2 & 1 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & -\dfrac{8}{13} \\ 0 & 1 & 0 & 0 & \dfrac{7}{13} \\ 0 & 0 & 1 & 0 & -\dfrac{62}{13} \\ 0 & 0 & 0 & 1 & \dfrac{56}{13} \end{bmatrix}$$

The nonzero rows of a row reduced echelon matrix are independent. There are four rows here, so I need a fifth vector which is independent of these four.

The row reduced echelon matrix has leading coefficients in columns 1 through 4. So I can get a vector which is independent of these rows by using the standard basis vector $(0,0,0,0,1)$ which has a 1 in the *fifth* position.

Therefore, the following set is a basis for $\mathbb{R}^5$:

$$\{(1,2,1,1,0),(2,-1,-1,0,3),(1,4,-1,-1,2),(3,2,2,1,-6),(0,0,0,0,1)\}.$$ □

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21. Find standard basis vectors $u, v \in \mathbb{Z}_3^4$ such that the following set forms a basis for $\mathbb{Z}_3^4$.

$$\{(1,2,2,1),(2,1,1,1),u,v\}$$

Construct a matrix with the given vectors as rows and row reduce:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix} \underset{r_2 \to r_2 + r_1}{\to} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \underset{r_1 \to r_1 + r_2}{\to} \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \underset{r_2 \to 2r_2}{\to} \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The row reduced echelon matrix has leading coefficients in the first and fourth columns. Therefore, I add standard basis vectors having 1's in the second the third columns. The basis is

$$\{(1,2,2,1),(2,1,1,1),(0,1,0,0),(0,0,1,0)\}.$$ □

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22. (a) Find a basis for the subspace of $\mathbb{Z}_2^4$ spanned by the set

$$\{(1,1,0,1),(1,0,1,1),(0,1,1,0)\}.$$

(b) Find a subset of the following set S which forms a basis for the subspace of $\mathbb{Z}_2^4$ spanned by S.

$$S = \{(1,1,0,1),(1,0,1,1),(0,1,1,0)\}.$$

(a) Note that I'm not required to use any of the original vectors. Construct a matrix with the vectors as rows and row reduce:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \underset{r_2 \to r_2 + r_1}{\to} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \underset{r_1 \to r_1 + r_2}{\to} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \underset{r_3 \to r_3 + r_2}{\to} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\{(1,0,1,1),(0,1,1,0)$ is a basis for the subspace spanned by the original set of vectors. $\square$

(b) In this case, I'm required to use (some of) the original vectors. Hence, I construct a matrix with the vectors as *columns* and row reduce:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \underset{r_2 \to r_2 + r_1}{\to} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \underset{r_4 \to r_4 + r_1}{\to} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_1 \to r_1 + r_2}{\to}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_3 \to r_3 + r_2}{\to} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The row reduced echelon matrix has leading coefficients in the first and second columns. Therefore, the first and second columns of the original matrix — i.e. the first and second vectors in the original set of vectors —forms a basis for the subspace of $\mathbb{Z}_2^4$ spanned by the set of vectors. The basis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}. \quad \square$$

---

23. Consider the following real matrices:

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 2 & 1 & -1 & 3 \\ 4 & 12 & 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 2 \end{bmatrix}.$$

Determine whether the row space of A is contained in the row space of B. If it is, prove it. If it isn't, find a vector in the row space of A which is not in the row space of B.

I will do this in two ways: A short way and a long way.
First, I'll do the short way. Row reducing A and B gives

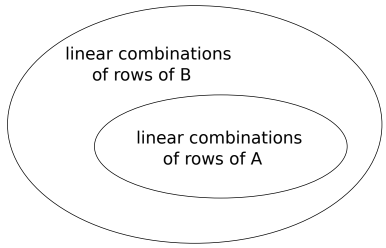
$$R = \begin{bmatrix} 1 & 0 & 0 & \frac{7}{2} \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{5}{2} \end{bmatrix}.$$

Since A and B have the same row-reduced echelon form, and since a matrix and its row-reduced echelon form have the same row space, the row spaces of A and B are in fact the same. (So trivially, the row space of A is contained in the row space of B.)

