

## Review Problems for Test 1

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

- Show that  $\neg(P \rightarrow Q)$  is equivalent to  $\neg(\neg P \vee Q)$ .
- Show that  $[(\neg P \vee Q) \wedge (\neg Q \vee R)] \rightarrow (P \rightarrow R)$  is a tautology.
- (a) Suppose  $P$  is false and  $P \vee \neg Q$  is true. Tell whether  $Q$  is true, false, or its truth or falsity can't be determined.  
(b) Suppose  $R$  is false and  $\neg P \wedge (Q \vee R)$  is true. Tell whether  $Q$  is true, false, or its truth or falsity can't be determined.  
(c) Suppose  $P \rightarrow (Q \vee R)$  is false. Tell whether  $R$  is true, false, or its truth or falsity can't be determined.
- In each case, determine whether the statement is true or false.
  - "If  $\frac{d}{dx} \cos x = \sin x$  for all  $x$ , then  $\frac{d}{dx} \tan x = \cot x$  for all  $x$ ."
  - "If Phoebe Small's dog needs a bath, then  $\pi \neq 3.14$ ."
  - "For all  $x$ , if  $|x - 7| = 9$ , then  $x = 16$ ."
- Translate each statement into logical notation, if  $A(x)$  means " $x$  is an abelian group" and  $C(x)$  means " $x$  is a cyclic group".

Note: You don't need to know what "abelian group" or "cyclic group" mean to do this.

- Every cyclic group is abelian.
  - There is an abelian group which is cyclic.
  - Not every abelian group is cyclic.
- Express the negation of each statement in words in such a way that only simple statements are negated.
    - Calvin is sleepy or Bonzo is late.
    - If Phoebe does not buy the cookies, then Calvin goes home.
    - Bonzo orders the stromboli if and only if Calvin does not eat the parmigiana.
  - Express the negation of each statement in words.
    - Some cakes and some pies enjoy bowling.
    - Every vegetable is puzzled by some calculus problem.
    - There's a roast beef sandwich which is admired by everyone.
  - Write the negation of each statement, simplifying your answer so that only simple statements are negated.
    - $(P \wedge \neg Q) \rightarrow R$
    - $P \leftrightarrow (Q \vee R)$

9. Write the negation of each statement, simplifying your answer so that only simple statements are negated.

(a)  $\forall x(P(x) \wedge Q(X) \wedge \neg R(x))$

(b)  $\forall x\exists y((P(x) \wedge P(y)) \rightarrow \neg Q(x, y))$

10. The square of a real number is nonnegative: If  $w \in \mathbb{R}$ , then  $w^2 \geq 0$ .

This fact is often useful in proving other inequalities.

(a) Explain what is wrong with this statement: “If  $w \in \mathbb{R}$ , then  $w^2$  is positive.”

(b) Prove that if  $x \in \mathbb{R}$ , then

$$x^2 + 25 \geq 10x.$$

(c) Prove that if  $x \in \mathbb{R}$ , then

$$(x^4 + 2x^2 + 1) + \frac{1}{x^4 + 2x^2 + 1} \geq 2.$$

11. Use the Triangle Inequality to prove:

(a) If  $x \in \mathbb{R}$ , then  $|(x - 1)^2| + |2x| \geq x^2 + 1$ .

(b) If  $x, y, z \in \mathbb{R}$ , then

$$|(x - y)(x + y)| + |(y - z)(y + z)| \geq |(x - z)(x + z)|.$$

12. A real number  $x$  is **rational** if it can be written in the form  $x = \frac{a}{b}$ , where  $a$  and  $b$  are integers. (It’s implicit in this that  $b \neq 0$ . For if  $b = 0$ , then  $\frac{a}{b}$  is undefined, and it can’t be equal to the real number  $x$ .)

(a) Prove that the sum of two rational numbers is rational.

(b) Prove that the product of two rational numbers is rational.

13. Prove that if  $a_1, a_2, b_1, b_2 \in \mathbb{R}$ , then

$$(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2).$$

(This is called the **Cauchy-Schwartz Inequality**.)

Hint: Start with  $(a_1b_2 - a_2b_1)^2 \geq 0$ .

14. Prove:  $B \wedge F$ .

$$\text{Premises: } \begin{cases} A \wedge B \\ (E \vee \neg F) \rightarrow C \\ (A \vee D) \rightarrow \neg C \end{cases}$$

15. Prove:  $D$                       Premises:  $\begin{cases} A \wedge \neg C \\ \neg D \rightarrow (B \vee C) \\ \neg A \vee \neg B \end{cases}$ .

16. Prove:  $C$                       Premises:  $\begin{cases} \neg A \rightarrow (C \wedge D) \\ A \rightarrow B \\ \neg B \end{cases}$ .

17. Prove:  $\neg D$                       Premises:  $\begin{cases} A \leftrightarrow B \\ \neg(A \rightarrow B) \vee (B \rightarrow \neg C) \\ D \rightarrow (A \wedge C) \end{cases}$ .

18. Criticize the following “proof” and write a correct proof. What fundamental logical error is being made?

“To prove:  $\sin x \tan x = \sec x - \cos x$ .

$$\begin{aligned}\sin x \tan x &= \sec x - \cos x \\ \sin x \cdot \frac{\sin x}{\cos x} &= \frac{1}{\cos x} - \cos x \\ \cos x \cdot \sin x \cdot \frac{\sin x}{\cos x} &= \cos x \cdot \left( \frac{1}{\cos x} - \cos x \right) \\ (\sin x)^2 &= 1 - (\cos x)^2\end{aligned}$$

The last statement is true, so the original statement is true.”

19. Is the following statement true or false?

“If  $2x + 1 = 7$ , then either  $x = 3$  or  $x = -3$ .”

20. In calculus you learn that a differentiable function is constant if and only if its derivative is identically 0. Use this fact to prove that  $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$  for all  $x \neq 0$ .

Hint: First show by computation that the derivative of  $\tan^{-1} x + \tan^{-1} \frac{1}{x}$  is 0. This shows that it equals a constant  $k$ . Find the value of  $k$  by setting  $x = 1$ .

21. An integer  $n$  is **divisible by 3** if  $n = 3m$  for some integer  $m$ .

(a) Using this definition, prove that if  $n$  is divisible by 3, then  $2n^2 + 7n + 9$  is divisible by 3.

