PHYS 321

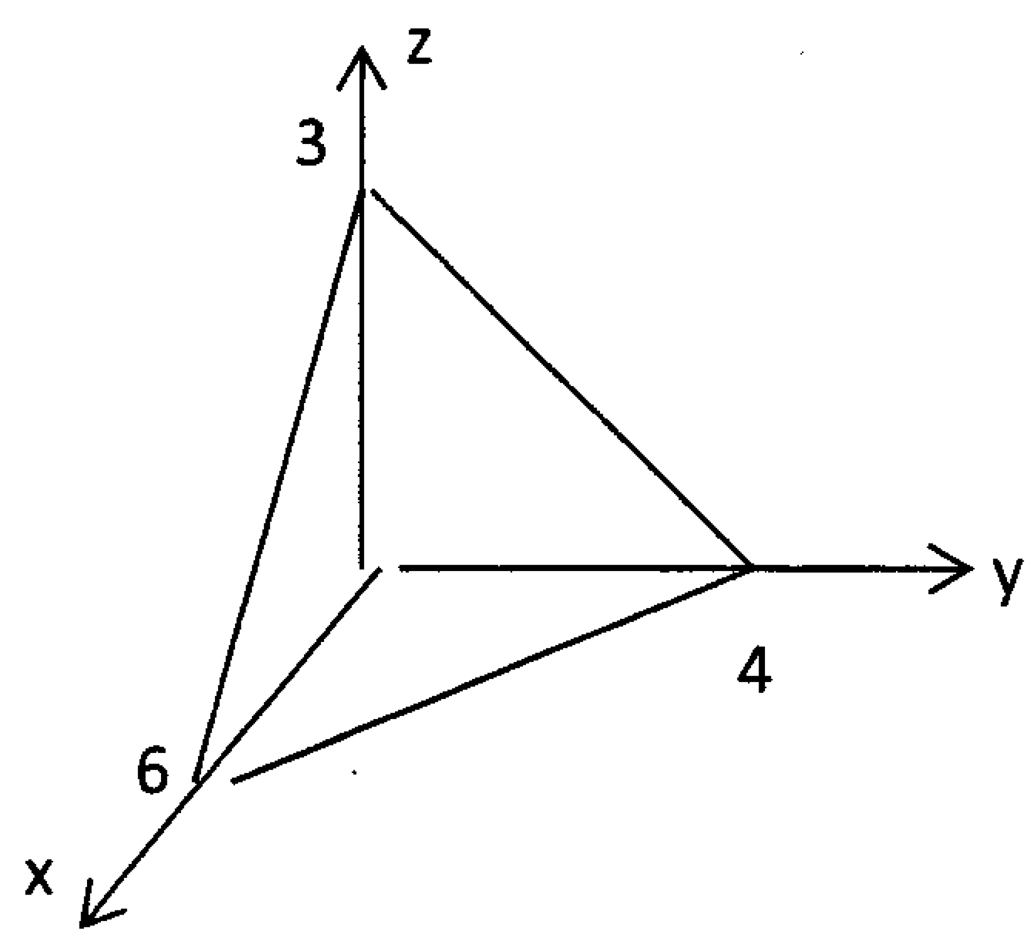
Due Monday Feb 8, 2016

1) Evaluate $\int \nabla \cdot \vec{F}\, d\tau$ over the volume region $x^2 + y^2 + z^2 \leq 25$ where $\vec{F} = (x^2 + y^2 + z^2)(x\hat{x} + y\hat{y} + z\hat{z})$.

2) Evaluate $\oint(2y\, dx - 3x\, dy)$ around the square bounded by x=3, x=5, y=1, y=3.

3) Evaluate $\int d\vec{a} \cdot \nabla \times (x^2\hat{x} + z^2\hat{y} - y^2\hat{z})$ over the open surface defined by $z = 4 - x^2 - y^2$ over the x,y plane.

4) Evaluate $\int d\vec{a} \cdot \nabla \times (y\hat{x} + 2\hat{y})$ over the open surface in the first octant made up of the plane $\frac{x}{6} + \frac{y}{4} + \frac{z}{3} = 1$ and the triangles in the x,z and y,z planes. See figure.



5) Evaluate $\oint \vec{A} \cdot d\vec{a}$ over the closed surface of the tin can shape bounded by $x^2 + y^2 = 9$, z= 0, z= 5 where $\vec{A} = 2xy\hat{x} - y^2\hat{y} + (z + xy)\hat{z}$

6) Evaluate $\oint d\vec{a} \cdot \nabla \times (2xy\hat{x} - xz\hat{z})$ over the closed surface of the ellipsoid defined by $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$