Next, here's a longer way which uses the definition of row space.

The row space of a matrix M is the subspace spanned by the rows of the matrix: that is, the set of all linear combinations of the rows of the matrix.

Hence, to show that the row space of A is contained in the row space of B I have to show that every linear combination of the rows of A is a linear combination of the rows of B.



It's enough to show that every row of A is a linear combination of rows of B. To see this, suppose that A_1, A_2, and A_3 are the rows of A and B_1, B_2, and B_3 are the rows of B. And suppose that the A_i's are linear combinations of the B_i's:

$$A_1 = x_{11}B_1 + x_{12}B_2 + x_{13}B_3,$$

$$A_2 = x_{21}B_1 + x_{22}B_2 + x_{23}B_3,$$

$$A_3 = x_{31}B_1 + x_{32}B_2 + x_{33}B_3.$$

Then if $rA_1 + sA_2 + tA_3$ is a linear combination of the A_i's, I have

$$\begin{aligned} rA_1 + sA_2 + tA_3 &= r(x_{11}B_1 + x_{12}B_2 + x_{13}B_3) + s(x_{21}B_1 + x_{22}B_2 + x_{23}B_3) + t(x_{31}B_1 + x_{32}B_2 + x_{33}B_3) \\ &= (rx_{11} + sx_{21} + tx_{31})B_1 + (rx_{12} + sx_{22} + tx_{32})B_2 + (rx_{13} + sx_{23} + tx_{33})B_3. \end{aligned}$$

That is, every linear combination of the A_i's is a linear combination of the B_i's.

So I've reduced the problem to showing that each row of A is a linear combination of rows of B. This is an "Is the vector in the span?" problem. For the first row of A, I want

$$(1, 3, 0, -1) = x \cdot (1, 4, 1, 0) + y \cdot (1, 0, -1, 1) + z \cdot (1, 1, 0, 2).$$

Rewriting the vectors as column vectors and using matrix form, I have

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}.$$

Row reduce to solve:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 0 & 1 & 3 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first row of A is a linear combination of the rows of B, with coefficients 1, 1, and -1.

For the second row, the system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}.$$

Row reduce to solve:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 4 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The second row of A is a linear combination of the rows of B, with coefficients 0, 1, and 1.
For the third row, the system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 2 \\ 1 \end{bmatrix}.$$

Row reduce to solve:

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 4 & 0 & 1 & 12 \\ 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third row of A is a linear combination of the rows of B, with coefficients 3, 1, and 0.

Since all the rows of A are linear combinations of the rows of B, the row space of A is contained in the row space of B.

If in any of the three cases the system had no solution, then the corresponding row of A would be a vector in the row space of A which was *not* in the row space of B. □

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24. Consider the following real matrix, where $x \in \mathbb{R}$:

$$A = \begin{bmatrix} 1 & 1 & 5 \\ x & 2 & 10 \end{bmatrix}.$$

(a) Find a value of x for which $\operatorname{rank} A = 1$.

(b) Find a value of x for which $\operatorname{rank} A = 2$.

(a) If $x = 2$, the second row is twice the first row.

$$\begin{bmatrix} 1 & 1 & 5 \\ 2 & 2 & 10 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

The row-reduced echelon form has one nonzero row, so its rank is 1. Hence, the rank of A is 1. □

(b) Let $x = 0$:

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & 2 & 10 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

The row-reduced echelon form has two nonzero rows, so its rank is 2. Hence, the rank of A is 2. □

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25. In a 5×19 real matrix A, the 5 rows are independent. What is nullity A?