(b) Calvin Butterball complains that  $2 \cdot 2^2 + 7 \cdot 2 + 9 = 31$ , which is not divisible by 3. Does he have a valid complaint?

(c) Show that the following statement is *false*: “If  $n$  is divisible by 3, then  $2n^2 + 7n + 1$  is divisible by 3.” (Notice that the statement is an if-then statement. When is an if-then statement false?)

22. Prove that if  $x^2 + 5x - 2 \not\geq 12$ , then  $x \not\geq 2$ .

23. Prove:  $P \rightarrow (Q \wedge R)$       Premises:  $\begin{cases} P \rightarrow S \\ S \rightarrow R \\ (\neg Q \vee \neg T) \rightarrow \neg S \end{cases}$ .

24. Prove:  $P \rightarrow (Q \wedge S)$       Premises:  $\begin{cases} (\neg Q \vee R) \rightarrow \neg P \\ R \vee S \end{cases}$ .

25. Prove:  $(A \vee B) \rightarrow E$ .

Premises:  $\begin{cases} A \rightarrow E \\ B \rightarrow (\neg C \rightarrow D) \\ \neg C \wedge \neg D \end{cases}$ .

26. Prove that if  $x > 4$ , then  $x^3 + 3x + 5 > 78$ .

27. (a) Prove that there is an integer which is divisible by both 4 and 6 but is not divisible by 24.

(b) Prove that there is an integer which is divisible by both 4 and 6 and is greater than 24 but is not divisible by 24.

(c) Prove that there is an  $x \in \mathbb{R}$  such that  $x^3 > 100x^2$ .

(d) Prove that there is an  $x \in \mathbb{R}$  such that

$$x^2 + 0.12 < 0.7x.$$

28. Prove that there is an  $x \in \mathbb{R}$  such that  $x = (\sin x)^2$ .

29. When the Mean Value Theorem is applied to the function  $f(x) = \frac{x^2}{2} + \sin x$  on the interval  $[0, \pi]$ , it says that there is a number  $c$  such that

$$\frac{\pi}{2} = c - \cos c, \quad \text{where } 0 < c < \pi.$$

Calvin Butterball complains: “There must be something wrong — you can’t solve this equation algebraically for  $c$ .” Does Calvin have a valid objection?

30. Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function, and

$$f(3) = 17 \quad \text{and} \quad f(7) = -11.$$

Prove that there is an  $x$  such that  $3 \leq x \leq 7$  and  $x \cdot f(x) = 0$ .

31. Suppose that  $f$  is a continuous function satisfying

$$f(3) = 100 \quad \text{and} \quad f(7) = 12.$$

Prove that  $f(x) - x^2 = 0$  for some  $x \in [3, 7]$ .

32. Give an  $\epsilon$ - $\delta$  proof that  $\lim_{x \rightarrow 2} (x^2 - 7x) = -10$ .

33. Give an  $\epsilon$ - $\delta$  proof that  $\lim_{x \rightarrow 3} \frac{x+7}{x-1} = 5$ .

34. Give an  $\epsilon$ - $\delta$  proof that  $\lim_{x \rightarrow 1} (3x^2 + 4x) = 7$ .

35. Give an  $\epsilon$ - $\delta$  proof that  $\lim_{x \rightarrow 1} \frac{1}{x} = 1$ .

## Solutions to the Review Problems for Test 1

1. Show that  $\neg(P \rightarrow Q)$  is equivalent to  $\neg(\neg P \vee Q)$ .

$P$	$Q$	$P \rightarrow Q$	$\neg(P \rightarrow Q)$	$\neg P$	$\neg P \vee Q$	$\neg(\neg P \vee Q)$
T	T	T	F	F	T	F
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	F	T	T	F

The columns for  $\neg(P \rightarrow Q)$  and  $\neg(\neg P \vee Q)$  are identical, so the statements are logically equivalent.  $\square$

2. Show that  $[(\neg P \vee Q) \wedge (\neg Q \vee R)] \rightarrow (P \rightarrow R)$  is a tautology.

$P$	$Q$	$R$	$\neg P$	$\neg Q$	$\neg P \vee Q$	$\neg Q \vee R$	$P \rightarrow R$
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	F	T	T
T	F	F	F	T	F	T	F
F	T	T	T	F	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$(\neg P \vee Q) \wedge (\neg Q \vee R)$	$[(\neg P \vee Q) \wedge (\neg Q \vee R)] \rightarrow (P \rightarrow R)$
T	T
F	T
F	T
F	T
T	T
F	T
T	T
T	T

The column for  $[(\neg P \vee Q) \wedge (\neg Q \vee R)] \rightarrow (P \rightarrow R)$  contains only T's, so the statement is a tautology.

□

3. (a) Suppose  $P$  is false and  $P \vee \neg Q$  is true. Tell whether  $Q$  is true, false, or its truth or falsity can't be determined.

(b) Suppose  $R$  is false and  $\neg P \wedge (Q \vee R)$  is true. Tell whether  $Q$  is true, false, or its truth or falsity can't be determined.

(c) Suppose  $P \rightarrow (Q \vee R)$  is false. Tell whether  $R$  is true, false, or its truth or falsity can't be determined.

(a) Since  $P \vee \neg Q$  is true,  $P$  is true or  $\neg Q$  is true. But  $P$  is false. Hence,  $\neg Q$  is true. Therefore,  $Q$  is false. □

(b) Since  $\neg P \wedge (Q \vee R)$  is true,  $\neg P$  is true and  $Q \vee R$  is true. (Only the second part is relevant.) Since  $Q \vee R$  is true,  $Q$  is true or  $R$  is true. Since  $R$  is false,  $Q$  must be true. □

(c) Since  $P \rightarrow (Q \vee R)$  is false,  $P$  is true and  $Q \vee R$  is false. (Only the second part is relevant.) Since  $Q \vee R$  is false, both  $Q$  and  $R$  must be false — and so in particular,  $R$  is false. □

4. In each case, determine whether the statement is true or false.

(a) “If  $\frac{d}{dx} \cos x = \sin x$  for all  $x$ , then  $\frac{d}{dx} \tan x = \cot x$  for all  $x$ .”

(b) “If Phoebe Small’s dog needs a bath, then  $\pi \neq 3.14$ .”

(c) “For all  $x$ , if  $|x - 7| = 9$ , then  $x = 16$ .”

