Review Sheet for Test 3

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. Consider the real matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the eigenvalues and a complete set of independent eigenvectors, and a matrix P such that $P^{-1}AP$ is diagonal, and the corresponding diagonal matrix.

2. Consider the real matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & -1 & 4 \end{bmatrix}.$$

Find the eigenvalues and a complete set of independent eigenvectors, and a matrix P such that $P^{-1}AP$ is diagonal, and the corresponding diagonal matrix.

3. Give an example of a nonzero 2×2 matrix over $\mathbb{R}$ which is not diagonalizable.

4. Suppose that θ is not a multiple of π. Prove that the 2×2 real matrix A which gives rotation counterclockwise through θ does not have any real eigenvalues.

(You can do this algebraically, but see if you can give a geometric argument.)

5. Suppose $A \in M(n, \mathbb{R})$ and *every* vector in $\mathbb{R}^n$ is an eigenvector of A. Prove that A is a multiple of the $n \times n$ identity matrix.

6. Let A be an $n \times n$ matrix, let v be an eigenvector corresponding to the eigenvalue λ, and let $c \neq 0$. Prove or disprove: cv is an eigenvector of A.

7. Let $a, b \in \mathbb{R}$, and let

$$M = \begin{bmatrix} a & b & b & a \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ a & a & a & a \end{bmatrix}.$$

Show that $(1, 0, 0, 1)$ is an eigenvector for M corresponding to the eigenvalue $2a$.

8. Find the general solution $y(x)$ to each of the following differential equations.

(a) $y'' - 8y' - 9y = 0$.

(b) $(D - 2)^3 Dy = 0$.

(c) $y'' - 4y' + 20y = 0$.

9. Solve the following linear system for x, y, and z in terms of t:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 3 & -4 & 8 \\ 0 & 23 & -40 \\ 0 & 12 & -21 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

10. Solve the following linear system for x and y in terms of t:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Your answer should be given entirely in terms of real numbers and functions.

11. Two tanks hold 100 gallons of liquid each. The first tank starts with 36 pounds of dissolved salt, while the second starts with pure water. Pure water flows into the first tank at 5 gallons per minute; the well-stirred mixture flows into tank 2 at 9 gallons per minute. The mixture in tank 2 is pumped back into tank 1 at 4 gallons per minute, and also drains out at 5 gallons per minute. Find the amount of salt in each tank after t minutes.

12. Suppose that A is a real 2×2 matrix and

$$e^{At} = \begin{bmatrix} \frac{7}{8}e^{6t} + \frac{1}{8}e^{-2t} & \frac{1}{8}e^{6t} - \frac{1}{8}e^{-2t} \\ \frac{7}{8}e^{6t} - \frac{7}{8}e^{-2t} & \frac{1}{8}e^{6t} + \frac{7}{8}e^{-2t} \end{bmatrix}.$$

Find A.

13. Find e^{At} for

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

14. Compute e^{At} for

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}.$$

15. Compute e^{At} for

$$A = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}.$$

16. Compute e^{At} for

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

17. Let $u \cdot v$ denote the standard inner product on $\mathbb{R}^4$.

(a) Find $(1, -2, 5, 14) \cdot (3, 1, 2, -1)$.

(b) Find the cosine of the angle between $(1, 1, -2, 5)$ and $(2, 2, 0, 7)$.

(c) Find a nonzero vector that is perpendicular to both $(1, -3, 6, 0)$ and $(2, 1, 11)$.

18. Suppose that u, v, and w are vectors in a real inner product space, and

$$\|u\| = 5, \quad \|v\| = 3, \quad \|w\| = 2,$$

$$\langle u, v \rangle = -2, \quad \langle u, w \rangle = 6, \quad \langle v, w \rangle = 10.$$

(a) Find $\langle 3u + v, v + 2w \rangle$.

(b) Find $\|u + w\|$.

19. Find an orthonormal basis relative to the standard dot product on $\mathbb{R}^4$ for the subspace spanned by the set

$$\{(1,0,1,1),(4,1,5,0),(4,45,1,7)\}.$$

20. Let $C[0,1]$ denote the real vector space of continuous real-valued functions on the interval $[0,1]$. An inner product is defined on $C[0,1]$ by

$$\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx.$$

(a) Compute $\langle f,g\rangle$, where $f(x)=x$ and $g(x)=\cos\dfrac{\pi}{2}x$.

(b) Find $\|h\|$, where $h(x)=2x+1$.

(c) For what value of k are the functions $f(x)=x+k$ and $g(x)=x^2$ orthogonal?

(d) Consider the set of functions $S=\{x+1,2x\}$. Find an orthonormal set which spans the same subspace of $C[0,1]$ as S.

21. The following set of vectors in $\mathbb{R}^3$ is orthonormal relative to the standard dot product:

$$u_1=\frac{1}{\sqrt{2}}(1,0,1),\quad u_2=\frac{1}{\sqrt{3}}(1,-1,-1),\quad u_3=\frac{1}{\sqrt{6}}(-1,-2,1).$$

Find the components of $(5,-4,2)$ relative to the basis $\{u_1,u_2,u_3\}$.