Since $19 = \operatorname{rank} A + \operatorname{nullity} A$ and $\operatorname{rank} A = 5$, it follows that $\operatorname{nullity} A = 19 - 5 = 14$. □

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26. (a) Let $A, B \in M(n, F)$, where F is a field. Prove that

$$\text{null space}(B) \subset \text{null space}(AB).$$

(b) Let $A, B \in M(n, F)$, where F is a field. Suppose that A is invertible. Prove that

$$\text{null space}(B) = \text{null space}(AB).$$

(c) Find a matrix $A \in M(2, \mathbb{R})$ such that

$$\text{null space}(A) \subsetneq \text{null space}(A^2).$$

(d) Find a matrix $A \in M(2, \mathbb{R})$ such that

$$\text{row space}(A^2) \subsetneq \text{row space}(A).$$

(a) Remember the definition: To say that x is in the null space of a matrix M means that $Mx = \vec{0}$.

Let $x \in \text{null space}(B)$. Then

$$\begin{aligned} Bx &= \vec{0} \\ A(Bx) &= A \cdot \vec{0} \\ (AB)x &= \vec{0} \end{aligned}$$

Therefore, $x \in \text{null space}(AB)$. Hence, $\text{null space}(B) \subset \text{null space}(AB)$. □

(b) In part (a), I showed that $\text{null space}(B) \subset \text{null space}(AB)$. So assuming that A is invertible, I only have to show the opposite inclusion: $\text{null space}(AB) \subset \text{null space}(B)$.

Let $x \in \text{null space}(AB)$. Then

$$\begin{aligned} (AB)x &= \vec{0} \\ A^{-1}(AB)x &= A^{-1} \cdot \vec{0} \\ IBx &= \vec{0} \\ Bx &= \vec{0} \end{aligned}$$

This means that $x \in \text{null space}(B)$. Hence, $\text{null space}(AB) \subset \text{null space}(B)$. Combined with the other inclusion, I've proved that $\text{null space}(B) = \text{null space}(AB)$. □

(c) Here's the idea. I want the null space — the set of vectors that are multiplied to $\vec{0}$ — to get *bigger* when I square A. One way to multiply more vectors to $\vec{0}$ is to have more all-zero rows in A^2 than in A. Experiment a bit with matrix multiplication. You might come up with something like this.

Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now $\text{null space}(A) \subset \text{null space}(A^2)$ — just take B to equal A in part (a) above. To show that this is a *proper* containment, I have to find a vector which is in $\text{null space}(A^2)$ but is not in $\text{null space}(A)$.

Now that null space of A consists of vectors (a, b) such that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Multiplying out the left side, I get $(0, b) = (0, 0)$, or $b = 0$. So vectors in the null space of A must have second component 0.

On the other hand, A^2 is the zero matrix, so it multiplies *everything* to $\vec{0}$ — the null space is $\mathbb{R}^2$.

So to get something in null space(A^2) which is not in null space(A), I just take anything vector which does *not* have second component 0 — such as $(0, 1)$. You can check that A^2 multiplies this vector to $\vec{0}$, but A does not. $\square$

(d) The row space is the set spanned by the rows. So you'd expect the row space to get *smaller* if there are more all-zero rows in A^2 than in A. But I just came up with an example that does this in part (c): Use

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The row space is all multiple of $(0, 1)$: Vectors of the form $(0, a)$, where $a \in \mathbb{R}$.
Now

$$A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Obviously, the row space of A^2 is just $(0, 0)$ — it's the only vector you can build using the rows of A^2. And since $(0, 0)$ has the form $(0, a)$ (with $a = 0$), the row space of A^2 is contained in the row space of A.

On the other hand, the vector $(0, 1)$ is in the row space of A, but it's not in the row space of A^2. Therefore, in this case, row space(A^2) $\subsetneq$ row space(A). $\square$

---

27. For the given function, check each axiom for a linear transformation. If the axiom holds, prove it. If the axiom does not hold, give a specific counterexample.

