Graphing Quadratic Functions: Parabolas

MATH 101 *College Algebra*

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Objectives

In this lesson we will learn to:

- graph a parabola (a quadratic function) and determine its vertex, domain, range, line of symmetry, and zeros, and
- solve applied problems by using quadratic functions and the concept of maximum and minimum.

Quadratic Functions

Definition

Any function that can be written in the form

$$y = ax^2 + bx + c$$

where a, b, and c are real numbers with $a \neq 0$ is a **quadratic function**.

Remarks:

- The graph of any quadratic function is a curve called a **parabola**.
- **Vertical parabolas** are parabolas which "open upward" or "open downward".
- **Horizontal parabolas** are parabolas which "open left" or "open right".

Example

Consider the function $y = x^2 - 6x + 5$. We can choose some values for x and compute the corresponding values for y.

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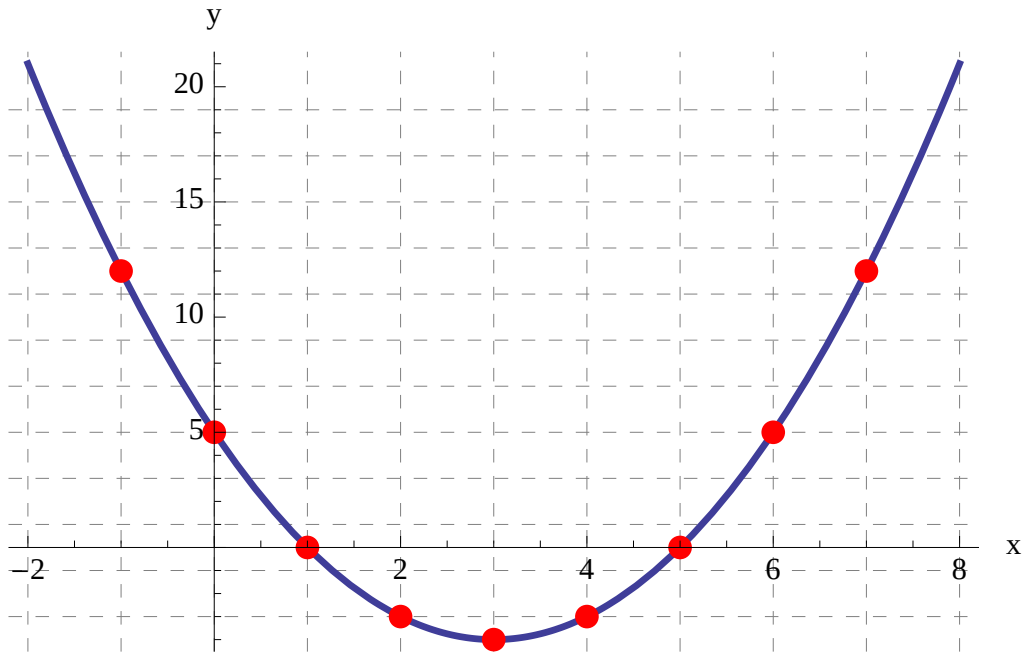
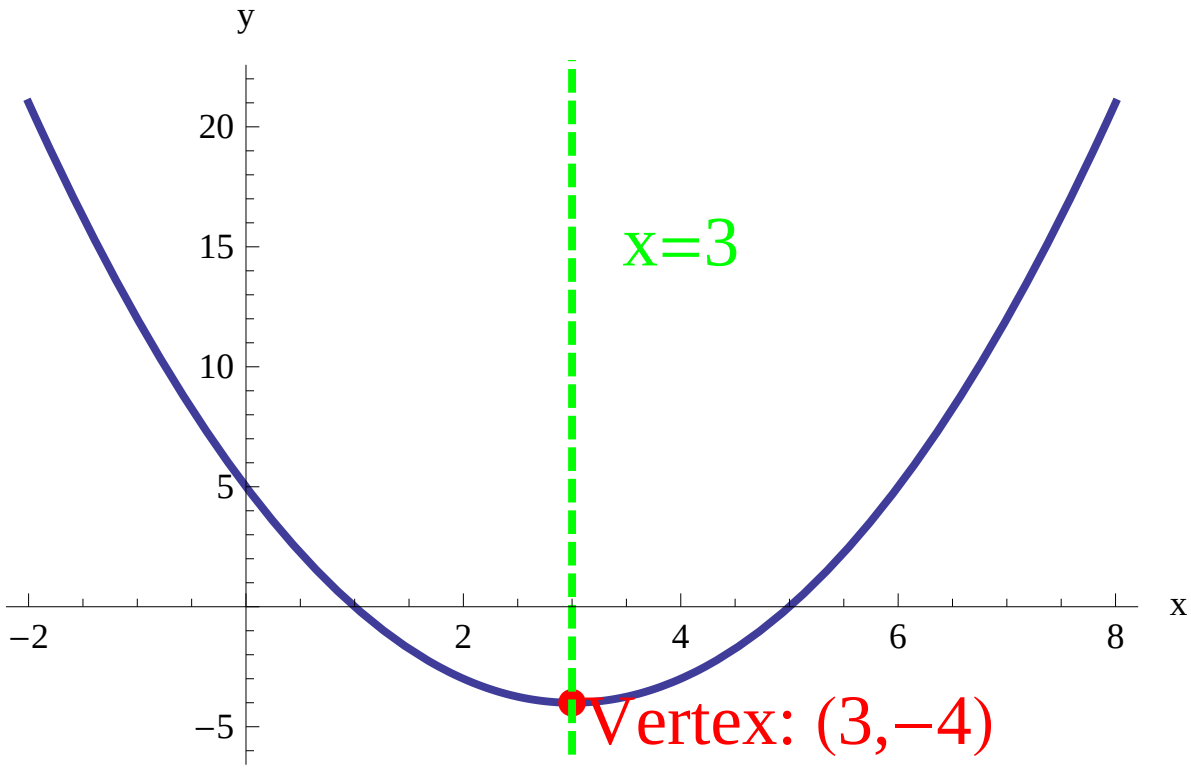
x	y
-1	12
0	5
1	0
2	-3
3	-4
4	-3
5	0
6	5
7	12