22. An inner product is defined on $\mathbb{R}^2$ by

$$\langle x,y\rangle = x^T\begin{bmatrix} 10 & -1 \\ -1 & 5\end{bmatrix} y.$$

(a) Find $\|(1,-1)\|$ relative to the given inner product.

(b) Find $\cos\theta$, where θ is the angle relative to the given inner product between $(1,-1)$ and $(1,1)$.

(c) Find a nonzero vector (a,b) which is orthogonal to $(3,1)$ relative to this inner product.

23. Let x be a fixed vector in a real inner product space V. Let

$$x^\perp = \{v\in V \mid \langle x,v\rangle = 0\}.$$

Prove that $x^\perp$ is a subspace of V. ($x^\perp$ is called the **orthogonal complement** of x.)

Solutions to the Review Sheet for Test 3

1. Consider the real matrix

$$A=\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1\end{bmatrix}.$$

Find the eigenvalues and a complete set of independent eigenvectors, and a matrix P such that $P^{-1}AP$ is diagonal, and the corresponding diagonal matrix.

The characteristic polynomial is

$$\det\begin{bmatrix} 2-x & 1 & 0 \\ 1 & 2-x & 0 \\ 0 & 0 & 1-x\end{bmatrix} = -(x-1)^2(x-3).$$

The eigenvalues are $\lambda = 1$ and $\lambda = 3$.

For $\lambda = 1$,

$$A - 1 \cdot I = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

With a, b, and c as variables, the corresponding homogeneous system is $a + b = 0$, or $a = -b$. The solution vector is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -b \\ b \\ c \end{bmatrix} = b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Taking $b = 0$ and $c = 1$, and then $b = 0$ and $c = 1$, I get the eigenvectors $(-1, 1, 0)$ and $(0, 0, 1)$.

For $\lambda = 3$,

$$A - 3 \cdot I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

With a, b, and c as variables, the corresponding homogeneous system is $a - b = 0$, or $a = b$, and $c = 0$. The solution vector is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Taking $b = 1$, I get the eigenvector $(1, 1, 0)$.

Using the eigenvectors as columns, I obtain

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \quad \square$$

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2. Consider the real matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & -1 & 4 \end{bmatrix}.$$

Find the eigenvalues and a complete set of independent eigenvectors, and a matrix P such that $P^{-1}AP$ is diagonal, and the corresponding diagonal matrix.

$$\det(A - xI) = \begin{vmatrix} 2 - x & 0 & 0 \\ 0 & -x & 5 \\ 0 & -1 & 4 - x \end{vmatrix} = (2 - x)[(-x)(4 - x) - (5)(-1)] = (2 - x)(x^2 - 4x + 5).$$

The eigenvalues are 2 and $2 \pm i$.

For $x = 2$,

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 5 \\ 0 & -1 & 2 \end{bmatrix} \quad \to \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

With eigenvector (a, b, c), the row reduced echelon matrix gives the equations

$$b = 0, \quad c = 0.$$

So

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, $(1, 0, 0)$ is an eigenvector for $x = 2$.

For $x = 2 + i$,

$$A - (2+i)I = \begin{bmatrix} -i & 0 & 0 \\ 0 & -2-i & 5 \\ 0 & -1 & 2-i \end{bmatrix}$$

The last two rows are clearly independent of the first, so they must be multiples (or all three rows would be independent, and there would be no nonzero eigenvectors). It follows that some row operation can be used to "wipe out" the third row (in fact, $r_3 \to r_3 - \dfrac{1}{2+i} r_2$ works), and

$$\begin{bmatrix} -i & 0 & 0 \\ 0 & -2-i & 5 \\ 0 & -1 & 2-i \end{bmatrix} \quad \to \quad \begin{bmatrix} -i & 0 & 0 \\ 0 & -2-i & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

(Note that you don't need to actually figure out the row operation that does this — you *know* that one exists, because the rows *must* be multiples.)

With eigenvector (a, b, c), the row reduced echelon matrix gives the equations

$$-ia = 0, \quad (-2-i)b + 5c = 0.$$

The first equation gives $a = 0$. The second equation is satisfied by $b = 5$ and $c = 2 + i$ (by swapping the coefficients "$-2-i$" and "5" and negating one of them, in this case the "$-2-i$"). Thus, $(0, 5, 2+i)$ is an eigenvector for $x = 2 + i$.

By conjugation, $(0, 5, 2-i)$ is an eigenvector for $2 - i$.

(At this point, note that you might get different looking, but correct, results if you had "wiped out" the second row of the matrix above rather than the third row, or if if you had chosen to negative the "5" rather than the "$-2-i$".)

A diagonalizing matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 5 \\ 0 & 2+i & 2-i \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}. \quad \square$$

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3. Give an example of a nonzero 2×2 matrix over $\mathbb{R}$ which is not diagonalizable.