(a) The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x + \sin x.$$

(b) The function $g : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$g(x, y) = (x - y, x + y, 2x + 3y).$$

(c) The function $h : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$h(x, y) = (x + y, xy).$$

(d) The function $p : \mathbb{R}^2 \to M(2, \mathbb{R})$ given by

$$p(x, y) = \begin{bmatrix} 2x & 3y \\ 0 & y^2 \end{bmatrix}.$$

(a) The sum axiom does not hold:

$$f\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = f(\pi) = \pi + \sin \pi = \pi.$$

$$f\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2} + \sin \frac{\pi}{2}\right) + \left(\frac{\pi}{2} + \sin \frac{\pi}{2}\right) = \pi + 2.$$

The scalar multiplication axiom does not hold:

$$\frac{1}{2} \cdot f(\pi) = \frac{1}{2}(\pi + \sin \pi) = \frac{\pi}{2}.$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \sin \frac{\pi}{2} = \frac{\pi}{2} + 1. \quad \square$$

(b) g may be written as

$$g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Since g is defined by multiplication by a matrix (of numbers), g must be a linear transformation. Here are direct checks of the axioms:

$$g[(a,b)+(c,d)=g(a+c,b+d)=\begin{bmatrix}1 & -1\\ 1 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}a+c\\ b+d\end{bmatrix}=$$

$$\begin{bmatrix}1 & -1\\ 1 & 1\\ 2 & 3\end{bmatrix}\left(\begin{bmatrix}a\\ b\end{bmatrix}+\begin{bmatrix}c\\ d\end{bmatrix}\right)=\begin{bmatrix}1 & -1\\ 1 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}+\begin{bmatrix}1 & -1\\ 1 & 1\\ 2 & 3\end{bmatrix}=$$

$$\begin{bmatrix}c\\ d\end{bmatrix}=g(a,b)+g(c,d).$$

(c) The sum axiom doesn't hold:

$$h(1,2)=(3,2),\quad h(2,3)=(5,6),\quad \text{so}\quad h(1,2)+h(2,3)=(3,2)+(5,6)=(8,8).$$

However,

$$h\left((1,2)+(2,3)\right)=h(3,5)=(8,15)\neq(8,8).$$

The scalar multiplication axiom doesn't hold:

$$4\cdot h(1,2)=4\cdot(3,2)=(12,8).$$

$$h[4\cdot(1,2)]=h(4,8)=(12,32).\quad \square$$

(d) The sum axiom doesn't hold:

$$p(1,2)+p(3,4)=\begin{bmatrix}2 & 6\\ 0 & 4\end{bmatrix}+\begin{bmatrix}6 & 12\\ 0 & 16\end{bmatrix}=\begin{bmatrix}8 & 18\\ 0 & 20\end{bmatrix}.$$

$$p[(1,2)+(3,4)]=p(4,6)=\begin{bmatrix}8 & 18\\ 0 & 36\end{bmatrix}.$$

The scalar multiplication axiom doesn't hold:

$$5\cdot p(1,2)=5\cdot\begin{bmatrix}2 & 6\\ 0 & 4\end{bmatrix}=\begin{bmatrix}10 & 30\\ 0 & 20\end{bmatrix}.$$

$$p[5\cdot(1,2)]=p(5,10)=\begin{bmatrix}10 & 30\\ 0 & 100\end{bmatrix}.\quad \square$$

---

28. Let A be a *fixed* matrix in $M(2,\mathbb{R})$. Define $T: M(2,\mathbb{R})\to M(2,\mathbb{R})$ by

$$T(X)=AX-XA\quad\text{for}\quad X\in M(2,\mathbb{R}).$$

Prove that T is a linear transformation.

