Alternating Series

MATH 211, *Calculus II*

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Fall 2021

Alternating Series

We have explored **positive term** series and their convergence/divergence properties.

Today we study **alternating series** which have one of two forms:

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

or

$$\sum_{k=1}^{\infty}(-1)^{k}a_k = -a_1 + a_2 - a_3 + a_4 - \cdots$$

where $a_k > 0$ for all $k = 1, 2, \ldots$.

Examples

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k}{k^2+2} = \frac{2}{3} - \frac{2}{3} + \frac{6}{11} - \frac{4}{9} + \cdots$$

$$\sum_{k=1}^{\infty}(-1)^{k}\frac{3k}{2+4k} = -\frac{1}{2} + \frac{3}{5} - \frac{9}{14} + \frac{2}{3} - \cdots$$

Alternating Series Test

Theorem (Alternating Series Test)

The alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ *is convergent if*

1. $0 < a_{k+1} \leq a_k$ *for all* $k \geq 1$, *and*
2. $\lim_{k\to\infty} a_k = 0$.

Remarks:

- This theorem is not true for positive-term series, *e.g.*, the **harmonic series**.
- Deleting or ignoring a finite number of terms from a series does not alter its convergence or divergence.

Proof

Even-indexed partial sums:

$$\begin{aligned} S_2 &= a_1 - a_2 \geq 0 \\ S_4 &= S_2 + a_3 - a_4 \geq S_2 \\ &\vdots \\ S_{2n} &= S_{2n-2} + a_{2n-1} - a_{2n} \geq S_{2n-2} \end{aligned}$$

Conclusion: the sequence of even-indexed partial sums $\{S_{2n}\}_{n=1}^{\infty}$ is **monotonically increasing.**

$$0 \leq S_{2n} = a_1 + (-a_2 + a_3) + (-a_4 + a_5) + \cdots - a_{2n} \leq a_1$$

Conclusion: the sequence of even-indexed partial sums $\{S_{2n}\}_{n=1}^{\infty}$ is **bounded**, therefore the sequence must converge to a limit L.

$$\begin{aligned} S_{2n+1} &= S_{2n} + a_{2n+1} \\ \lim_{n\to\infty} S_{2n+1} &= \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = L + 0 = L \end{aligned}$$

Examples

Determine if the following alternating series converge or diverge.

1. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k}{k^2+2}$
2. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k}\frac{3k}{2+4k}$
3. $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ (the **alternating harmonic series**)

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k}{k^2+2}$$

To use the Alternating Series Test, we must check

1. $0 < a_{k+1} \le a_k$,
   Let $f(x) = 2x/(x^2+2)$, then $f'(x) = (4-x^2)/(x^2+2)^2 < 0$ for $x > 2$, thus the sequence $\{a_k\}_{k=1}^{\infty}$ is monotone decreasing for $k = 2, 3, \ldots$. Since we can ignore a finite number of terms of the infinite series without affecting its convergence or divergence, we can conclude the first hypothesis of the AST holds.
2. $\lim_{k\to\infty} a_k = 0$,

$$\lim_{k\to\infty}\frac{2k}{k^2+2} = \lim_{k\to\infty}\frac{(2k)/k^2}{(k^2+2)/k^2} = \lim_{k\to\infty}\frac{2/k}{1+2/k^2} = 0.$$

Thus the infinite series converges by the Alternating Series Test.

$$\sum_{k=1}^{\infty}(-1)^k\frac{3k}{2+4k}$$

Note that

$$\lim_{k\to\infty}\frac{3k}{2+4k}=\lim_{k\to\infty}\frac{(3k)/k}{(2+4k)/k}=\lim_{k\to\infty}\frac{3}{2/k+4}=\frac{3}{4}\neq 0$$

and thus the infinite series diverges by the kth Term Test.

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$$

To use the Alternating Series Test, we must check

1. $0 < a_{k+1} \leq a_k$,

$$0 < \frac{1}{k+1} \leq \frac{1}{k} \quad \text{for all } k = 1, 2, \ldots.$$

2. $\lim_{k\to\infty} a_k = 0$,

$$\lim_{k\to\infty}\frac{1}{k} = 0$$

Thus the infinite series converges by the Alternating Series Test.