Anatomy of a Parabola

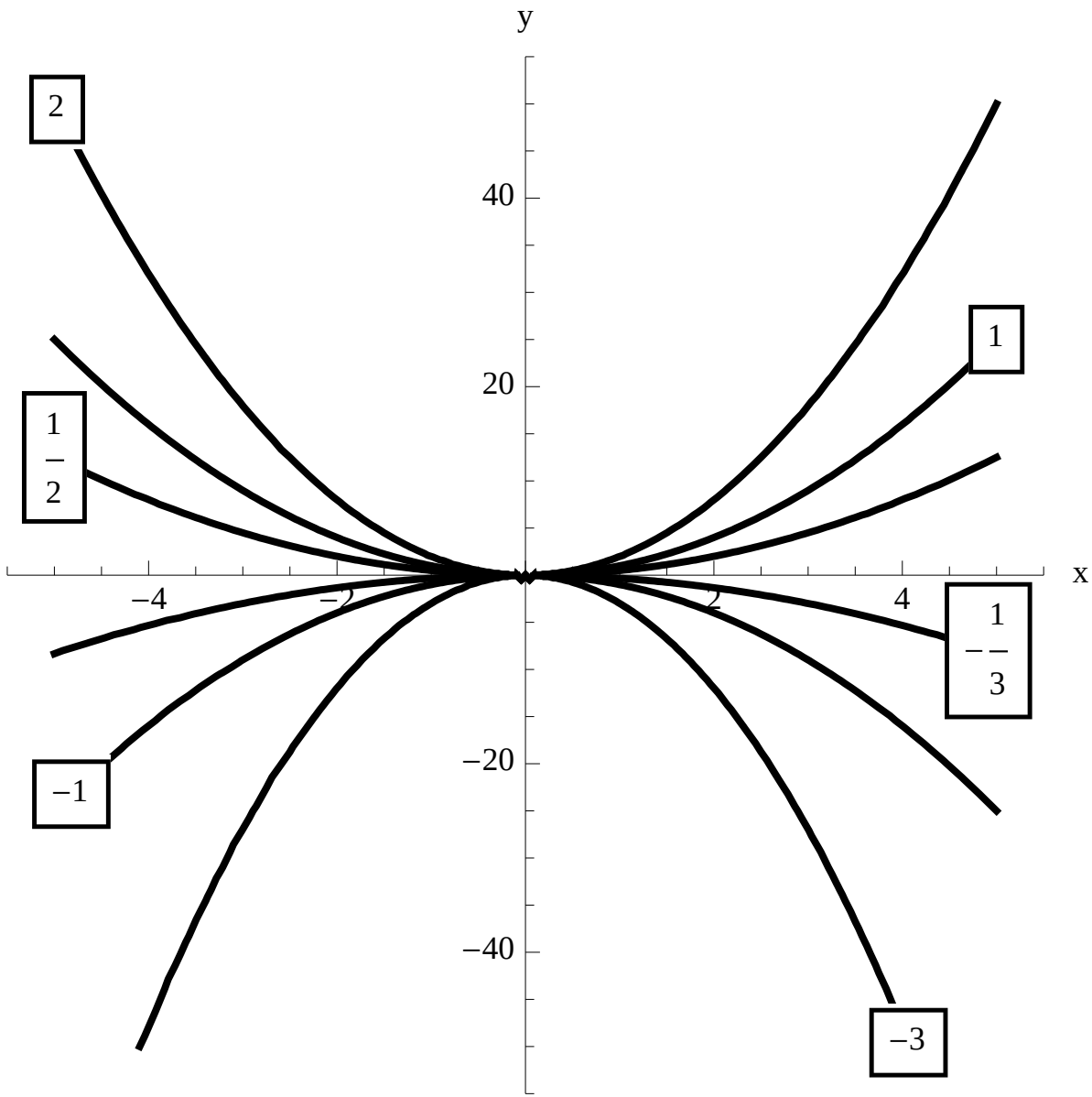


Observations:

- The turning point at $(3, -4)$ is called the **vertex** of the parabola.
- The vertical line at $x = 3$ is called the **line of symmetry** of the parabola.

Functions of the Form $y = ax^2$

Consider the graphs of the following parabolas of the form $y = ax^2$.