Let $X,Y\in M(2,\mathbb{R})$. Then

$$T(X+Y)=A(X+Y)-(X+Y)A=AX+AY-XA-YA=(AX-XA)+(AY-YA)=T(X)+T(Y).$$

Next, let $X\in M(2,\mathbb{R})$ and let $k\in\mathbb{R}$. Then

$$T(kX)=A(kX)-(kX)A=k(AX)-k(XA)=k(AX-XA)=kT(X).$$

Therefore, T is a linear transformation. $\square$

---

29. Define $\operatorname{tr} : M(2, \mathbb{R}) \to \mathbb{R}$ by

$$\operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

tr is called the **trace function**.

(a) Show that tr is a linear transformation.

(b) Find a basis for the null space of tr.

(a)

$$\operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}\right) = (a+a') + (d+d') = (a+d) + (a'+d') =$$

$$\operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \operatorname{tr}\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right).$$

Let $k \in \mathbb{R}$. Then

$$\operatorname{tr}\left(k \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = ka + kd = k(a+d) = k \cdot \operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).$$

Hence, tr is a linear transformation. $\square$

(b) The kernel of tr consists of the elements of $M(2, \mathbb{R})$ which tr maps to 0.

$$\operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 \quad \text{implies} \quad a + d = 0.$$

Therefore, $d = -a$, so

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

Now

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Consider the set of matrices

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

All the matrices in S are in the kernel of tr. Moreover, the equation above shows that they span $\ker T$. To show that they're independent, suppose that

$$a \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{so} \quad a = b = c = 0.$$

Therefore, the matrices are independent.
Therefore, the set S is a basis for $\ker T$. $\square$

30. Define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(u,v) = (u, (u^2+1)\cos v, (u^2+1)\sin v).$$

(a) Compute the derivative $Df(u,v)$.

(b) Show that there no values (u,v) for which $Df(u,v)$ is the zero matrix.

(a)

$$Df(u,v) = \begin{bmatrix} 1 & 0 \\ 2u\cos v & -(u^2+1)\sin v \\ 2u\sin v & (u^2+1)\cos v \end{bmatrix}. \quad \square$$

(b) Set $Df(u,v) = 0$:

$$\begin{bmatrix} 1 & 0 \\ 2u\cos v & -(u^2+1)\sin v \\ 2u\sin v & (u^2+1)\cos v \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Consider the second and third entries in the second columns. They give the equations

$$-(u^2+1)\cos v = 0, \quad (u^2+1)\sin v = 0.$$

Since $u^2+1 \neq 0$, I may cancel it to get

$$\cos v = 0, \quad \sin v = 0.$$

There is no v for which these equations are both true. You can think about the graphs of sine and cosine to see this; alternatively, if both equations were true, you could square both and add:

$$\begin{array}{r} (\cos v)^2 = 0 \\ (\sin v)^2 = 0 \\ \hline 1 = 0 \end{array}$$

This contradiction shows that the equations can't both hold simultaneously. $\square$

---

31. Find a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ that takes the parallelogram determined by the vectors $(2,3)$ and $(2,4)$ to the parallelogram determined by the vectors $(-2,1)$ and $(-1,-3)$.

The following transformation takes the square determined by the vectors $(1,0)$ and $(0,1)$ to the parallelogram determined by the vectors $(2,3)$ and $(2,4)$:

$$f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, the inverse takes the parallelogram determined by the vectors $(2,3)$ and $(2,4)$ to the square determined by the vectors $(1,0)$ and $(0,1)$:

$$f^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

The following transformation takes the square determined by the vectors $(1,0)$ and $(0,1)$ to the parallelogram determined by the vectors $(-2,1)$ and $(-1,-3)$:

$$g\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Therefore, the composite $g \cdot f^{-1}$ takes the parallelogram determined by the vectors $(2,3)$ and $(2,4)$ to the parallelogram determined by the vectors $(-2,1)$ and $(-1,-3)$:

$$(g \cdot f^{-1}) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & -3 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -5 & 2 \\ 13 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \quad \square$$

---

32. Construct a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects points in $\mathbb{R}^2$ across the line $y = 9x$.