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The characteristic polynomial is $(x-1)^2$, so $\lambda = 1$ is the only eigenvalue. Now

$$A - I = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

With (a, b) as a solution vector, the corresponding homogeneous system is $b = 0$. Thus,

$$(a, b) = (a, 0) = a(1, 0).$$

Taking $a = 1$, I get the eigenvector $(1, 0)$.

Since there is only one eigenvector, the matrix is not diagonalizable. □

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4. Suppose that θ is not a multiple of π. Prove that the 2×2 real matrix A which gives rotation counterclockwise through θ does not have any real eigenvalues.

First, rotation through θ is an invertible operation (the inverse is rotation through $-\theta$). Hence, A is invertible, and 0 can't be an eigenvalue of A.

Next, if c is a nonzero eigenvalue of A with eigenvector v, then $Av = cv$. This equation says that Av and cv are parallel; since cv and v are parallel, this means that Av and v are parallel. But this is impossible, because Av is just v rotated by θ, which is not a multiple of π.

Hence, A has no nonzero eigenvalues. □

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5. Suppose $A \in M(n, \mathbb{R})$ and *every* vector in $\mathbb{R}^n$ is an eigenvector of A. Prove that A is a multiple of the $n \times n$ identity matrix.

I need a little fact about matrix multiplication which you should check for yourself: If e_i is the i^{th} standard basis vector, then Ae_i is the i^{th} column of A. (Try it out for a 3×3 matrix to see the idea.)

First, I'll show that A is a diagonal matrix. Every vector is an eigenvector, so in particular the standard basis vectors are eigenvectors. So

$$Ae_1 = \lambda_1 e_1 \quad \text{for some} \quad \lambda_1.$$

But Ae_1 is the first column of A, so this says that the first column of A is

$$\begin{bmatrix} \lambda_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Likewise,

$$Ae_2 = \lambda_2 e_2 \quad \text{for some} \quad \lambda_2.$$

But Ae_2 is the second column of A, so this says that the second column of A is

$$\begin{bmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Continue with e_3, ..., e_n. Then stringing the columns of A together, I find that

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

But the vector $(1, 1, 1, \ldots, 1)$ is also an eigenvector of A, so for some λ,

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \lambda \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Multiplying out the two sides of this equation, I get

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \\ \vdots \\ \lambda \end{bmatrix}.$$

So λ_1, λ_2, $\ldots \lambda_n$ are all equal to λ, and

$$A = \begin{bmatrix} \lambda & 0 & 0 & \ldots & 0 \\ 0 & \lambda & 0 & \ldots & 0 \\ 0 & 0 & \lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda \end{bmatrix}.$$

This is just λ times the identity matrix. $\square$

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6. Let A be an $n \times n$ matrix, let v be an eigenvector corresponding to the eigenvalue λ, and let $c \neq 0$. Prove or disprove: cv is an eigenvector of A.

Since v is an eigenvector corresponding to the eigenvalue λ,

$$Av = \lambda v.$$

Multiply both sides by c:

$$A(cv) = \lambda(cv).$$

Since $c \neq 0$ and $v \neq \vec{0}$ (since v is an eigenvector), it follows that $cv \neq \vec{0}$. Hence, cv is an eigenvector for A corresponding to the eigenvalue λ. $\square$

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7. Let $a, b \in \mathbb{R}$, and let

$$M = \begin{bmatrix} a & b & b & a \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ a & a & a & a \end{bmatrix}.$$

Show that $(1, 0, 0, 1)$ is an eigenvector for M corresponding to the eigenvalue $2a$.

$$\begin{bmatrix} a & b & b & a \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ a & a & a & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2a \\ 0 \\ 0 \\ 2a \end{bmatrix} = 2a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The definition says that x is an eigenvector for a matrix A with eigenvalue λ if $Ax = \lambda x$, and that is what I've shown for M, $(1, 0, 0, 1)$, and $2a$. I *don't* need to go through a lot of trouble computing the characteristic polynomial of M and finding eigenvalues and eigenvectors. Did you try to do that? $\square$

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8. Find the general solution $y(x)$ to each of the following differential equations.

(a) $y'' - 8y' - 9y = 0$.

(b) $(D-2)^3 Dy = 0$.

(c) $y'' - 4y' + 20y = 0$.

(a) In operator form, this is $(D^2 - 8D - 9)y = 0$, or $(D-9)(D+1)y = 0$. The solution is

$$y = c_1 e^{9x} + c_2 e^{-x}. \quad \square$$

(b) The root 2 is repeated 3 times; the "D" term corresponds to a root of 0. The general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} + c_4. \quad \square$$

(c) In operator form, this is $(D^2 - 4D + 20)y = 0$. The equation $m^2 - 4m + 20 = 0$ has roots $m = 2 \pm 4i$. The general solution is

$$y = c_1 e^{2x} \cos 4x + c_2 e^{2x} \sin 4x. \quad \square$$

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9. Solve the following linear system for x, y, and z in terms of t:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 3 & -4 & 8 \\ 0 & 23 & -40 \\ 0 & 12 & -21 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 3 & -4 & 8 \\ 0 & 23 & -40 \\ 0 & 12 & -21 \end{bmatrix}.$$

The characteristic polynomial is

$$|A - xI| = \begin{vmatrix} 3-x & -4 & 8 \\ 0 & 23-x & -40 \\ 0 & 12 & -21-x \end{vmatrix} = (3-x) \begin{vmatrix} 23-x & -40 \\ 12 & -21-x \end{vmatrix} = (3-x)[(23-x)(-21-x)-(-40)(12)] =$$

$$(3-x)(x^2 - 2x - 3) = -(x-3)^2(x+1).$$

The eigenvalues are $x = 3$ and $x = -1$.

