Applications to Physics and Engineering

MATH 211, *Calculus II*

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Objectives

In this lesson we will learn to calculate:

- the center of mass of an object occupying a segment of the x-axis,
- the hydrostatic pressure and force exerted by a fluid (usually water) on the walls of a container.

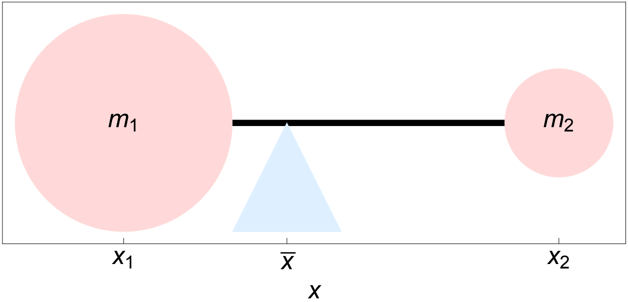
Moments and Center of Mass

In mathematics, the physical sciences, and engineering it is convenient to replace a rigid object of mass m by an idealized point-mass (also of m).

Question: where do we place the mass?

Example (1 of 2)

Suppose two masses m_1 and m_2 are attached to opposite ends of a rod.



Where can we support the rod so that the system is balanced?

Let $\overline{x}$ be the location of the balance point.

Example (2 of 2)

$$\begin{aligned} m_1(\overline{x} - x_1) &= m_2(x_2 - \overline{x}) \\ \overline{x}(m_1 + m_2) &= m_1 x_1 + m_2 x_2 \\ \overline{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \end{aligned}$$

Remark: the coordinate $\overline{x}$ is called the **center of mass** (or **center of gravity**) of the system.

General Case

Definition

If S denotes a set of point masses $\{m_1, m_2, \ldots, m_n\}$ located at the points $\{x_1, x_2, \ldots, x_n\}$ respectively along the x-axis, then

- the **total mass** is $m = \sum_{i=1}^{n} m_i$,
- the **moment about the origin** is $M_0 = \sum_{i=1}^{n} m_i x_i$,
- the **center of mass** is $\overline{x} = \dfrac{M_0}{m}$.

Example

Suppose S consists of masses $\{5, 7, 11, 13\}$ kg located at $\{-3, -1, 1, 2\}$ respectively along the x-axis. Find the center of mass of S.

$$
\begin{aligned}
m &= 5 + 7 + 11 + 13 = 36 \\
M_0 &= (5)(-3) + (7)(-1) + (11)(1) + (13)(2) = 15 \\
\overline{x} &= \frac{15}{36} = \frac{5}{12}
\end{aligned}
$$

Distributed Case

Suppose an object is continuously distributed along the x-axis in the interval $[a, b]$ and the **density** (mass/length) of the object is given by $\rho(x)$.

Question: how can we find the center of mass of such an object?

Answer: a Riemann Sum! Let $n \in \mathbb{N}$ and define $\Delta x = (b-a)/n$ and $x_i = a + i\Delta x$. The mass of the portion of the object in the interval $[x_{k-1}, x_k]$ is $\Delta m_k \approx \rho(x_k)\Delta x$. Thus the total mass of the object is

$$\begin{aligned} m &\approx \sum_{k=1}^{n} \rho(x_k)\Delta x \\ m &= \lim_{n\to\infty} \sum_{k=1}^{n} \rho(x_k)\Delta x = \int_a^b \rho(x)\, dx. \end{aligned}$$

Moment

We can use a Riemann sum to find the moment about the origin of the distributed object.

$$\begin{aligned} M_0 &\approx \sum_{k=1}^{n} x_k \rho(x_k) \Delta x \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} x_k \rho(x_k) \Delta x \\ M_0 &= \int_a^b x\, \rho(x)\, dx \end{aligned}$$

Thus the center of mass is

$$\overline{x} = \frac{M_0}{m} = \frac{\int_a^b x\, \rho(x)\, dx}{\int_a^b \rho(x)\, dx}.$$

