Equations in Quadratic Form

MATH 101 *College Algebra*

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Objectives

In this lesson we will learn to:

- make substitutions that allow equations to be written in quadratic form,
- solve equations that can be written in quadratic form, and
- solve equations that contain rational expressions.

Equations in Quadratic Form

Recall: the standard form of a quadratic equation is

$$ax^2 + bx + c = 0$$

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$$\begin{aligned} u - 2 &= 0 \\ u &= 2 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned} \quad \text{or} \quad \begin{aligned} u - 1 &= 0 \\ u &= 1 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

Strategy for Substitution

1. Look for the middle term.
2. Substitute a first-degree variable, such as u, for the variable expression in the middle term.
3. Substitute the square of this variable, u^2, for the variable expression in the first term.
4. Solve the resulting quadratic equation for u.
5. Substitute the results “back” for u in the beginning substitution and solve for the original variable.

Example

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$$(x^{-1})^2 - 12x^{-1} + 35 = 0$$
$$u^2 - 12u + 35 = 0 \text{ (where } u = x^{-1}\text{)}$$
$$(u-7)(u-5) = 0$$

$$\begin{aligned} u - 7 &= 0 \\ u &= 7 \\ x^{-1} &= 7 \\ x &= \tfrac{1}{7} \end{aligned} \quad \text{or} \quad \begin{aligned} u - 5 &= 0 \\ u &= 5 \\ x^{-1} &= 5 \\ x &= \tfrac{1}{5} \end{aligned}$$

Solving Equations with Rational Expressions

Recall: to solve an equation involving rational expressions multiply both sides of the equation by the Least Common Multiple (LCM) of all the denominators in the equation.

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Solution

$$(x+3)(x-2)\left[\frac{x+1}{x+3}+\frac{2x-1}{x-2}\right]=(x+3)(x-2)\left[\frac{12x-2}{(x+3)(x-2)}\right]$$

$$(x+1)(x-2)+(2x-1)(x+3)=12x-2$$

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$$(x+1)(x-2)+(2x-1)(x+3)=12x-2$$

$$(x^2-x-2)+(2x^2+5x-3)=12x-2$$

$$3x^2-8x-3=0$$

$$(3x+1)(x-3)=0$$

$$\begin{array}{rcl} 3x+1 & = & 0 \\ x & = & -\frac{1}{3} \end{array} \quad \text{or} \quad \begin{array}{rcl} x-3 & = & 0 \\ x & = & 3 \end{array}$$

Solving Higher Degree Equations

Some polynomial equations with degrees greater than 2 can be solved by factoring and/or using the square root property.

Recall: the sum and difference of two cubes can be factored as follows.

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$
$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

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Solve the following polynomial equation.

$$x^3 + 27 = 0$$

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Solve the following polynomial equation.

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$$(x+3)(x^2 - 3x + 9) = 0$$

$$\begin{aligned} x + 3 &= 0 \\ x &= -3 \end{aligned} \quad \text{or} \quad \begin{aligned} x^2 - 3x + 9 &= 0 \\ x &= \tfrac{3 \pm \sqrt{(-3)^2 - 4(1)(9)}}{2} \\ x &= \tfrac{3 \pm \sqrt{-27}}{2} \\ x &= \tfrac{3 \pm 3i\sqrt{3}}{2} \end{aligned}$$