(a) Since  $\frac{d}{dx} \cos x \neq \sin x$  — in fact,  $\frac{d}{dx} \cos x = -\sin x$  — the if-part of the conditional is false. Hence, the conditional is true (regardless of the fact that the then-part is false).  $\square$

(b) The then-part of the conditional is true:  $\pi$  is only *approximately* equal to 3.14. Hence, the conditional is true (regardless of whether the if-part is true or false).  $\square$

(c) In this case, the conditional is inside a quantifier. I can't determine the truth or falsity of the if-part or the then-part separately. So I have to think about whether the statement is true *mathematically*.

$$\begin{array}{ccc}
 & |x - 7| = 9 & \\
 & x - 7 = \pm 9 & \\
 \swarrow & & \searrow \\
 x - 7 = 9 & & x - 7 = -9 \\
 x = 16 & & x = -2
 \end{array}$$

I can see that the statement is not universally true — it is not true *for all*  $x$ . If  $x = -2$ , then the if-part is true, but the then-part is false, making the conditional false. (The statement is true if  $x = 16$ .)  $\square$

5. Translate each statement into logical notation, if  $A(x)$  means “ $x$  is an abelian group” and  $C(x)$  means “ $x$  is a cyclic group”.

(a) Every cyclic group is abelian.

(b) There is an abelian group which is cyclic.

(c) Not every abelian group is cyclic.

(a)  $\forall x(C(x) \rightarrow A(x)) \quad \square$

(b)  $\exists x(A(x) \wedge C(x)) \quad \square$

(c)  $\neg \forall x(A(x) \rightarrow C(x)) \quad \square$

6. Express the negation of each statement in words in such a way that only simple statements are negated.

(a) Calvin is sleepy or Bonzo is late.

(b) If Phoebe does not buy the cookies, then Calvin goes home.

(c) Bonzo orders the stromboli if and only if Calvin does not eat the parmigiana.

(a) The negation is “Calvin isn't sleepy and Bonzo isn't late”.  $\square$

(b) The original statement is equivalent to “Phoebe buys the cookies or Calvin goes home”. Hence, the negation is “Phoebe doesn't buy the cookies and Calvin doesn't go home”.  $\square$

(c) You could translate the original statement into a conjunction of two conditionals, each of which can be translated as a disjunction as in part (b). However, it's simpler to note that the original biconditional is true exactly when both pieces have the same truth values, and false otherwise. Hence, the negation should be true exactly when the two pieces have opposite truth values, and false otherwise.

If I negate one piece of the biconditional, it has the effect of making the biconditional true exactly when the original pieces have opposite truth values. It follows that negating one piece of the biconditional gives the negation of the original biconditional:  $\neg(P \leftrightarrow Q) \leftrightarrow (\neg P \leftrightarrow Q)$ .

Hence, the negation of the original statement is “Bonzo does not order the stromboli if and only if Calvin does not eat the parmigiana”.  $\square$

7. Express the negation of each statement in words.

- (a) Some cakes and some pies enjoy bowling.  
 (b) Every vegetable is puzzled by some calculus problem.  
 (c) There’s a roast beef sandwich which is admired by everyone.  
 (a) Let

$$C(x) = \text{“x is a cake”}$$

$$P(x) = \text{“x is a pie”}$$

$$B(x) = \text{“x enjoys bowling”}$$

The given statement is

$$[\exists x(C(x) \wedge B(x))] \wedge [\exists y(P(y) \wedge B(y))]$$

Negate and simplify:

$$\begin{aligned} \neg([\exists x(C(x) \wedge B(x))] \wedge [\exists y(P(y) \wedge B(y))]) &\leftrightarrow \text{(DeMorgan)} \\ [\neg\exists x(C(x) \wedge B(x))] \vee [\neg\exists y(P(y) \wedge B(y))] &\leftrightarrow \text{(Negate a quantifier)} \\ [\forall x\neg(C(x) \wedge B(x))] \vee [\forall y\neg(P(y) \wedge B(y))] &\leftrightarrow \text{(DeMorgan)} \\ [\forall x(\neg C(x) \vee \neg B(x))] \vee [\forall y(\neg P(y) \vee \neg B(y))] & \end{aligned}$$

The last expression is hard to say in words. It’s easier to take the second expression, which would be: “No cakes enjoy bowling or no pies enjoy bowling”.  $\square$

(b) Let

$$V(x) = \text{“x is a vegetable”}$$

$$C(x) = \text{“x is a calculus problem”}$$

$$P(x, y) = \text{“x is puzzled by y”}$$

The given statement is

$$\forall x(V(x) \rightarrow \exists y(C(y) \wedge P(x, y)))$$

Negate and simplify:

$$\begin{aligned} \neg\forall x(V(x) \rightarrow \exists y(C(y) \wedge P(x, y))) &\leftrightarrow \text{(Negate a quantifier)} \\ \exists x\neg(V(x) \rightarrow \exists y(C(y) \wedge P(x, y))) &\leftrightarrow \text{(Cond. disjunction)} \\ \exists x\neg(\neg V(x) \vee \exists y(C(y) \wedge P(x, y))) &\leftrightarrow \text{(DeMorgan)} \\ \exists x(V(x) \wedge \neg\exists y(C(y) \wedge P(x, y))) &\leftrightarrow \text{(Negate a quantifier)} \\ \exists x(V(x) \wedge \forall y\neg(C(y) \wedge P(x, y))) &\leftrightarrow \text{(DeMorgan)} \\ \exists x(V(x) \wedge \forall y(\neg C(y) \vee \neg P(x, y))) & \end{aligned}$$

The negation is “There’s a vegetable which isn’t puzzled by any calculus problem”. (Presumably, that vegetable got an **A** in calculus.)  $\square$

(c) Let

$$R(x) = \text{“x is a roast beef sandwich”}$$

$$A(x, y) = \text{“x is admired by y”}$$

The given statement is

$$\exists x[R(x) \wedge \forall yA(x, y)]$$

Negate and simplify:

$$\begin{aligned} \neg \exists x [R(x) \wedge \forall y A(x, y)] &\leftrightarrow \text{(Negate a quantifier)} \\ \forall x \neg [R(x) \wedge \forall y A(x, y)] &\leftrightarrow \text{(DeMorgan)} \\ \forall x [\neg R(x) \vee \neg \forall y A(x, y)] &\leftrightarrow \text{(Negate a quantifier)} \\ \forall x [\neg R(x) \vee \exists y \neg A(x, y)] & \end{aligned}$$