For $x = 3$:

$$A - 3I = \begin{bmatrix} 0 & -4 & 8 \\ 0 & 20 & -40 \\ 0 & 12 & -24 \end{bmatrix} \quad \to \quad \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

With variables a, b, and c, the homogeneous system is $b - 2c = 0$, or $b = 2c$. So

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

This gives the independent eigenvectors $(1, 0, 0)$ and $(0, 2, 1)$.

For $x = -1$:

$$A + I = \begin{bmatrix} 4 & -4 & 8 \\ 0 & 24 & -40 \\ 0 & 12 & -20 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

With variables a, b, and c, the homogeneous system is

$$a+\frac{1}{3}c=0 \quad \text{and} \quad b-\frac{5}{3}c=0.$$

Then $a=-\dfrac{1}{3}c$ and $b=\dfrac{5}{3}c$. So

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c\cdot\begin{bmatrix} -\dfrac{1}{3} \\ \dfrac{5}{3} \\ 1 \end{bmatrix}.$$

Taking $c=3$, I get the eigenvector $(-1,5,3)$.
The general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 e^{3t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{-t}\begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}. \quad \square$$

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10. Solve the following linear system for x and y in terms of t:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Your answer should be given entirely in terms of real numbers and functions.

Let

$$A=\begin{bmatrix} 1 & 5 \\ -2 & 7 \end{bmatrix}.$$

The characteristic polynomial is

$$|A-xI|=\begin{vmatrix} 1-x & 5 \\ -2 & 7-x \end{vmatrix} = (1-x)(7-x)-(5)(-2)=x^2-8x+17.$$

The roots are $x=4\pm i$.
For $x=4+i$, I have

$$A-(4+i)I=\begin{bmatrix} -3-i & 5 \\ -2 & 3-i \end{bmatrix}.$$

Since the rows must be multiples, I can drop the second row. With variables a and b, the first row yields the homogeneous system

$$(-3-i)a+5b=0.$$

By inspection, an eigenvector is $(5,3+i)$.
The solution corresponding to this eigenvector is

$$e^{(4+i)t}\begin{bmatrix} 5 \\ 3+i \end{bmatrix} = e^{4t}(\cos t+i\sin t)\begin{bmatrix} 5 \\ 3+i \end{bmatrix} = e^{4t}\begin{bmatrix} 5\cos t+5i\sin t \\ (3\cos t-\sin t)+i(\cos t+3\sin t) \end{bmatrix}.$$

The real and imaginary parts give two independent solutions:

$$e^{4t}\begin{bmatrix} 5\cos t \\ 3\cos t-\sin t \end{bmatrix} \quad \text{and} \quad e^{4t}\begin{bmatrix} 5\sin t \\ \cos t+3\sin t \end{bmatrix}.$$

They give the general solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{4t} \begin{bmatrix} 5\cos t \\ 3\cos t - \sin t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 5\sin t \\ \cos t + 3\sin t \end{bmatrix}. \quad \square$$

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11. Two tanks hold 100 gallons of liquid each. The first tank starts with 36 pounds of dissolved salt, while the second starts with pure water. Pure water flows into the first tank at 5 gallons per minute; the well-stirred mixture flows into tank 2 at 9 gallons per minute. The mixture in tank 2 is pumped back into tank 1 at 4 gallons per minute, and also drains out at 5 gallons per minute. Find the amount of salt in each tank after t minutes.

Let x be the number of pounds of salt dissolved in the first tank at time t and let y be the number of pounds of salt dissolved in the second tank at time t. The rate equations are

$$\frac{dx}{dt} = \left(5\ \frac{\text{gal}}{\text{min}}\right)\left(0\ \frac{\text{lbs}}{\text{gal}}\right) + \left(4\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{y\ \text{lbs}}{100\text{gal}}\right) - \left(9\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{x\ \text{lbs}}{100\text{gal}}\right),$$

$$\frac{dy}{dt} = \left(9\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{x\ \text{lbs}}{100\text{gal}}\right) - \left(4\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{y\ \text{lbs}}{100\text{gal}}\right) - \left(5\ \frac{\text{gal}}{\text{min}}\right)\left(\frac{y\ \text{lbs}}{100\text{gal}}\right).$$