Estimating the Sum of an Alternating Series

Theorem

Suppose $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ *is a convergent alternating series. If* S *is the sum of the series and* S_n *is the* n^{th} *partial sum of the series, then*

$$|S - S_n| \leq a_{n+1}$$

for all n*.*

Proof

Consider the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ and let n be an even integer,

$$\begin{aligned} S_n &\leq S \leq S_{n+1} \\ 0 &\leq S - S_n \leq S_{n+1} - S_n = a_{n+1} \\ -a_{n+1} &\leq S - S_n \leq a_{n+1} \\ |S - S_n| &\leq a_{n+1}. \end{aligned}$$

Similarly we can show that $|S - S_n| \leq a_{n+1}$ when n is odd.

Examples

Determine the smallest value of n sufficient to estimate the sum of the following convergent alternating series to within an error of 10^{-4}.

1. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ (the **alternating harmonic series**)
2. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k}{k^2+2}$
3. $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3k}{2+4k^4}$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$$

We have already shown that this series converges.

$$\begin{aligned} |S - S_n| \leq a_{n+1} &< 10^{-4} \\ \frac{1}{n+1} &< 10^{-4} \\ n+1 &> 10^4 \\ n &> 9999 \implies \text{(smallest) } n = 10,000 \end{aligned}$$

$$\sum_{k=1}^{10{,}000}(-1)^{k+1}\frac{1}{k} \approx 0.693097 \approx \ln 2$$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k}{k^2+2}$$

We have already shown that this series converges.

$$\begin{aligned}
|S - S_n| \le a_{n+1} &< 10^{-4} \\
\frac{2(n+1)}{(n+1)^2+2} &< 10^{-4} \\
2n+2 &< 10^{-4}(n^2+2n+3) \\
20,000(n+1) &< n^2+2n+3 \\
0 &< n^2 - 19,998n - 19,997 \\
n &> 19,999 \implies \text{(smallest) } n = 20,000
\end{aligned}$$

$$\sum_{k=1}^{20,000}(-1)^{k+1}\frac{2k}{k^2+2} \approx 0.301765$$

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{3k}{2+4k^4}$$

We can verify via the Alternating Series Test that this infinite series converges.

$$\begin{aligned}
|S-S_n| \le a_{n+1} &< 10^{-4} \\
\frac{3(n+1)}{2+4(n+1)^4} &< 10^{-4} \\
n &> 18.5743 \implies \text{(smallest) } n = 19
\end{aligned}$$

$$\sum_{k=1}^{19}(-1)^{k+1}\frac{3k}{2+4k^4} \approx 0.428897$$

Comment

Checking the condition mentioned in the Alternating Series Test that $0 < a_{k+1} \leq a_k$ is essential. If this hypothesis is not satisfied, then the series may converge or diverge even if it is alternating and $a_k \to 0$ as $k \to \infty$.

Example: Convergent

Consider the alternating series,

$$1-2+1-\frac{1}{2}+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{4}-\frac{1}{8}+\cdots$$

We can see that $a_{k+1} > a_k$ for some terms violating the first condition of the AST. However the sequence of partial sums is

$$\left\{1,-1,0,\frac{-1}{2},\frac{1}{2},0,\frac{1}{4},\frac{-1}{4},0,\frac{-1}{8},\frac{1}{8},0,\ldots\right\}$$

and thus $S_n \to 0$ as $n \to \infty$. Hence

$$1-2+1-\frac{1}{2}+1-\frac{1}{2}+\frac{1}{4}-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{4}-\frac{1}{8}+\cdots=0$$

Example: Divergent

Consider the alternating series,

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} + \frac{1}{4} - \frac{1}{16} + \frac{1}{5} - \frac{1}{32} + \frac{1}{6} - \frac{1}{64} + \cdots$$

We can see that $a_{k+1} > a_k$ for some terms violating the first condition of the AST.

The sum of the negative terms is -1 (geometric series) while the positive terms are the Harmonic series (divergent). Hence the original alternating series diverges.

Homework

- Read Section 5.5
- Exercises: 251, 255, 259, . . . , 295/handout