The negation is “Every roast beef sandwich is not admired by someone”. (Personally, I never met a roast beef sandwich I didn’t like.)  $\square$

8. Write the negation of each statement, simplifying your answer so that only simple statements are negated.

(a)  $(P \wedge \neg Q) \rightarrow R$

(b)  $P \leftrightarrow (Q \vee R)$

(a)

$$\begin{aligned} \neg [(P \wedge \neg Q) \rightarrow R] &\leftrightarrow \neg [\neg (P \wedge \neg Q) \vee R] && \text{(Conditional disjunction)} \\ &\leftrightarrow \neg \neg (P \wedge \neg Q) \wedge \neg R && \text{(DeMorgan)} \\ &\leftrightarrow (P \wedge \neg Q) \wedge \neg R && \text{(Double negation)} \quad \square \end{aligned}$$

(b)

$$\begin{aligned} \neg [P \leftrightarrow (Q \vee R)] &\leftrightarrow \neg [(P \rightarrow (Q \vee R)) \wedge ((Q \vee R) \rightarrow P)] && \text{(Biconditional)} \\ &\leftrightarrow \neg (P \rightarrow (Q \vee R)) \vee \neg ((Q \vee R) \rightarrow P) && \text{(DeMorgan)} \\ &\leftrightarrow \neg (\neg P \vee (Q \vee R)) \vee \neg (\neg(Q \vee R) \vee P) && \text{(Cond. disjunction)} \\ &\leftrightarrow (\neg \neg P \wedge \neg(Q \vee R)) \vee (\neg \neg(Q \vee R) \wedge \neg P) && \text{(DeMorgan)} \\ &\leftrightarrow (P \wedge \neg(Q \vee R)) \vee ((Q \vee R) \wedge \neg P) && \text{(Double negation)} \\ &\leftrightarrow (P \wedge (\neg Q \wedge \neg R)) \vee ((Q \vee R) \wedge \neg P) && \text{(DeMorgan)} \quad \square \end{aligned}$$

9. Write the negation of each statement, simplifying your answer so that only simple statements are negated.

(a)  $\forall x (P(x) \wedge Q(x) \wedge \neg R(x))$

(b)  $\forall x \exists y ((P(x) \wedge P(y)) \rightarrow \neg Q(x, y))$

(a)

$$\begin{aligned} \neg \forall x (P(x) \wedge Q(x) \wedge \neg R(x)) &\leftrightarrow \exists x \neg (P(x) \wedge Q(x) \wedge \neg R(x)) && \text{(Negate a quantifier)} \\ &\leftrightarrow \exists x (\neg P(x) \vee \neg Q(x) \vee \neg \neg R(x)) && \text{(DeMorgan)} \\ &\leftrightarrow \exists x (\neg P(x) \vee \neg Q(x) \vee R(x)) && \text{(Double negation)} \quad \square \end{aligned}$$

(b)

$$\begin{aligned} \neg \forall x \exists y ((P(x) \wedge P(y)) \rightarrow \neg Q(x, y)) &\leftrightarrow \exists x \neg \exists y ((P(x) \wedge P(y)) \rightarrow \neg Q(x, y)) && \text{(Negate a quantifier)} \\ &\leftrightarrow \exists x \forall y \neg ((P(x) \wedge P(y)) \rightarrow \neg Q(x, y)) && \text{(Negate a quantifier)} \\ &\leftrightarrow \exists x \forall y \neg (\neg (P(x) \wedge P(y)) \vee \neg Q(x, y)) && \text{(Cond. disjunction)} \\ &\leftrightarrow \exists x \forall y (\neg \neg (P(x) \wedge P(y)) \wedge \neg \neg Q(x, y)) && \text{(DeMorgan)} \\ &\leftrightarrow \exists x \forall y ((P(x) \wedge P(y)) \wedge Q(x, y)) && \text{(Double negation)} \quad \square \end{aligned}$$

10. The square of a real number is nonnegative: If  $w \in \mathbb{R}$ , then  $w^2 \geq 0$ .

This fact is often useful in proving other inequalities.

(a) Explain what is wrong with this statement: “If  $w \in \mathbb{R}$ , then  $w^2$  is positive.”

(b) Prove that if  $x \in \mathbb{R}$ , then

$$x^2 + 25 \geq 10x.$$



(c) Prove that if  $x \in \mathbb{R}$ , then

$$(x^4 + 2x^2 + 1) + \frac{1}{x^4 + 2x^2 + 1} \geq 2.$$

(a) “If  $w \in \mathbb{R}$ , then  $w^2$  is positive” is false: If  $w = 0$ , then  $w^2 = 0$ , and 0 is *not* positive.  $\square$

(b)

$$\begin{aligned}(x - 5)^2 &\geq 0 \\ x^2 - 10x + 25 &\geq 0 \quad \square \\ x^2 + 25 &\geq 10x\end{aligned}$$

(c) Note that  $(x^2 + 1)^2 = x^4 + 2x^2 + 1$ . So

$$\begin{aligned}\left((x^2 + 1) - \frac{1}{x^2 + 1}\right)^2 &\geq 0 \\ (x^2 + 1)^2 - 2 + \frac{1}{(x^2 + 1)^2} &\geq 0 \quad \square \\ (x^2 + 1)^2 + \frac{1}{(x^2 + 1)^2} &\geq 2 \\ (x^4 + 2x^2 + 1)^2 + \frac{1}{x^4 + 2x^2 + 1} &\geq 2\end{aligned}$$

11. Use the Triangle Inequality to prove:

(a)

$$|(x - 1)^2| + |2x| \geq x^2 + 1.$$

(b) If  $x, y, z \in \mathbb{R}$ , then

$$|(x - y)(x + y)| + |(y - z)(y + z)| \geq |(x - z)(x + z)|.$$

Recall that the Triangle Inequality says that if  $x, y \in \mathbb{R}$ , then

$$|x| + |y| \geq |x + y|.$$

(a) If  $x \in \mathbb{R}$ , then  $|(x - 1)^2| + |2x| \geq x^2 + 1$ .

$$|(x - 1)^2| + |2x| = |x^2 - 2x + 1| + |2x| \geq |(x^2 - 2x + 1) + 2x| = |x^2 + 1| = x^2 + 1.$$

(I can remove the absolute values in the last step because  $x^2 + 1$  is always positive.)  $\square$

(b)

$$|(x - y)(x + y)| + |(y - z)(y + z)| = |x^2 - y^2| + |y^2 - z^2| \geq |(x^2 - y^2) + (y^2 - z^2)| = |x^2 - z^2| = |(x - z)(x + z)|. \quad \square$$

12. A real number  $x$  is **rational** if it can be written in the form  $x = \frac{a}{b}$ , where  $a$  and  $b$  are integers. (It's implicit in this that  $b \neq 0$ . For if  $b = 0$ , then  $\frac{a}{b}$  is undefined, and it can't be equal to the real number  $x$ .)