7) Problem 1.61

1) $\vec{F} = (x^2+y^2+z^2)(x\,\hat{x} + y\,\hat{y} + z\,\hat{z})$

$$= r^2\,\vec{r} = r^3\,\hat{r}$$

$$\int \nabla\cdot\vec{F}\,d\tau = \oint \vec{F}\cdot d\vec{a} = \oint r^3\,\hat{r}\cdot d\vec{a}$$

where the area is the surface of sphere radius $R=5$

$$\int \nabla\cdot\vec{F}\,d\tau = R^3(4\pi R^2) = 4\pi R^5$$

2) $\oint (2y\,dx - 3x\,dy) = \oint \vec{A}\cdot d\vec{l}$

with $\vec{A} = 2y\,\hat{x} - 3x\,\hat{y}$ $\quad d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -3x & 0 \end{vmatrix} = -5\,\hat{z}$$

$$\oint \vec{A}\cdot d\vec{l} = \int (\nabla\times\vec{A})\cdot d\vec{a} = -5\int \hat{z}\cdot d\vec{a}$$

$= -5A$ where A is the area bounded by $x=3$, $x=5$
$y=1$, $y=3$ $\quad A = 4.0$

$\oint (2y\,dx - 3x\,dy) = -20$ if we go around
counterclockwise

3) $\int d\bar{a} \cdot \nabla \times (x^2 \hat{x} + z^2 \hat{y} - y^2 \hat{z})$ over the open surface defined by $z = 4 - x^2 - y^2$ over x-y plane $z \geq 0$

$= \oint_c d\bar{l} \cdot (x^2 \hat{x} + z^2 \hat{y} - y^2 \hat{z})$ with C defined by the intersection of the surface with the x-y plane; that is $z = 0 \Rightarrow x^2 + y^2 = 4$, a circle of radius 2 in the x-y plane.

Evaluate $\oint d\bar{l} \cdot (x^2 \hat{x} + z^2 \hat{y} - y^2 \hat{z})$. But $z = 0$ along C and $d\bar{l} \cdot \hat{z} = 0$ since $d\bar{l}$ lies in the x-y plane

$= \oint d\bar{l} \cdot (x^2 \hat{x})$ But $\oint_c d\bar{l} \cdot (x^2 \hat{x}) = \int_s d\bar{a} \cdot \nabla \times (x^2 \hat{x})$

where S is any surface with C as its perimeter

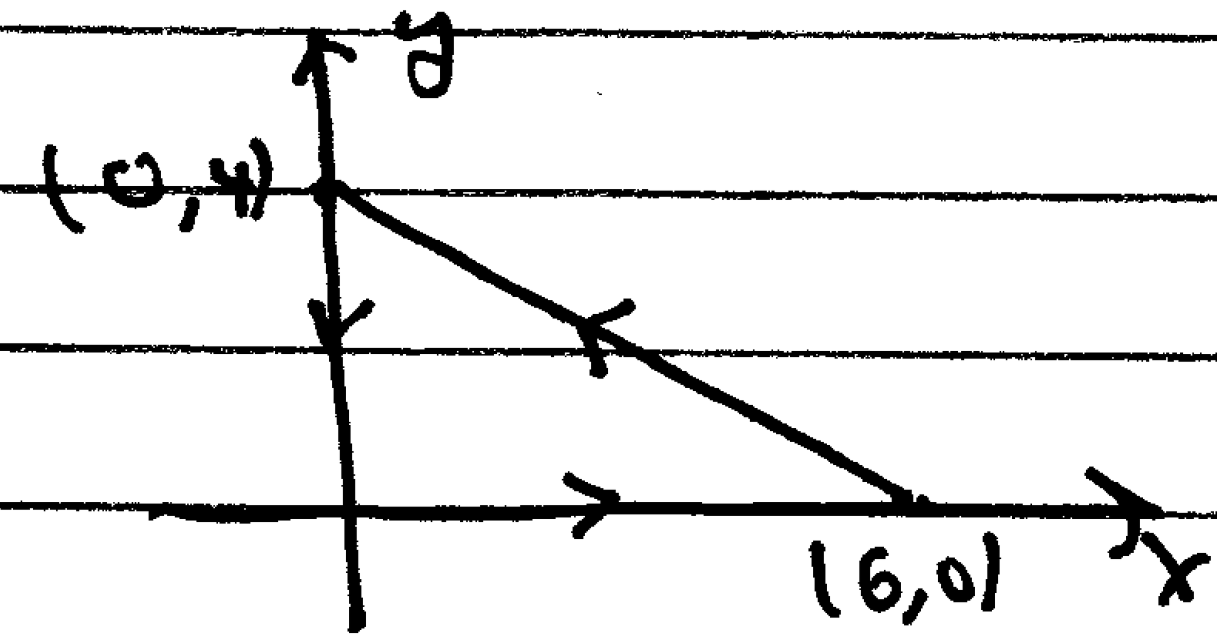
$\nabla \times (x^2 \hat{x}) = 0$

So $\int_s d\bar{a} \cdot \nabla (x^2 \hat{x} + z^2 \hat{y} - y^2 \hat{z}) = 0$

4)

$$\int_S d\bar{a} \cdot \nabla \times (y\,\hat{x} + 2\,\hat{y}) = \oint_C d\bar{l} \cdot (y\,\hat{x} + 2\,\hat{y})$$