Example (1 of 2)

Find the mass and center of mass of an object whose density is given by the $\rho(x) = x/5 + 1$ for $0 \leq x \leq 7$.

$$
\begin{aligned}
m &= \int_0^7 \left(\frac{x}{5} + 1 \right) dx = \left[\frac{x^2}{10} + x \right]_{x=0}^{x=7} = \frac{119}{10} \\
M_0 &= \int_0^7 \left(\frac{x^2}{5} + x \right) dx = \left[\frac{x^3}{15} + \frac{x^2}{2} \right]_{x=0}^{x=7} = \frac{1421}{30} \\
\overline{x} &= \frac{\frac{1421}{30}}{\frac{119}{10}} = \frac{203}{51} \approx 3.98039
\end{aligned}
$$

Example (2 of 2)

Find the mass and center of mass of an object whose density is given by the $\rho(x) = -x^2 - x + 6$ for $-3 \leq x \leq 2$.

$$\begin{aligned} m &= \int_{-3}^{2} \left(-x^2 - x + 6\right)\, dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 6x\right]_{x=-3}^{x=2} = \frac{125}{6} \\ M_0 &= \int_{-3}^{2} \left(-x^3 - x^2 + 6x\right)\, dx = \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2\right]_{x=-3}^{x=2} = -\frac{125}{12} \\ \overline{x} &= \frac{-\frac{125}{12}}{\frac{125}{6}} = -\frac{1}{2} \end{aligned}$$

Masses in the *xy*-plane

A system of n particles with masses $m_1, m_2, \ldots, m_n$ are located at the points with coordinates $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

The **moment of the system about the *y*-axis** is

$$M_y = \sum_{i=1}^{n} m_i x_i.$$

The **moment of the system about the *x*-axis** is

$$M_x = \sum_{i=1}^{n} m_i y_i.$$

The **centroid** or **center of mass** is found at coordinates

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

where $m = \sum_{i=1}^{n} m_i$.

Example

The masses m_i are located at the points P_i.

$$\begin{array}{cccc} m_1 = 5 & m_2 = 4 & m_3 = 3 & m_4 = 6 \\ P_1(-4, 2) & P_2(0, 5) & P_3(3, 2) & P_4(1, -2) \end{array}$$

Find the moments and the center of mass of the system.

Solution

$$\begin{aligned} m &= 5 + 4 + 3 + 6 = 18 \\ M_y &= 5(-4) + 4(0) + 3(3) + 6(1) = -5 \\ M_x &= 5(2) + 4(5) + 3(2) + 6(-2) = 24 \\ (\overline{x}, \overline{y}) &= \left(\frac{-5}{18}, \frac{24}{18}\right) = \left(\frac{-5}{18}, \frac{4}{3}\right) \end{aligned}$$

Center of Mass of a Plate

Suppose the region below the graph of $y = f(x)$ and above the x-axis for $a \leq x \leq b$ has a constant density ρ.

The moment of region about the y-axis is

$$M_y = \rho \int_a^b x\, f(x)\, dx.$$

The moment of region about the x-axis is

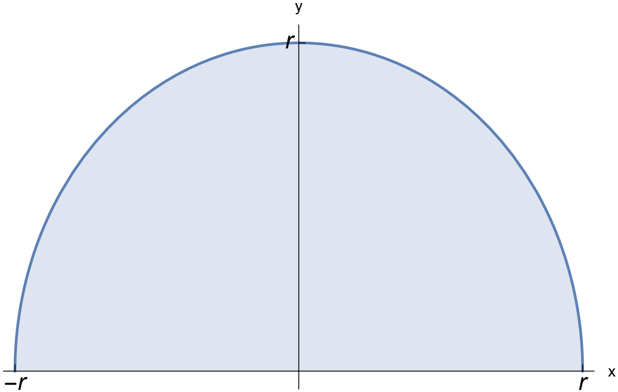
$$M_x = \rho \int_a^b \frac{1}{2}(f(x))^2\, dx.$$

The center of mass of the region is located at the point with coordinates,

$$(\overline{x}, \overline{y}) = \left(\frac{\rho \int_a^b x\, f(x)\, dx}{\rho \int_a^b f(x)\, dx}, \frac{\rho \int_a^b \frac{1}{2}(f(x))^2\, dx}{\rho \int_a^b f(x)\, dx} \right).$$

Example

Find the center of mass of a semicircular plate of radius $r > 0$.