Simplify:

$$\begin{aligned} x' &= -0.09x + 0.04y \\ y' &= 0.09x - 0.09y \end{aligned}$$

Next, find the characteristic polynomial:

$$\begin{vmatrix} -0.09 - \lambda & 0.04 \\ 0.09 & -0.09 - \lambda \end{vmatrix} = \lambda^2 + 0.18\lambda + 0.0045 = (\lambda + 0.15)(\lambda + 0.03).$$

The eigenvalues are $\lambda = -0.15$, $\lambda = -0.03$.
Consider $\lambda = -0.15$:

$$A + 0.15I = \begin{bmatrix} 0.06 & 0.04 \\ 0.09 & 0.06 \end{bmatrix} \quad \to \quad \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$$

$(2, -3)$ is an eigenvector.
Now consider $\lambda = -0.03$:

$$A + 0.03I = \begin{bmatrix} -0.06 & 0.04 \\ 0.09 & -0.06 \end{bmatrix} \quad \to \quad \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix}$$

$(2, 3)$ is an eigenvector.
The solution is

$$\vec{x} = c_1 e^{-0.15t} \begin{bmatrix} 2 \\ -3 \end{bmatrix} + c_2 e^{-0.03t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

When $t = 0$, $x = 36$ and $y = 0$. Plug in:

$$\begin{bmatrix} 36 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Solving for the constants, I obtain $c_1 = 9$, $c_2 = 9$. Thus,

$$\vec{x} = 9e^{-0.15t} \begin{bmatrix} 2 \\ -3 \end{bmatrix} + 9e^{-0.03t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \quad \square$$

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12. Suppose that A is a real 2×2 matrix and

$$e^{At} = \begin{bmatrix} \frac{7}{8}e^{6t} + \frac{1}{8}e^{-2t} & \frac{1}{8}e^{6t} - \frac{1}{8}e^{-2t} \\ \frac{7}{8}e^{6t} - \frac{7}{8}e^{-2t} & \frac{1}{8}e^{6t} + \frac{7}{8}e^{-2t} \end{bmatrix}.$$

Find A.

$$\frac{d}{dt}e^{At} = \begin{bmatrix} \frac{21}{4}e^{6t} - \frac{1}{4}e^{-2t} & \frac{3}{4}e^{6t} + \frac{1}{4}e^{-2t} \\ \frac{21}{4}e^{6t} + \frac{7}{4}e^{-2t} & \frac{3}{4}e^{6t} - \frac{7}{4}e^{-2t} \end{bmatrix}.$$

Setting $t = 0$, I have

$$A = \left[\frac{d}{dt}e^{At}\right]_{t=0} = \begin{bmatrix} 5 & 1 \\ 7 & -1 \end{bmatrix}. \quad \square$$

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13. Find e^{At} for

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

$$e^{At} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-5t} & 0 \\ 0 & 0 & e^{7t} \end{bmatrix}. \quad \square$$

Here's a summary of the algorithm for computing e^{At}.

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the eigenvalues of A.

(If the characteristic polynomial has a multiple root, you list the eigenvalue multiple times. For instance, if the characteristic polynomial has a factor of $(x-5)^3$, you list the eigenvalue 5 three times.)

Define

$$B_1 = I, \quad B_k = (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_{k-1} I) \quad \text{for} \quad k = 2, \ldots, n.$$

$$a_1(t) = e^{\lambda_1 t}, \quad a_k(t) = \int_0^t e^{\lambda_k (t-u)} a_{k-1}(u)\, du = e^{\lambda_k t} \int_0^t e^{-\lambda_k u} a_{k-1}(u)\, du.$$

Then

$$e^{At} = a_1(t) B_1 + a_2(t) B_2 + \cdots + a_n(t) B_n.$$

Note that in defining the $B's$, you never get to the last eigenvalue λ_n. But λ_n *does* occur in $a_n(t)$.

If you have complex roots, the result may have complex numbers in it until you do some simplification.

---

14. Compute e^{At} for

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}.$$

The characteristic polynomial is

$$|A - xI| = \begin{vmatrix} 2-x & -1 \\ 1 & 4-x \end{vmatrix} = (2-x)(4-x) - (-1)(1) = x^2 - 6x + 9 = (x-3)^2.$$

The eigenvalue is $x = 3$ (double), so my list of eigenvalues is $\{3, 3\}$.
First,

$$B_1 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_2 = A - 3I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Next, $a_1(t) = e^{3t}$ and

$$a_2(t) = e^{3t} \int_0^t e^{-3u} a_1(u)\, du = e^{3t} \int_0^t e^{-3u} e^{3u}\, du = e^{3t} \int_0^t du = te^{3t}.$$

Thus,

$$e^{At} = e^{3t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{3t} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} e^{3t} - te^{3t} & -te^{3t} \\ te^{3t} & e^{3t} + te^{3t} \end{bmatrix}. \quad \square$$

---

15. Compute e^{At} for

$$A = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}.$$

Since the matrix is upper triangular, the eigenvalues are the diagonal elements 3 and -2.
I have

$$B_1 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B_2 = A - 3I = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}.$$