where C is defined by the perimeter of the surface



$$\nabla \times (y\,\hat{x} + 2\,\hat{y}) = -\hat{z}$$

$$\oint d\bar{l} \cdot (y\,\hat{x} + 2\,\hat{y}) = -\int_S d\bar{a} \cdot \hat{z}$$ where S is the surface of the triangle. Area $= 12$

$$\int_S d\bar{a} \cdot \nabla \times (y\,\hat{x} + 2\,\hat{y}) = -12$$

5) $$\oint \bar{A} \cdot d\bar{a} = \int \nabla \cdot \bar{A}\, d\tau$$

$$\bar{A} = 2xy\,\hat{x} - y^2\,\hat{y} + (z + xy)\,\hat{z}$$

$$\nabla \cdot \vec{A} = 2y - 2y + 1 = 1$$

$$\oint d\bar{A} \cdot d\bar{a} = \int d\tau = \text{vol} = \pi R^2 h = 45\pi$$

6) $\oint d\vec{a}\cdot(\nabla\times\vec{A}) = \int d\tau\, \nabla\cdot(\nabla\times\vec{A}) = 0$ for any $\vec{A}$.

1.61)

a) $\int(\nabla T)\,d\tau = \oint_S T\,d\vec{a}$ Let $\vec{v} = \vec{c}\,T$ and

use the divergency thm.

$$\int(\nabla\cdot\vec{v})\,d\tau = \oint \vec{v}\cdot d\vec{a}$$

$$\nabla\cdot(\vec{c}\,T) = \vec{c}\cdot\nabla T$$

$\vec{c}\cdot\int \nabla T\,d\tau = \vec{c}\cdot\oint T\,d\vec{a}$ But $\vec{c}$ is

arbitrary $\Rightarrow$ $\int \nabla T\,d\tau = \oint T\,d\vec{a}$

b) $\int(\nabla\times\vec{v})\,d\tau = -\oint_S \vec{v}\times d\vec{a}$

use $\vec{v}\times\vec{c}$ in the divergency thm.

$$\int \nabla\cdot(\vec{v}\times\vec{c})\,d\tau = \oint(\vec{v}\times\vec{c})\cdot d\vec{a}$$

$\nabla\cdot(\vec{v}\times\vec{c}) = \vec{c}\cdot(\nabla\times\vec{v})$ $\quad (\vec{v}\times\vec{c})\cdot d\vec{a} = \vec{v}\cdot(\vec{c}\times d\vec{a})$

$(\vec{v}\times\vec{c})\cdot d\vec{a} = d\vec{a}\cdot(\vec{v}\times\vec{c}) = d\vec{a}\times\vec{v}\cdot\vec{c} = -(\vec{v}\times d\vec{a})\cdot\vec{c}$

So $\vec{c}\cdot\int(\nabla\times\vec{v})\,d\tau = -\vec{c}\cdot\oint\vec{v}\times d\vec{a}$ Drop the $\vec{c}$.

c)

$$\int_V (T\nabla^2 U + \nabla T\cdot\nabla U)\,d\tau = \int_S T\nabla U$$

Let $\vec{v} = T\nabla U$ in the divergence thm.

$$\int \nabla\cdot(T\nabla U)\,d\tau = \oint d\vec{a}\cdot(T\nabla U)$$

$$\nabla\cdot(T\nabla U) = T\,\nabla\cdot(\nabla U) + (\nabla U)\cdot(\nabla T)$$
$$= T\nabla^2 U + \nabla U\cdot\nabla T$$

d) $$\int_V (T\nabla^2 U - U\nabla^2 T)\,dV = \oint_S (T\nabla U - U\nabla T)\cdot d\vec{a}$$

Note $\nabla\cdot(T\nabla U - U\nabla T) = T\nabla^2 U + \nabla T\cdot\nabla U$
$- U\nabla^2 T - \nabla U\,\nabla T$
$= T\nabla^2 U - U\nabla^2 T$

the identity immediately follows from the divergence thm.

e) $\int_S \nabla T \times d\vec{a} = -\oint_C T\, d\vec{l}$

Let $\vec{v} = \vec{c}\,T$ in Stokes thm.

$$\oint (\vec{c}\,T)\cdot d\vec{l} = \int (\nabla \times \vec{c}\,T)\cdot d\vec{a}$$

$$\nabla \times (\vec{c}\,T) = \vec{c} \times \nabla T = -\nabla T \times \vec{c}$$

$$\vec{c} \cdot \oint T\, d\vec{l} = -\int (\nabla T \times \vec{c})\cdot d\vec{a} =$$

$$= -\int d\vec{a}\cdot(\nabla T \times \vec{c}) = -\int (d\vec{a} \times \nabla T)\cdot \vec{c}$$

so $\oint T\, d\vec{l} = -\int d\vec{a} \times \nabla T = \int \nabla T \times d\vec{a}$