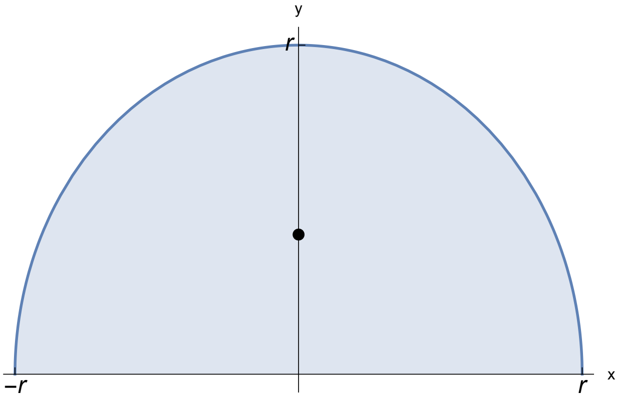


Solution

Let $f(x) = \sqrt{r^2 - x^2}$ for $-r \leq x \leq r$. Since the density is not mentioned, assume $\rho = 1$.

$$
\begin{aligned}
m &= \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \frac{1}{2}\pi r^2 \\
M_y &= \int_{-r}^{r} x\sqrt{r^2 - x^2}\,dx = \int_0^0 -\frac{1}{2}u^{1/2}\,du = 0 \\
M_x &= \int_{-r}^{r} \frac{1}{2}(\sqrt{r^2 - x^2})^2\,dx = \left[\frac{1}{2}\left(r^2 x - \frac{x^3}{3}\right)\right]_{x=-r}^{x=r} = \frac{2r^3}{3} \\
(\overline{x}, \overline{y}) &= \left(0, \frac{\frac{2r^3}{3}}{\frac{1}{2}\pi r^2}\right) = \left(0, \frac{4r}{3\pi}\right)
\end{aligned}
$$

Illustration of Centroid



Hydrostatic Force

Terminology:

pressure: force exerted per unit area (notation, p)

gravity: gravitational acceleration (notation, g)

- Metric units $g = 9.8$ m/s^2 or $g = 980$ cm/s^2.
- English units, $g = 32$ ft/s^2.

density: mass per unit volume (notation, ρ)

- for water $\rho = 1000$ kg/m^3 or $\rho = 1$ g/cm^3.
- English units, $\rho\, g = 62.4$ lb/ft^3.

depth: distance to the surface of a fluid (notation, h)

Pascal's Principle

Pascal's Principle: the pressure exerted at a depth h in a fluid is the same in every direction.

If the area of a plate is A then the force on the plate is $\rho\, g\, h\, A$, provided the plate is entirely at depth h.

Question: what if the plate is not oriented horizontally?

Riemann Sum Approach

- Suppose the plate is oriented vertically (parallel to the xy-plane) so that the width of the plate is a function of y, call it $w(y)$.
- Suppose the plate lies in the interval (on the y-axis) $[a, b]$.
- Let $n \in \mathbb{N}$ and $\Delta y = (b-a)/n$ and $y_i = a + i\Delta y$, then the force on the portion of the plate between y_{k-1} and y_k is