Next, $a_1(t) = e^{3t}$ and

$$a_2(t) = e^{-2t} \int_0^t e^{2u} a_1(u)\, du = e^{-2t} \int_0^t e^{2u} e^{3u}\, du = e^{-2t} \int_0^t e^{5u}\, du = e^{-2t} \cdot \frac{1}{5}(e^{5t} - 1) = \frac{1}{5} e^{3t} - \frac{1}{5} e^{-2t}.$$

So

$$e^{At} = e^{3t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{1}{5} e^{3t} - \frac{1}{5} e^{-2t} \right) \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} e^{3t} & \frac{1}{5}(e^{3t} - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}. \quad \square$$

---

16. Compute e^{At} for

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

Since the matrix is lower triangular, the eigenvalues are the diagonal entries 1 (double) and 2. I will list the eigenvalues in this order: $\{1, 1, 2\}$.
I have

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_2 = A - I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \quad B_3 = (A - I)(A - I) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}.$$

Moreover,

$$a_1(t) = e^t,$$

$$a_2(t) = \int_0^t e^{t-u} e^u\, du = e^t \int_0^t e^{-u} e^u\, du = e^t \int_0^t du = e^t \left[u\right]_0^t = te^t,$$

$$a_3(t) = \int_0^t e^{2(t-u)} ue^u\, du = e^{2t} \int_0^t e^{-2u} ue^u\, du = e^{2t} \int_0^t ue^{-u}\, du = e^{2t} \left[-ue^{-u} - e^{-u}\right]_0^t = e^{2t} - te^t - e^t.$$

Here's the work for the third integral, which is done by parts:

$$\begin{array}{ccc} & \dfrac{d}{du} & \displaystyle\int du \\ + & u & e^{-u} \\ - & 1 & -e^{-u} \\ + & 0 & e^{-u} \end{array}$$

$$\int ue^{-u}\,du = -ue^{-u} - e^{-u} + C.$$

Hence,

$$e^{At} = e^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} + (e^{2t} - te^t - e^t) \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} e^t & 0 & 0 \\ e^{2t} - e^t & e^{2t} & 0 \\ 2te^t + e^t - e^{2t} & e^t - e^{2t} & e^t \end{bmatrix}. \quad \square$$

---

17. Let $u \cdot v$ denote the standard inner product on $\mathbb{R}^4$.

(a) Find $(1, -2, 5, 14) \cdot (3, 1, 2, -1)$.

(b) Find the cosine of the angle between $(1, 1, -2, 5)$ and $(2, 2, 0, 7)$.

(c) Find a nonzero vector that is perpendicular to both $(1, -3, 6, 0)$ and $(2, 1, 11)$.

(a)

$$(1, -2, 5, 14) \cdot (3, 1, 2, -1) = 3 - 2 + 10 - 14 = -3. \quad \square$$

(b)

$$\cos\theta = \frac{(1, 1, -2, 5) \cdot (2, 2, 0, 7)}{\|(1, 1, -2, 5)\| \|(2, 2, 0, 7)\|} = \frac{2 + 2 + 0 + 35}{\sqrt{31}\sqrt{57}} = \frac{39}{\sqrt{1767}}. \quad \square$$

(c) I want (a, b, c, d) such that

$$(1, -3, 6, 0) \cdot (a, b, c, d) = 0 \quad \text{and} \quad (2, 1, 11) \cdot (a, b, c, d) = 0.$$

This gives the equations

$$\begin{aligned} a - 3b + 6c &= 0 \\ 2a + b + c + d &= 0 \end{aligned}$$

Row reduce to solve:

$$[1 \quad -3 \quad 6 \quad 0 \quad 02 \quad 1 \quad 1 \quad 1 \quad 0] \quad \to \quad \begin{bmatrix} 1 & 0 & \frac{9}{7} & \frac{3}{7} & 0 \\ 0 & 1 & -\frac{11}{7} & \frac{1}{7} & 0 \end{bmatrix}$$

The parametric solutions are

$$\begin{aligned} a &= -\frac{9}{7}s - \frac{3}{7}t \\ b &= \frac{11}{7}s - \frac{1}{7}t \\ c &= s \\ d &= t \end{aligned}$$

I can get a nonzero solution by setting at least one of s, t equal to a nonzero value. For example, taking $s = 7$ and $t = 7$, I get

$$(a, b, c, d) = (-12, 10, 7, 7). \quad \square$$

---

18. Suppose that u, v, and w are vectors in a real inner product space, and

$$\|u\| = 5, \quad \|v\| = 3, \quad \|w\| = 2,$$

$$\langle u, v \rangle = -2, \quad \langle u, w \rangle = 6, \quad \langle v, w \rangle = 10.$$

(a) Find $\langle 3u + v, v + 2w \rangle$.

(b) Find $\|u + w\|$.