Let $\theta = \tan^{-1} 9$. The following transformation rotates $\mathbb{R}^2$ counterclockwise by θ:

$$g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, g^{-1} rotates $\mathbb{R}^2$ clockwise by θ:

$$g^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The following transformation reflects $\mathbb{R}^2$ across the x-axis:

$$h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The following composite first uses g^{-1} to rotate $\mathbb{R}^2$ clockwise by θ, moving the line $y = 9x$ onto the x-axis. Next, h reflects points across the x-axis. Finally, g rotates $\mathbb{R}^2$ counterclockwise by θ, which moves the x-axis back up to $y = 9x$.

$$(g \cdot h \cdot g^{-1})\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \quad \square$$

---

33. Find an affine transformation which takes the unit square $\{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$ to the parallelogram with vertices $A(2,5)$, $B(3,7)$, $C(2,6)$, $D(1,4)$ in such a way that the origin $(0,0)$ goes to A, the vector $(1,0)$ goes to $\overrightarrow{AB}$, and the vector $(0,1)$ goes to $\overrightarrow{AD}$.

First, find vectors for the sides of the parallelogram: $\overrightarrow{AB} = (1,2)$ and $\overrightarrow{AD} = (-1,-1)$.
Define

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

The matrix carries the unit square to the parallelogram determined by the vectors, but with $(0,0)$ mapped to $(0,0)$. In order to map the square to the parallelogram $ABCD$, I added a vector $(2,5)$ to translate $(0,0)$ to $(2,5)$. (The rest of the parallelogram is translated as well.) $\square$

---

34. Consider the following bases for the real vector space $\mathbb{R}^2$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{bmatrix} 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

(a) Construct the translation matrices

$$[\mathcal{B} \to \text{std}], \quad [\text{std} \to \mathcal{B}], \quad [\mathcal{C} \to \text{std}], \quad [\text{std} \to \mathcal{C}], \quad [\mathcal{B} \to \mathcal{C}], \quad \text{and} \quad [\mathcal{C} \to \mathcal{B}].$$

(b) Write the vector $(-5, 4)$ in terms of $\mathcal{B}$.

(c) Write the vector $(3, 2)_{\mathcal{C}}$ in terms of the standard basis.

(d) Write the vector $(-1, 4)_{\mathcal{B}}$ in terms of $\mathcal{C}$.

(e) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = (2x - y, 3x + 8y).$$

Find $[f]_{\mathcal{B},\mathcal{C}}$.

(a)

$$[\mathcal{B} \to \text{std}] = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}, \quad [\text{std} \to \mathcal{B}] = [\mathcal{B} \to \text{std}]^{-1} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix},$$

$$[\mathcal{C} \to \text{std}] = \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}, \quad [\text{std} \to \mathcal{C}] = [\mathcal{C} \to \text{std}]^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix},$$

$$[\mathcal{B} \to \mathcal{C}] = [\text{std} \to \mathcal{C}][\mathcal{B} \to \text{std}] = \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 9 & 5 \end{bmatrix},$$

$$[\mathcal{C} \to \mathcal{B}] = [\text{std} \to \mathcal{B}][\mathcal{C} \to \text{std}] = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 9 & 2 \end{bmatrix}.$$

Note that $[\mathcal{C} \to \mathcal{B}] = [\mathcal{B} \to \mathcal{C}]^{-1}$. □

(b)

$$[\text{std} \to \mathcal{B}] \begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} -17 \\ 21 \end{bmatrix}_{\mathcal{B}}. \quad \square$$

(c)

$$[\mathcal{C} \to \text{std}] \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{\mathcal{C}} = \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 25 \\ 14 \end{bmatrix}. \quad \square$$

