Logarithmic and Exponential Equations and Change-of-Base

MATH 101 *College Algebra*

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Objectives

In this lesson we will learn to

- solve exponential equations in which the bases are the same,
- solve exponential equations in which the bases are not the same,
- solve equations involving logarithms, and
- use the change-of-base formula and a calculator to evaluate logarithmic expressions.

Review: Properties of Exponents

Theorem

If a and b are positive real numbers and if x and y are any real numbers, then:

1. $b^0 = 1$
2. $b^{-x} = \dfrac{1}{b^x}$
3. $b^x \cdot b^y = b^{x+y}$
4. $\dfrac{b^x}{b^y} = b^{x-y}$
5. $(b^x)^y = b^{xy}$
6. $(a\,b)^x = a^x\, b^x$
7. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Properties of Equations with Exponents and Logarithms

Theorem

For $b > 0$ and $b \neq 1$,

1. *If $b^x = b^y$, then $x = y$.*
2. *If $x = y$, then $b^x = b^y$.*
3. *If $\log_b x = \log_b y$, then $x = y$ (for $x > 0$ and $y > 0$)*
4. *If $x = y$, then $\log_b x = \log_b y$ (for $x > 0$ and $y > 0$).*

Solving an Equation with Same Base

If an equation involves exponential expressions on both sides of the equals symbol and the bases are the same, then the exponents must be the same.

Example

$$3^{2x^2+3} = 3^{x+6}$$

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$$3^{2x^2+3} = 3^{x+6}$$

$$2x^2 + 3 = x + 6$$

$$2x^2 - x - 3 = 0$$

$$(2x - 3)(x + 1) = 0$$

$$x = \frac{3}{2} \quad \text{or} \quad x = -1$$

Solving Exponential Equations with Different Bases

If the bases are different on the two sides of an equation involving exponential expressions, we may take the logarithm of both sides to solve the equation.

Example

$$3 \cdot 10^{-2.1x} = 83.5$$

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Example

$$
\begin{aligned}
3 \cdot 10^{-2.1x} &= 83.5 \\
10^{-2.1x} &= 27.8\overline{3} \\
\log 10^{-2.1x} &= \log 27.8\overline{3} \\
-2.1x &= 1.44457 \\
x &= -0.687888
\end{aligned}
$$

Solving Equations with Logarithms

We may use the properties of logarithms to solve equations involving logarithms.

Example

$$2\log x - 15 = 0$$

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Example

$$
\begin{aligned}
2\log x - 15 &= 0 \\
2\log x &= 15 \\
\log x &= \frac{15}{2} \\
10^{\log x} &= 10^{15/2} \\
x &= 10^{15/2} \approx 31,622,776.6
\end{aligned}
$$

Change-of-Base Formula

Question: how do we use a calculator to evaluate logarithms with bases other than e or 10?

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Theorem

For a, b, $x > 0$ with $a \neq 1$ and $b \neq 1$ then

$$\log_b x = \frac{\log_a x}{\log_a b}.$$

Proof.

$$\begin{aligned} b^{\log_b x} &= x \\ \log_a b^{\log_b x} &= \log_a x \\ \log_b x\,(\log_a b) &= \log_a x \\ \log_b x &= \frac{\log_a x}{\log_a b} \end{aligned}$$

□

Examples

Since nearly every scientific calculator includes a natural logarithm **LN** key, we may use it to help us calculate logarithms in any other base.

Example

$$\log_4 36 = \frac{\ln 36}{\ln 4} = \frac{3.58352}{1.38629} = 2.58496$$

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We can check our work:

$$4^{2.58496} = 36$$