(a) Prove that the sum of two rational numbers is rational.

(b) Prove that the product of two rational numbers is rational.

(a) Let  $x$  and  $y$  be rational numbers. I must prove that  $x + y$  is rational.

Since  $x$  is rational, I can write  $x = \frac{a}{b}$ , where  $a$  and  $b$  are integers. Since  $y$  is rational, I can write  $y = \frac{c}{d}$ , where  $c$  and  $d$  are integers. Then

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Since  $a$ ,  $b$ ,  $c$ , and  $d$  are integers,  $ad + bc$  and  $bd$  are integers. Note that  $bd \neq 0$  — for otherwise, either  $b = 0$  and  $\frac{a}{b}$  is undefined, or  $d = 0$  and  $\frac{c}{d}$  is undefined.

Since  $x + y$  has been expressed as a quotient of two integers,  $x + y$  is rational.  $\square$

(b) Since  $x$  is rational, I can write  $x = \frac{a}{b}$ , where  $a$  and  $b$  are integers. Since  $y$  is rational, I can write  $y = \frac{c}{d}$ , where  $c$  and  $d$  are integers. Then

$$xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Since  $a$ ,  $b$ ,  $c$ , and  $d$  are integers,  $ac$  and  $bd$  are integers. Note that  $bd \neq 0$  — for otherwise, either  $b = 0$  and  $\frac{a}{b}$  is undefined, or  $d = 0$  and  $\frac{c}{d}$  is undefined.

Since  $xy$  has been expressed as a quotient of two integers,  $xy$  is rational.  $\square$

13. Prove that if  $a_1, a_2, b_1, b_2 \in \mathbb{R}$ , then

$$(a_1b_1 + a_2b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2).$$

(This is called the **Cauchy-Schwartz Inequality**.)

Since a square is always nonnegative, I have

$$\begin{aligned} (a_1b_2 - a_2b_1)^2 &\geq 0 \\ a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 &\geq 0 \\ a_1^2b_2^2 + a_2^2b_1^2 &\geq 2a_1a_2b_1b_2 \\ a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2 &\geq a_1^2b_1^2 + 2a_1a_2b_1b_2 + a_2^2b_2^2 \\ (a_1^2 + a_2^2)(b_1^2 + b_2^2) &\geq (a_1b_1 + a_2b_2)^2 \end{aligned}$$

I found this proof by starting with the inequality and working backwards *on scratch paper*. Be careful when you're writing a proof that you *do not start with what you're trying to prove*.  $\square$

14. Prove:  $B \wedge F$ .

$$\text{Premises: } \begin{cases} A \wedge B \\ (E \vee \neg F) \rightarrow C \\ (A \vee D) \rightarrow \neg C \end{cases}$$

- |     |                                 |                                    |           |
|-----|---------------------------------|------------------------------------|-----------|
| 1.  | $A \wedge B$                    | Premise                            |           |
| 2.  | $(E \vee \neg F) \rightarrow C$ | Premise                            |           |
| 3.  | $(A \vee D) \rightarrow \neg C$ | Premise                            |           |
| 4.  | $A$                             | Decomposing a conjunction (1)      |           |
| 5.  | $A \vee D$                      | Constructing a disjunction (4)     |           |
| 6.  | $\neg C$                        | Modus ponens (3, 5)                | $\square$ |
| 7.  | $\neg(E \vee \neg F)$           | Modus tollens (2, 6)               |           |
| 8.  | $\neg E \wedge F$               | DeMorgan (8)                       |           |
| 9.  | $F$                             | Decomposing a conjunction (8)      |           |
| 10. | $B$                             | Decomposing a conjunction (1)      |           |
| 11. | $B \wedge F$                    | Constructing a conjunction (9, 10) |           |

---

15. Prove:  $D$ .

$$\text{Premises: } \left\{ \begin{array}{l} A \wedge \neg C \\ \neg D \rightarrow (B \vee C) . \\ \neg A \vee \neg B \end{array} \right.$$

1.  $A \wedge \neg C$  Premise
2.  $\neg D \rightarrow (B \vee C)$  Premise
3.  $\neg A \vee \neg B$  Premise
4.  $A$  Decomposing a conjunction (1)
5.  $\neg C$  Decomposing a conjunction (1)
6.  $\neg B$  Disjunctive syllogism (3,4)
7.  $\neg B \wedge \neg C$  Constructing a conjunction (5,6)
8.  $\neg(B \vee C)$  DeMorgan (7)
9.  $D$  Modus tollens (2,8)  $\square$

---

16. Prove:  $C$ .

$$\text{Premises: } \left\{ \begin{array}{l} \neg A \rightarrow (C \wedge D) \\ A \rightarrow B \\ \neg B \end{array} \right. .$$

1.  $A \rightarrow B$  Premise
2.  $\neg B$  Premise
3.  $\neg A$  Modus tollens (1,3)
4.  $\neg A \rightarrow (C \wedge D)$  Premise
5.  $C \wedge D$  Modus ponens (3,4)
6.  $C$  Decomposing a conjunction (5)  $\square$

---

17. Prove:  $\neg D$ .

$$\text{Premises: } \left\{ \begin{array}{l} A \leftrightarrow B \\ \neg(A \rightarrow B) \vee (B \rightarrow \neg C) . \\ D \rightarrow (A \wedge C) \end{array} \right.$$

1.  $A \leftrightarrow B$  Premise
2.  $\neg(A \rightarrow B) \vee (B \rightarrow \neg C)$  Premise
3.  $D \rightarrow (A \wedge C)$  Premise
4.  $A \rightarrow B$  Definition of biconditional (1)
5.  $B \rightarrow \neg C$  Disjunctive syllogism (2,4)
6.  $A \rightarrow \neg C$  Rule of syllogism (4,5)
7.  $\neg A \vee \neg C$  Conditional disjunction (6)
8.  $\neg(A \wedge C)$  DeMorgan (7)
9.  $\neg D$  Modus tollens (3,8)  $\square$

---

18. Criticize the following “proof” and write a correct proof. What fundamental logical error is being made?

“To prove:  $\sin x \tan x = \sec x - \cos x$ .

$$\begin{aligned} \sin x \tan x &= \sec x - \cos x \\ \sin x \cdot \frac{\sin x}{\cos x} &= \frac{1}{\cos x} - \cos x \\ \cos x \cdot \sin x \cdot \frac{\sin x}{\cos x} &= \cos x \cdot \left( \frac{1}{\cos x} - \cos x \right) \\ (\sin x)^2 &= 1 - (\cos x)^2 \end{aligned}$$

The last statement is true, so the original statement is true.”