(d)

$$[\mathcal{B} \to \mathcal{C}] \begin{bmatrix} -1 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -2 & -1 \\ 9 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -2 \\ 11 \end{bmatrix}_{\mathcal{C}}. \quad \square$$

(e) First,

$$f \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

That is,

$$[f]_{\text{std},\text{std}} = \begin{bmatrix} 2 & -1 \\ 3 & 8 \end{bmatrix}.$$

Then

$$[f]_{\mathcal{B},\mathcal{C}} = [\text{std} \to \mathcal{C}][f]_{\text{std},\text{std}}[\mathcal{B} \to \text{std}] = \begin{bmatrix} -1 & 2 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 33 & 29 \\ -112 & -99 \end{bmatrix}. \quad \square$$

---

35. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y + 3z \\ x + y - z \end{bmatrix}.$$

Consider the following bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(a) Find $[\mathcal{B} \to \text{std}]$, $[\mathcal{C} \to \text{std}]$, and $[\text{std} \to \mathcal{C}]$.

(b) Express $(3, 4)$ relative to the basis $\mathcal{C}$.

(c) Express $(3, 1, 0)_{\mathcal{B}}$ relative to the standard basis.

(d) Find $[T]_{\mathcal{B},\mathcal{C}}$.

(a)

$$[\mathcal{B} \to \text{std}] = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$[\mathcal{C} \to \text{std}] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},$$

$$[\text{std} \to \mathcal{C}] = [\mathcal{C} \to \text{std}]^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \quad \square$$

(b)

$$\frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ 7 \end{bmatrix}_{\mathcal{C}}. \quad \square$$

(c)

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}. \quad \square$$

(d) First,

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Thus,

$$[T]_{\text{std},\text{std}} = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & -1 \end{bmatrix}.$$

Therefore,

$$[T]_{\mathcal{B},\mathcal{C}} = [\text{std} \to \mathcal{C}][T]_{\text{std},\text{std}}[\mathcal{B} \to \text{std}] =$$

$$\frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -6 & 3 \\ 3 & -2 & -1 \end{bmatrix}. \quad \square$$

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36. What is i^{101}?

$$i^{101} = i^{100} \cdot i = (i^2)^{50} \cdot i = (-1)^{50} \cdot i = 1 \cdot i = i. \quad \square$$

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37. Simplify $\dfrac{1+2i}{5-3i}$.

$$\frac{1+2i}{5-3i} = \frac{1+2i}{5-3i}\cdot\frac{5+3i}{5+3i} = \frac{-1+13i}{34}. \quad \square$$

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38. Compute $(1+i\sqrt{3})^{10}$.

Use DeMoivre's Formula:

$$(1+i\sqrt{3})^{10} = \left[2\cdot\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right]^{10} = 2^{10}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{10} = 1024\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{10} = 1024\left(e^{\pi i/3}\right)^{10} =$$

$$1024e^{10\pi i/3} = 1024\left(\cos\frac{10\pi}{3}+i\sin\frac{10\pi}{3}\right) = 1024\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right) = 1024\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) =$$

$$-512-512i\sqrt{3}. \quad \square$$

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39. Find the inverse of the following matrix. (Multiply any constants outside the matrix into the matrix and simplify.)

$$\begin{bmatrix} 2-3i & 4 \\ 1 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} 2-3i & 4 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{(4-6i)-4}\begin{bmatrix} 2 & -4 \\ -1 & 2-3i \end{bmatrix} = \frac{-1}{6i}\begin{bmatrix} 2 & -4 \\ -1 & 2-3i \end{bmatrix} =$$

$$\frac{i}{6}\begin{bmatrix} 2 & -4 \\ -1 & 2-3i \end{bmatrix} = \begin{bmatrix} \dfrac{i}{3} & -\dfrac{2i}{3} \\ -\dfrac{i}{6} & \dfrac{1}{2}+\dfrac{i}{3} \end{bmatrix}. \quad \square$$

To do two things at once is to do neither. - Publilius Syrus