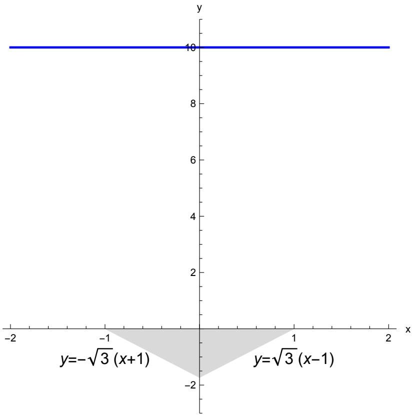
$$\Delta F_k \approx \rho\, g\, h(y_k) w(y_k) \Delta y.$$

- Total **hydrostatic force** is

$$\begin{aligned} F &\approx \sum_{k=1}^{n} \rho\, g\, h(y_k) w(y_k) \Delta y \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \rho\, g\, h(y_k) w(y_k) \Delta y \\ F &= \rho\, g \int_a^b h(y) w(y)\, dy \end{aligned}$$

Example

A dam has a submerged gate in the shape of an equilateral triangle, two feet on a side with the horizontal base nearest the surface of the water and ten feet below it. Find the force on the gate.



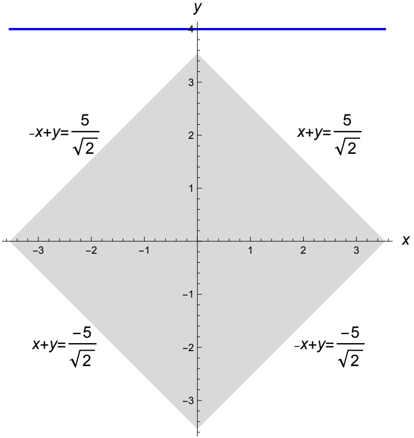
Solution

- The edges of the plate are described by the lines with equations $y = \sqrt{3}(x-1)$ and $y = -\sqrt{3}(x+1)$.
- The width of the plate is $w(y) = 2 + 2y/\sqrt{3}$
- The hydrostatic force is

$$
\begin{aligned}
F &= 62.4 \int_{-\sqrt{3}}^{0} (10-y)(2+2y/\sqrt{3})\,dy \\
&= 124.8 \int_{-\sqrt{3}}^{0} \left(10 + \frac{10}{\sqrt{3}}y - y - \frac{1}{\sqrt{3}}y^2\right) dy \\
&= 124.8 \left[10y + \frac{5}{\sqrt{3}}y^2 - \frac{1}{2}y^2 - \frac{1}{3\sqrt{3}}y^3\right]_{y=-\sqrt{3}}^{y=0} \\
&\approx 1143.2 \text{ lb.}
\end{aligned}
$$

Example

A square plate of sides 5 feet is submerged vertically in water such that one of the diagonals is parallel to the surface of the water. If the distance from the surface of the water to the center of the plate is 4 feet, find the force exerted by the water on one side of the plate.



Solution

- The edges of the plate are described by the lines with equations $y - x = 5/\sqrt{2}$ and $x + y = 5/\sqrt{2}$ (for $y \geq 0$) and by $x + y = -5/\sqrt{2}$ and $y - x = -5/\sqrt{2}$ (for $y < 0$).
- The width of the plate is

$$w(y) = \begin{cases} \frac{10}{\sqrt{2}} - 2y & \text{if } y \geq 0, \\ \frac{10}{\sqrt{2}} + 2y & \text{if } y < 0. \end{cases}$$

- The hydrostatic force is

$$\begin{aligned} F &= 62.4 \int_0^{5/\sqrt{2}} (4 - y) \left(\frac{10}{\sqrt{2}} - 2y \right) dy \\ &\quad + 62.4 \int_{-5/\sqrt{2}}^{0} (4 - y) \left(\frac{10}{\sqrt{2}} + 2y \right) dy \\ &= 62.4 \left(50 - \frac{125}{6\sqrt{2}} \right) + 62.4 \left(50 + \frac{125}{6\sqrt{2}} \right) = 6240 \text{ lb.} \end{aligned}$$

Homework

- Read Section 8.3
- Exercises: see WebAssign/D2L