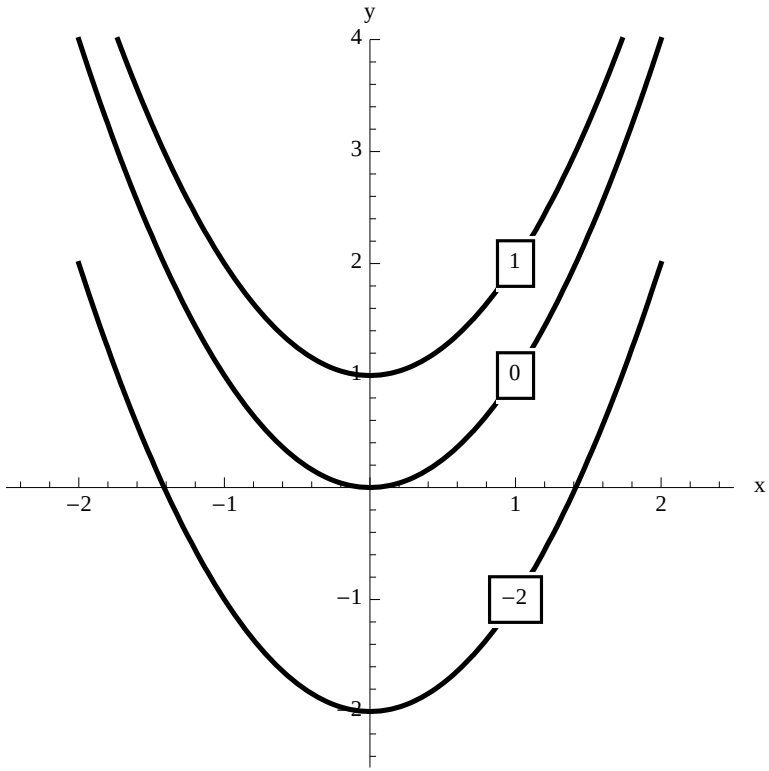


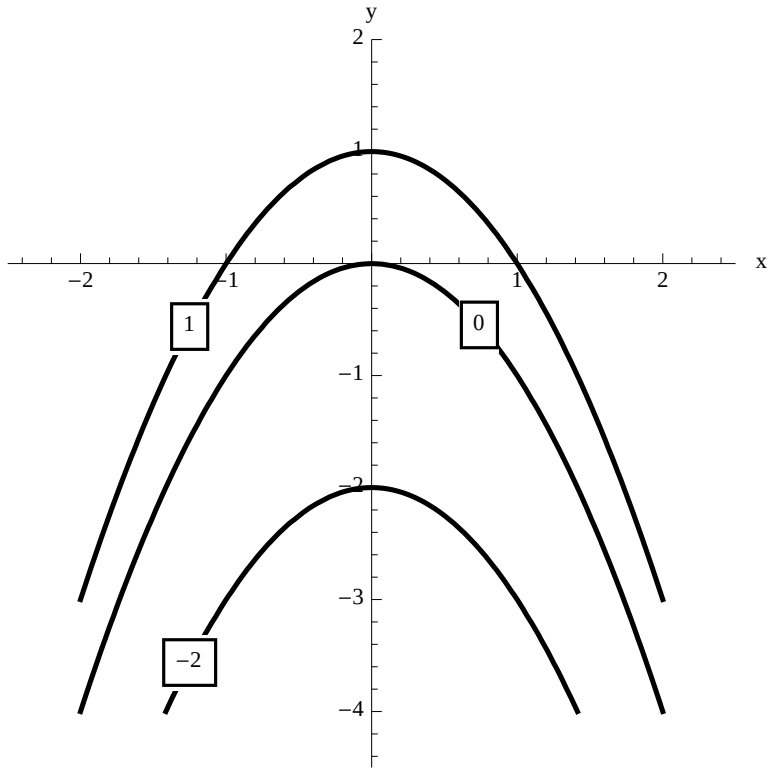
Remarks

- For all quadratic functions, the domain is the set of all real numbers.
- For parabolas of the form $y = ax^2$,
  - the parabola opens upward if $a > 0$ and the range is $y \geq 0$,
  - the parabola opens downward if $a < 0$ and the range is $y \leq 0$,
  - the vertex is the point $(0, 0)$,
  - the line of symmetry is $x = 0$,
  - the bigger $|a|$ is, the narrower the opening of the parabola, and
  - the smaller $|a|$ is, the wider the opening of the parabola.

Functions of the Form $y = ax^2 + k$

Consider the graphs of the following parabolas of the form $y = ax^2 + k$ (for these examples $a = \pm 1$).