You can't prove something by assuming — starting with — what you want to prove. This is called *begging the question*, and it is a fundamental and very serious error of logic.

(The fact that the derivation ended in a true statement proves nothing, because a true conclusion can be deduced from a false premise.)

Here's a correct proof of the identity:

$$\sin x \tan x = \sin x \cdot \frac{\sin x}{\cos x} = \frac{(\sin x)^2}{\cos x} = \frac{1 - (\cos x)^2}{\cos x} = \frac{1}{\cos x} - \frac{(\cos x)^2}{\cos x} = \sec x - \cos x. \quad \square$$

19. Is the following statement true or false?

“If  $2x + 1 = 7$ , then either  $x = 3$  or  $x = -3$ .”

If “ $2x + 1 = 7$ ” is false, then the conditional is true.

On the other hand, if “ $2x + 1 = 7$ ” is true, then it follows by basic algebra that  $x = 3$ . Therefore, the disjunction “ $x = 3$  or  $x = -3$ ” is true. Hence, the conditional is true.

Therefore, the conditional is true.  $\square$

20. In calculus you learn that a differentiable function is constant if and only if its derivative is identically

0. Use this fact to prove that  $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$  for all  $x \neq 0$ .

Let  $f(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x}$ . Then

$$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1}{1+x^2} + \frac{-1}{x^2+1} = 0.$$

Hence,  $f(x)$  is constant — say  $f(x) = \tan^{-1} x + \tan^{-1} \frac{1}{x} = k$ . Setting  $x = 1$ , I have

$$\tan^{-1} 1 + \tan^{-1} 1 = k, \quad \text{so} \quad \frac{\pi}{2} = k.$$

Hence,  $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$ .  $\square$

21. An integer  $n$  is **divisible by 3** if  $n = 3m$  for some integer  $m$ .

(a) Using this definition, prove that if  $n$  is divisible by 3, then  $2n^2 + 7n + 9$  is divisible by 3.

(b) Calvin Butterball complains that  $2 \cdot 2^2 + 7 \cdot 2 + 9 = 31$ , which is not divisible by 3. Does he have a valid complaint?

(c) Show that the following statement is *false*: “If  $n$  is divisible by 3, then  $2n^2 + 7n + 1$  is divisible by 3.” (Notice that the statement is an if-then statement. When is an if-then statement false?)

(a) Suppose  $n$  is divisible by 3. Let  $n = 3m$ , where  $m$  is an integer. Then

$$2n^2 + 7n + 9 = 2(3m)^2 + 7(3m) + 9 = 18m^2 + 21m + 9 = 3(6m^2 + 7m + 3).$$

Since  $2n^2 + 7n + 9$  has been expressed as 3 times an integer,  $2n^2 + 7n + 9$  is divisible by 3.  $\square$

(b) The statement in (a) said that *if*  $n$  is divisible by 3, then  $2n^2 + 7n + 9$  is divisible by 3. Calvin is using  $n = 2$ , which is not divisible by 3. So he has no right to expect that  $2 \cdot 2^2 + 7 \cdot 2 + 9 = 31$  should be divisible by 3.  $\square$

(c) An if-then statement is false provided that the “if” part is true and the “then” part is false. So I need a value of  $n$  which is divisible by 3, for which  $2n^2 + 7n + 1$  is *not* divisible by 3.

By trial and error, 3 is divisible by 3, but  $2 \cdot 3^2 + 7 \cdot 3 + 1 = 40$  is not divisible by 3. This shows that statement is false.

Lots of values will work — for instance, 6 is divisible by 3, but  $2 \cdot 6^2 + 7 \cdot 6 + 1 = 115$  is not divisible by 3.  $\square$

22. Prove that if  $x^2 + 5x - 2 \not> 12$ , then  $x \not> 2$ .

I will prove the contrapositive: If  $x > 2$ , then  $x^2 + 5x - 2 > 12$ .

If  $x > 2$ , then

$$x^2 > 4 \quad \text{and} \quad 5x > 10.$$

Adding these inequalities, I obtain

$$x^2 + 5x > 14.$$

Subtracting 2 from both sides yields

$$x^2 + 5x - 2 > 12. \quad \square$$

23. Prove:  $P \rightarrow (Q \wedge R)$ .

$$\text{Premises: } \begin{cases} P \rightarrow S \\ S \rightarrow R \\ (\neg Q \vee \neg T) \rightarrow \neg S \end{cases}.$$

This is a conditional proof; I'll begin by assuming  $P$ , and I'll try to prove  $Q \wedge R$ .

1.	$P$	Premise for conditional proof
2.	$P \rightarrow S$	Premise
3.	$S$	Modus ponens (1,2)
4.	$S \rightarrow R$	Premise
5.	$R$	Modus ponens (3,4)
6.	$(\neg Q \vee \neg T) \rightarrow \neg S$	Premise
7.	$\neg(\neg Q \vee \neg T)$	Modus tollens (3,6)
8.	$Q \wedge T$	DeMorgan (7)
9.	$Q$	Decomposing a conjunction (8)
10.	$Q \wedge R$	Constructing a conjunction (5,9)
11.	$P \rightarrow (Q \wedge R)$	Conditional proof (1,10) $\square$

24. Prove:  $P \rightarrow (Q \wedge S)$ .

$$\text{Premises: } \begin{cases} (\neg Q \vee R) \rightarrow \neg P \\ R \vee S \end{cases} .$$