(a)

$$\langle 3u + v, v + 2w \rangle = 3 \langle u, v \rangle + 6 \langle u, w \rangle + \langle v, v \rangle + 2 \langle v, w \rangle = 3(-2) + 6(6) + 3^2 + 2(10) = -6 + 36 + 9 + 20 = 59. \quad \square$$

(b) Note that $\|u + w\|^2 = \langle u + w, u + w \rangle$. Now

$$\langle u + w, u + w \rangle = \langle u, u \rangle + 2 \langle u, w \rangle + \langle w, w \rangle = 5^2 + 2(6) + 2^2 = 25 + 12 + 4 = 41.$$

Hence, $\|u + w\| = \sqrt{41}$. $\square$

---

19. Find an orthonormal basis relative to the standard dot product on $\mathbb{R}^4$ for the subspace spanned by the set

$$\{(1, 0, 1, 1), (4, 1, 5, 0), (4, 45, 1, 7)\}.$$

First, I'll get an orthogonal basis for the subspace. At the end, I'll divide each vector by its length to get an orthonormal set. To get the orthogonal basis, apply Gram-Schmidt to the original set of vectors.

The first vector in the orthogonal set will be $w_1 = (1, 0, 1, 1)$. The second vector is

$$w_2 = (4, 1, 5, 0) - \frac{(4, 1, 5, 0) \cdot (1, 0, 1, 1)}{(1, 0, 1, 1) \cdot (1, 0, 1, 1)} (1, 0, 1, 1) = (1, 1, 2, -3).$$

The third vector is

$$w_3 = (4, 45, 1, 7) - \frac{(4, 45, 1, 7) \cdot (1, 0, 1, 1)}{(1, 0, 1, 1) \cdot (1, 0, 1, 1)} (1, 0, 1, 1) - \frac{(4, 45, 1, 7) \cdot (1, 1, 2, -3)}{(1, 1, 2, -3) \cdot (1, 1, 2, -3)} (1, 1, 2, -3) =$$

$$(4, 45, 1, 7) - (4, 0, 4, 4) - (2, 2, 4, -6) = (-2, 43, -7, 9).$$

To get the orthonormal basis, divide w_1, w_2, and w_3 by their lengths. The orthonormal basis is

$$\left\{ \frac{1}{\sqrt{3}} (1, 0, 1, 1), \frac{1}{\sqrt{15}} (1, 1, 2, -3), \frac{1}{\sqrt{1983}} (-2, 43, -7, 9) \right\}. \quad \square$$

---

20. Let $C[0, 1]$ denote the real vector space of continuous real-valued functions on the interval $[0, 1]$. An inner product is defined on $C[0, 1]$ by

$$\langle f, g \rangle = \int_0^1 f(x) g(x) \, dx.$$

(a) Compute $\langle f, g\rangle$, where $f(x) = x$ and $g(x) = \cos \frac{\pi}{2} x$.

(b) Find $\|h\|$, where $h(x) = 2x + 1$.

(c) For what value of k are the functions $f(x) = x + k$ and $g(x) = x^2$ orthogonal?

(d) Consider the set of functions $S = \{x + 1, 2x\}$. Find an orthonormal set which spans the same subspace of $C[0, 1]$ as S.

(a)

$$\langle f, g\rangle = \int_0^1 x \cos \frac{\pi}{2} x \, dx = \left[\frac{2}{\pi} x \sin \frac{\pi}{2} x + \frac{4}{\pi^2} \cos \frac{\pi}{2} x\right]_0^1 = \frac{2}{\pi} - \frac{4}{\pi^2}. \quad \square$$

(b)

$$\|h\|^2 = \langle h, h\rangle = \int_0^1 (2x + 1)^2 \, dx = \int_0^1 (4x^2 + 4x + 1) \, dx = \left[\frac{4}{3} x^3 + 2x^2 + x\right]_0^1 = \frac{13}{3}.$$

Hence, $\|h\| = \sqrt{\frac{13}{3}}$. $\square$

(c)

$$\langle f, g\rangle = \int_0^1 x^2 (x + k) \, dx = \int_0^1 (x^3 + kx^2) \, dx = \left[\frac{1}{4} x^4 + \frac{1}{3} kx^3\right]_0^1 = \frac{1}{4} + \frac{1}{3} k.$$

Setting $\langle f, g\rangle = 0$, I have

$$\frac{1}{4} + \frac{1}{3} k = 0, \quad \text{so} \quad k = -\frac{3}{4}. \quad \square$$

(d) I have to do Gram-Schmidt on the set S. The first vector will be $v_1 = x + 1$.
For the second vector, I have

$$v_2 = 2x - \frac{\langle x + 1, 2x\rangle}{\langle x + 1, x + 1\rangle} (x + 1).$$

I need to compute the integrals in the top and bottom of the fraction:

$$\langle x + 1, 2x\rangle = \int_0^1 (x + 1)(2x) \, dx = \int_0^1 (2x^2 + 2x) \, dx = \left[\frac{2}{3} x^3 + x^2\right]_0^1 = \frac{5}{3}.$$

$$\langle x + 1, x + 1\rangle = \int_0^1 (x + 1)(x + 1) \, dx = \int_0^1 (x + 1)^2 \, dx = \left[\frac{1}{3} (x + 1)^3\right]_0^1 = \frac{7}{3}.$$

Returning to v_2, I have

$$v_2 = 2x - \frac{\frac{5}{3}}{\frac{7}{3}} (x + 1) = 2x - \frac{5}{7} (x + 1) = \frac{9}{7} x - \frac{5}{7}.$$

I may multiply by 7 to clear denominators to obtain $v_2' = 9x - 5$.