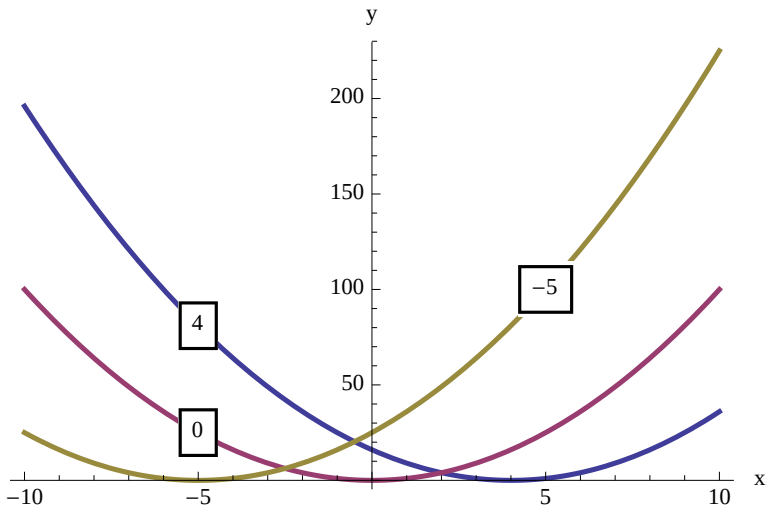


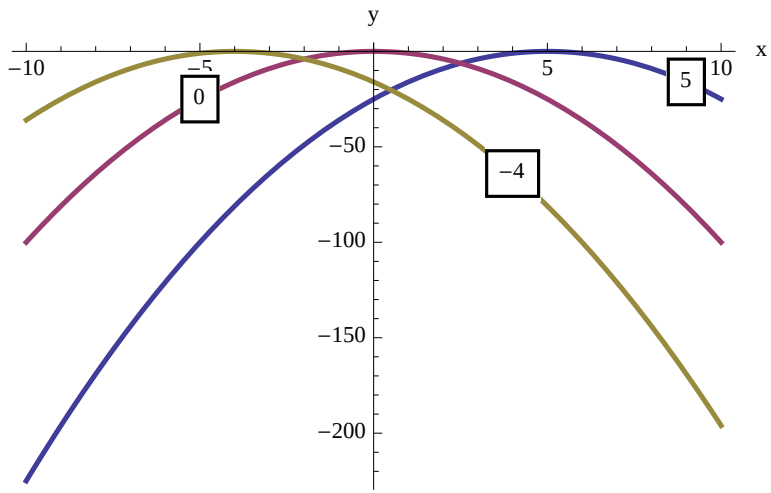
Remarks

- Adding k to ax^2 **shifts** or **translates** the graph **vertically** by k units.
- For parabolas of the form $y = ax^2 + k$,
  - the parabola opens upward if $a > 0$ and the range is $y \geq k$,
  - the parabola opens downward if $a < 0$ and the range is $y \leq k$,
  - the vertex is the point $(0, k)$,
  - the line of symmetry is $x = 0$,
  - the bigger $|a|$ is, the narrower the opening of the parabola, and
  - the smaller $|a|$ is, the wider the opening of the parabola.

Functions of the Form $y = a(x - h)^2$

Consider the graphs of the following parabolas of the form $y = a(x - h)^2$ (for these examples $a = \pm 1$).





Remarks

- Subtracting h from x in $a(x-h)^2$ **shifts** or **translates** the graph **horizontally** by h units.
- For parabolas of the form $y = a(x-h)^2$,
  - the parabola opens upward if $a > 0$ and the range is $y \geq 0$,
  - the parabola opens downward if $a < 0$ and the range is $y \leq 0$,
  - the vertex is the point $(h, 0)$,
  - the line of symmetry is $x = h$,
  - the bigger $|a|$ is, the narrower the opening of the parabola, and
  - the smaller $|a|$ is, the wider the opening of the parabola.

Functions of the Form $y = ax^2 + bx + c$

$$
\begin{aligned}
y &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \\
&= a(x - h)^2 + k
\end{aligned}
$$