- |     |                                      |                                   |
|-----|--------------------------------------|-----------------------------------|
| 1.  | $P$                                  | Premise for conditional proof     |
| 2.  | $(\neg Q \vee R) \rightarrow \neg P$ | Premise                           |
| 3.  | $R \vee S$                           | Premise                           |
| 4.  | $\neg(\neg Q \vee R)$                | Modus tollens (1,2)               |
| 5.  | $Q \wedge \neg R$                    | DeMorgan (4)                      |
| 6.  | $Q$                                  | Decomposing a conjunction (5)     |
| 7.  | $\neg R$                             | Decomposing a conjunction (5)     |
| 8.  | $S$                                  | Disjunctive syllogism (3,7)       |
| 9.  | $Q \wedge S$                         | Constructing a conjunction (5,8)  |
| 10. | $P \rightarrow (Q \wedge S)$         | Conditional proof (1,9) $\square$ |
- 

25. Prove:  $(A \vee B) \rightarrow E$ .

$$\text{Premises: } \begin{cases} A \rightarrow E \\ B \rightarrow (\neg C \rightarrow D) \\ \neg C \wedge \neg D \end{cases} .$$

- |     |  |                                    |
|-----|--|------------------------------------|
| 1.  | $A \rightarrow E$                      | Premise                            |
| 2.  | $B \rightarrow (\neg C \rightarrow D)$ | Premise                            |
| 3.  | $\neg C \wedge \neg D$                 | Premise                            |
| 4.  | $A \vee B$                             | Premise for conditional proof      |
| 5.  | $\neg(C \vee D)$                       | DeMorgan (3)                       |
| 6.  | $\neg(\neg C \rightarrow D)$           | Conditional disjunction (5)        |
| 7.  | $\neg B$                               | Modus tollens (2, 6)               |
| 8.  | $A$                                    | Disjunctive syllogism (4, 7)       |
| 9.  | $E$                                    | Modus ponens (1, 8)                |
| 10. | $(A \vee B) \rightarrow E$             | Conditional proof (4, 9) $\square$ |
- 

26. Prove that if  $x > 4$ , then  $x^3 + 3x + 5 > 78$ .

Since  $x > 4$ , I have

$$x^3 > 64 \quad \text{and} \quad 3x > 12.$$

In addition,  $5 > 2$ . Adding these three inequalities, I obtain

$$x^3 + 3x + 5 > 64 + 12 + 2 = 78. \quad \square$$

---

27. (a) Prove that there is an integer which is divisible by both 4 and 6 but is not divisible by 24.

(b) Prove that there is an integer which is divisible by both 4 and 6 and is greater than 24 but is not divisible by 24.

(c) Prove that there is an  $x \in \mathbb{R}$  such that  $x^3 > 100x^2$ .

(d) Prove that there is an  $x \in \mathbb{R}$  such that

$$x^2 + 0.12 < 0.7x.$$

(a) One way to do this is to take multiples of 6 and stop at the first multiple which is also divisible by 4. I get 6, 12, 18, and so on, and I notice that 12 is divisible by 4.

12 is divisible by both 4 and 6, but is not divisible by 24.  $\square$

(b) This time I take multiples of 6 *greater than* 24 and stop at the first multiple which is also divisible by 4. I get 30, 36, 42, and so on, and I notice that 36 is divisible by 4.

36 is divisible by both 4 and 6, but is not divisible by 24.  $\square$

(c) Lots of  $x$ 's work. I could figure out informally how to find one like this: " $x^3 > 100x^2$ " can be written as " $x \cdot x^2 > 100x^2$ ". The  $x^2$ -terms match on the two sides, so for this to be true the  $x$  on the left should be bigger than the 100 on the right. So I should take  $x$  to be a number bigger than 100. For instance, take  $x = 200$ . Then

$$x^3 = 200^3 = 8\,000\,000, \quad \text{and} \quad 100x^2 = 100 \cdot 200^2 = 4\,000\,000.$$

So  $200^3 > 100 \cdot 200^2$ , and  $x = 200$  works.  $\square$

(d) Note that

$$x^2 + 0.12 < 0.7x \quad \text{implies} \quad x^2 - 0.7x + 0.12 < 0, \quad \text{or} \quad (x - 0.3)(x - 0.4) < 0.$$

The solution to this inequality is  $0.3 < x < 0.4$ . Thus, for example,  $x = 0.35$  makes the inequality true. In fact, if  $x = 0.35$ ,

$$x^2 + 0.12 = 0.2425 \quad \text{and} \quad 0.7x = 0.245, \quad \text{so} \quad x^2 + 0.12x < 0.7x. \quad \square$$

28. Prove that there is an  $x \in \mathbb{R}$  such that  $x = (\sin x)^2$ .

Let  $f(x) = x - (\sin x)^2$ . Then

$$f(\pi) = \pi - (\sin \pi)^2 = \pi \quad \text{and} \quad f(-\pi) = -\pi - (\sin(-\pi))^2 = -\pi.$$

$f$  is continuous, it is positive when  $x = \pi$ , and it is negative when  $x = -\pi$ . By the Intermediate Value Theorem, there is an  $x \in [-\pi, \pi]$  such that  $f(x) = 0$ . Then  $x - (\sin x)^2 = 0$ , or  $x = (\sin x)^2$ .  $\square$

29. When the Mean Value Theorem is applied to the function  $f(x) = \frac{x^2}{2} + \sin x$  on the interval  $[0, \pi]$ , it says that there is a number  $c$  such that

$$\frac{\pi}{2} = c - \cos c, \quad \text{where} \quad 0 < c < \pi.$$

Calvin Butterball complains: "There must be something wrong — you can't solve this equation algebraically for  $c$ ." Does Calvin have a valid objection?

The Mean Value Theorem merely asserts that *there is* a number  $c$  satisfying the stated conditions; it does *not* say that it will be easy to *find*  $c$ . While Calvin is correct that the equation can't be solved algebraically for  $c$ , the theorem didn't promise that. *Existence theorems* often only assert that something *exists*; that is not the same thing as saying how to find or construct it.  $\square$

30. Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function, and

$$f(3) = 17 \quad \text{and} \quad f(7) = -11.$$

Prove that there is an  $x$  such that  $3 \leq x \leq 7$  and  $x \cdot f(x) = 0$ .