Finally, I have to divide $x+1$ and $9x-5$ by their lengths. Since $\langle x + 1, x + 1\rangle = \frac{7}{3}$, I know $\|x+1\| = \sqrt{\frac{7}{3}}$.
Also,

$$\langle 9x - 5, 9x - 5\rangle = \int_0^1 (9x - 5)(9x - 5) \, dx = \int_0^1 (9x - 5)^2 \, dx = \left[\frac{1}{27} (9x - 5)^3\right]_0^1 = 7.$$

So $\|9x-5\| = \sqrt{7}$. The orthonormal set is

$$\left\{\frac{\sqrt{3}}{\sqrt{7}}(x+1), \frac{1}{\sqrt{7}}(9x-5)\right\}. \quad \square$$

---

21. The following set of vectors in $\mathbb{R}^3$ is orthonormal relative to the standard dot product:

$$u_1 = \frac{1}{\sqrt{2}}(1,0,1), \quad u_2 = \frac{1}{\sqrt{3}}(1,-1,-1), \quad u_3 = \frac{1}{\sqrt{6}}(-1,-2,1).$$

Find the components of $(5,-4,2)$ relative to the basis $\{u_1, u_2, u_3\}$.

Since the basis is orthonormal, I can find the components by taking the dot product of $(5,-4,2)$ with each of the vectors:

$$(5,-4,2) \cdot \frac{1}{\sqrt{2}}(1,0,1) = \frac{7}{\sqrt{2}}.$$

$$(5,-4,2) \cdot \frac{1}{\sqrt{3}}(1,-1,-1) = \frac{7}{\sqrt{3}}.$$

$$(5,-4,2) \cdot \frac{1}{\sqrt{6}}(-1,-2,1) = \frac{5}{\sqrt{6}}.$$

That is,

$$(5,-4,2) = \frac{7}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(1,0,1) + \frac{7}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}(1,-1,-1) + \frac{5}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}(-1,-2,1). \quad \square$$

---

22. An inner product is defined on $\mathbb{R}^2$ by

$$\langle x, y\rangle = x^T \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix} y.$$

(a) Find $\|(1,-1)\|$ relative to the given inner product.

(b) Find $\cos\theta$, where θ is the angle relative to the given inner product between $(1,-1)$ and $(1,1)$.

(c) Find a nonzero vector (a,b) which is orthogonal to $(3,1)$ relative to this inner product.

(a)

$$\|(1,-1)\| = \langle (1,-1),(1,-1)\rangle^{1/2} = \left(\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)^{1/2} = \sqrt{17}. \quad \square$$

(b)

$$\cos\theta = \frac{\langle (1,-1),(1,1)\rangle}{\|(1,-1)\|\|(1,1)\|}.$$

Now

$$\langle (1,-1),(1,1)\rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5,$$

$$\|(1,1)\| = \langle (1,1),(1,1)\rangle^{1/2} = \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^{1/2} = \sqrt{13}.$$

Hence,

$$\cos\theta = \frac{5}{\sqrt{17}\sqrt{13}} = \frac{5}{\sqrt{221}}. \quad \square$$

(c) Find a nonzero vector (a, b) which is orthogonal to $(3, 1)$ relative to this inner product.

I want

$$\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 10 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

Multiplying out gives the equation

$$29a + 2b = 0.$$

A nonzero solution is given by $(a, b) = (2, -29)$. $\square$

---

23. Let x be a fixed vector in a real inner product space V. Let

$$x^\perp = \{v \in V \mid \langle x, v\rangle = 0\}.$$

Prove that $x^\perp$ is a subspace of V. ($x^\perp$ is called the **orthogonal complement** of x.)

I must show that $x^\perp$ is closed under sums and under scalar multiplication.
Let $v, w \in x^\perp$. Then

$$\begin{array}{rcll} \langle x, v + w\rangle & = & = \langle x, v\rangle + \langle x, w\rangle & \text{(Linearity of inner products)} \\ & = & 0 + 0 & (v, w \in x^\perp) \\ & = & 0 & \end{array}$$

Therefore, $v + w \in x^\perp$.
Let $v \in x^\perp$ and let $k \in \mathbb{R}$. Then

$$\begin{array}{rcll} \langle x, kv\rangle & = & k \cdot \langle x, v\rangle & \text{(Linearity of inner products)} \\ & = & k \cdot 0 & (v \in x^\perp) \\ & = & 0 & \end{array}$$

Therefore, $kv \in x^\perp$.
Hence, $x^\perp$ is a subspace of V. $\square$

If we can really understand the problem, the answer will come out of it, because the answer is not separate from the problem. - Jiddu Krishnamurti

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