where

$$
\begin{aligned}
h &= -\frac{b}{2a} \\
k &= \frac{4ac - b^2}{4a}
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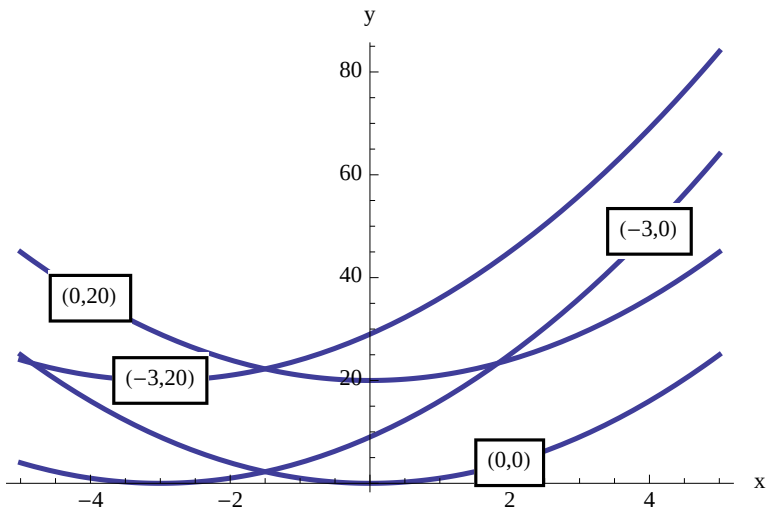
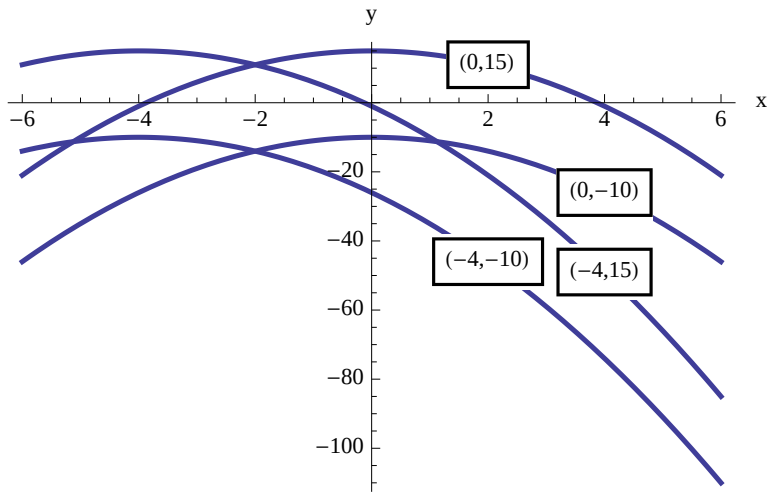
Remark: we can obtain the graph of $y = ax^2 + bx + c$ through a horizontal shift of h units and a vertical shift of k units.

Examples

The following graphs show parabolas of $y = a(x - h)^2 + k$ for different choices of h and k. The values of a used are ± 1.





Remarks

Writing the standard form of a quadratic function

$$y = ax^2 + bx + c = a(x - h)^2 + k$$

$$\text{where} \quad h = -\frac{b}{2a} \quad \text{and} \quad k = \frac{4ac - b^2}{4a}$$

- Subtracting h from x in $a(x - h)^2$ **shifts** or **translates** the graph **horizontally** by h units.
- Adding k to $a(x - h)^2$ **shifts** or **translates** the graph **vertically** by k units.
- The parabola opens upward if $a > 0$ and the range is $y \geq k$,
- The parabola opens downward if $a < 0$ and the range is $y \leq k$,
- The vertex is the point (h, k), and
- The line of symmetry is $x = h$.

Zeros of a Quadratic Function

If the graph of the quadratic function $y = ax^2 + bx + c$ crosses the x-axis, these points are called the **zeros of the function**. They can be found by solving the equation:

$$ax^2 + bx + c = 0.$$

Example

Find the zeros of the function, the line of symmetry, the coordinates of the vertex, the range, and the graph of the parabola for

$$y = x^2 - 7x + 10$$

Solution

Zeros of the function:

$$0 = x^2 - 7x + 10 = (x - 2)(x - 5)$$

$$x = 2 \text{ or } x = 5$$

Solution

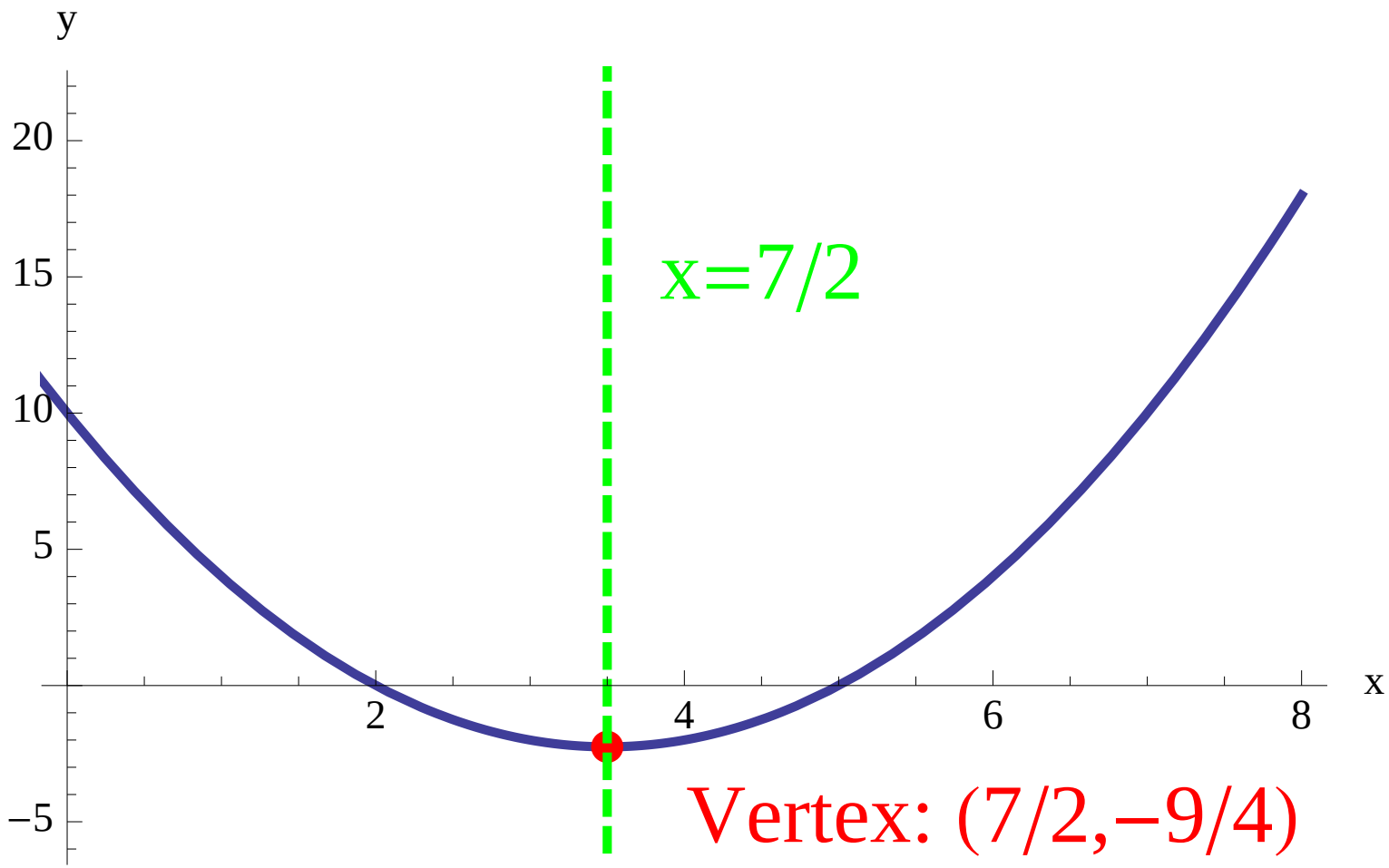
Zeros of the function:

$$0 = x^2 - 7x + 10 = (x-2)(x-5)$$
$$x = 2 \text{ or } x = 5$$

$$y = x^2 - 7x + 10 = x^2 - 7x + \frac{49}{4} - \frac{49}{4} + 10$$
$$= \left(x - \frac{7}{2}\right)^2 - \frac{49}{4} + 10 = \left(x - \frac{7}{2}\right)^2 - \frac{9}{4}$$

- **Line of symmetry:** $x = \frac{7}{2}$
- **Vertex:** $\left(\frac{7}{2}, -\frac{9}{4}\right)$
- **Domain:** all real numbers $(-\infty, \infty)$
- **Range:** $y \geq -\frac{9}{4}$ since $a = 1 > 0$.

Graph



Maximum and Minimum Values

Observations: for parabolas which open upward, the vertex is the lowest point on the parabola, while for parabolas which open downward, the vertex is the highest point on the parabola.

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Theorem

For a parabola with its equation in the form $y = a(x - h)^2 + k$,

1. *if* $a > 0$, *the parabola opens upward and* (h, k) *is the lowest point and* $y = k$ *is called the* ***minimum value*** *of the function,*
2. *if* $a < 0$, *the parabola opens downward and* (h, k) *is the highest point and* $y = k$ *is called the* ***maximum value*** *of the function,*