I have

$$\begin{array}{r} x \\ 3 \\ 7 \end{array} \quad \begin{array}{l} x \cdot f(x) \\ 3 \cdot 17 = 51 \\ 7 \cdot (-11) = -77 \end{array}$$

Since  $x \cdot f(x)$  is continuous, and since  $51 > 0 > -77$ , the Intermediate Value Theorem implies that there is an  $x$  such that  $3 \leq x \leq 7$  and  $x \cdot f(x) = 0$ .  $\square$

---

31. Suppose that  $f$  is a continuous function satisfying

$$f(3) = 100 \quad \text{and} \quad f(7) = 12.$$

Prove that  $f(x) - x^2 = 0$  for some  $x \in [3, 7]$ .

The function  $g(x) = f(x) - x^2$  is continuous. Moreover,

$$g(3) = f(3) - 3^2 = 91 \quad \text{and} \quad g(7) = f(7) - 7^2 = -37.$$

By the Intermediate Value Theorem,  $g(x) = 0$  for some  $x \in [3, 7]$ . Hence,  $f(x) - x^2 = 0$  for some  $x \in [3, 7]$ .  $\square$

---

32. Give an  $\epsilon$ - $\delta$  proof that  $\lim_{x \rightarrow 2} (x^2 - 7x) = -10$ .

Let  $\epsilon > 0$ . Set  $\delta = \min\left(1, \frac{\epsilon}{4}\right)$ . Assume  $\delta > |x - 2| > 0$ .

First,

$$\begin{aligned} 1 &\geq \delta > |x - 2| \\ 1 &< x < 2 \\ -4 &< x - 5 < -3 \end{aligned}$$

Therefore,  $|x - 5| < 4$ .

Moreover,  $\frac{\epsilon}{4} \geq \delta > |x - 2|$ . Multiplying the last two inequalities, I get

$$\epsilon = 4 \cdot \frac{\epsilon}{4} > |x - 5||x - 2| = |x^2 - 7x - (-10)|.$$

This proves that  $\lim_{x \rightarrow 2} (x^2 - 7x) = -10$ .  $\square$

---

33. Give an  $\epsilon$ - $\delta$  proof that  $\lim_{x \rightarrow 3} \frac{x+7}{x-1} = 5$ .

Let  $\epsilon > 0$ . Set  $\delta = \min\left(1, \frac{\epsilon}{4}\right)$ . Then

$$1 \geq \delta \quad \text{and} \quad \frac{\epsilon}{4} \geq \delta.$$

First,

$$\begin{aligned} 1 &\geq \delta > |x - 3| \\ 2 &< x < 4 \\ 1 &< x - 1 < 3 \\ 1 &> \frac{1}{x - 1} > \frac{1}{3} \\ 4 &> \frac{4}{x - 1} > \frac{4}{3} \end{aligned}$$



Hence,

$$4 > \left| \frac{4}{x-1} \right| = \left| \frac{-4}{x-1} \right|.$$

Also,

$$\frac{\epsilon}{4} \geq \delta > |x-3|.$$

Multiplying the last two inequalities, I obtain

$$\begin{aligned} \epsilon &= 4 \cdot \frac{\epsilon}{4} > \left| \frac{-4}{x-1} \right| |x-3| \\ &= \left| \frac{-4x+12}{x-1} \right| \\ &= \left| \frac{(x+7)-5(x-1)}{x-1} \right| \\ &= \left| \frac{x+7}{x-1} - 5 \right| \end{aligned}$$

This proves that  $\lim_{x \rightarrow 3} \frac{x+7}{x-1} = 5$ .  $\square$

---

34. Give an  $\epsilon$ - $\delta$  proof that  $\lim_{x \rightarrow 1} (3x^2 + 4x) = 7$ .

Let  $\epsilon > 0$ . Set  $\delta = \min\left(1, \frac{\epsilon}{13}\right)$ . Assume that  $\delta > |x-1| > 0$ .

Since  $1 \geq \delta > |x-1|$ , I have  $0 < x < 2$ . Then

$$0 < 3x < 6, \quad \text{so} \quad 7 < 3x+7 < 13.$$

Therefore,  $|3x+7| < 13$ .

In addition,  $\frac{\epsilon}{13} \geq \delta > |x-1|$ .

Multiplying the inequalities, I get

$$\epsilon = 13 \cdot \frac{\epsilon}{13} > |3x+7||x-1| = |3x^2+4x-7|.$$

This proves that  $\lim_{x \rightarrow 1} (3x^2 + 4x) = 7$ .  $\square$

---

35. Give an  $\epsilon$ - $\delta$  proof that  $\lim_{x \rightarrow 1} \frac{1}{x} = 1$ .

I'll do some scratchwork first. I want

$$\epsilon > \left| \frac{1}{x} - 1 \right| = \left| \frac{1-x}{x} \right| = |x-1| \cdot \frac{1}{|x|}.$$

In this case, I can't make the usual assumption that  $1 \geq \delta$ . This would give  $1 > |x-1|$ , or  $0 < x < 2$ . The problem is that the second term  $\frac{1}{|x|}$  becomes *arbitrarily large* near  $x = 0$ , because there is a vertical asymptote there.

Thus, I have to make  $\delta$  small enough to keep  $x$  away from  $x = 0$ . I'll assume  $\frac{1}{2} \geq \delta$ . Then  $\frac{1}{2} > |x-1|$ , so  $\frac{1}{2} < x < \frac{3}{2}$ , and  $2 > \frac{1}{x} > \frac{2}{3}$ . Now  $\frac{1}{|x|}$  is bounded by 2.

---

Here's the proof.

Let  $\epsilon > 0$ , and set  $\delta = \min\left(\frac{1}{2}, \frac{\epsilon}{2}\right)$ . Assume  $\delta > |x - 1| > 0$ .

Since  $\frac{1}{2} \geq \delta > |x - 1|$ , I have

$$\frac{1}{2} < x < \frac{3}{2}, \quad \text{and} \quad 2 > \frac{1}{x} > \frac{2}{3}.$$

Therefore,  $\frac{1}{|x|} < 2$ .

In addition, I have  $\frac{\epsilon}{2} \geq \delta > |x - 1|$ .

Multiplying the inequalities, I get

$$\epsilon = 2 \cdot \frac{\epsilon}{2} > \frac{1}{|x|} \cdot |x - 1| = \left| \frac{1 - x}{x} \right| = \left| \frac{1}{x} - 1 \right|.$$

This proves that  $\lim_{x \rightarrow 1} \frac{1}{x} = 1$ .  $\square$

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*Perhaps the most valuable result of all education is the ability to make yourself do the thing you have to do, when it ought to be done, whether you like it or not.* - THOMAS HUXLEY