Calculus and Parametric Equations

MATH 211, *Calculus II*

J. Robert Buchanan

Department of Mathematics

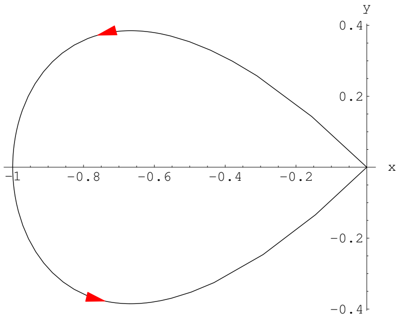
Fall 2021

Introduction

Given a pair a parametric equations

$$\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}$$

for $a \leq t \leq b$ we know how to graph the parametric curve.



Objectives

Today we will focus our attention on finding:

- the slope of the tangent line to the graph of a parametric curve,
- the area enclosed by a simple closed curve,
- the arc length of a parametric curve, and
- the surface area of a surface of revolution.

Slope of the Tangent Line

Suppose $x = f(t)$ and $y = g(t)$, by the Chain Rule for Derivatives

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Let $(x_0, y_0) = (x(t_0), y(t_0))$ then so long as $\frac{dx}{dt}(t_0) \neq 0$ then

$$\frac{dy}{dx}(x_0) = \frac{\frac{dy}{dt}(t_0)}{\frac{dx}{dt}(t_0)}.$$

Remark: If $x'(t_0) = y'(t_0) = 0$ then

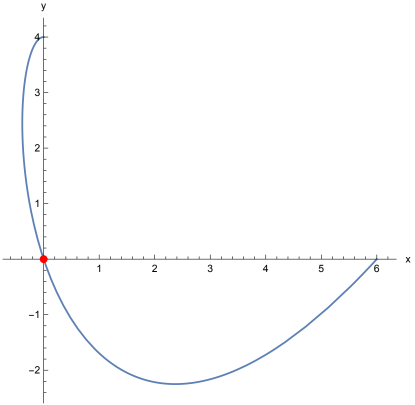
$$\frac{dy}{dx}(x_0) = \lim_{t \to t_0} \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \lim_{t \to t_0} \frac{y'(t)}{x'(t)},$$

provided the limit exists.

Example

Find the slope and equation of the tangent line for the following parametric equations at $t = 1$.

$$\begin{aligned} x &= t^3 - t \\ y &= t^4 - 5t^2 + 4 \end{aligned}$$



Solution

$$\begin{aligned}
\frac{dx}{dt} &= 3t^2 - 1 \\
\frac{dy}{dt} &= 4t^3 - 10t \\
\left.\frac{dy}{dx}\right|_{t=1} &= \frac{4-10}{3-1} = -3
\end{aligned}$$

Since $(x(1), y(1)) = (0, 0)$ then the equation of the tangent line is

$$y = -3x$$

Find the Second Derivative (Concavity)

The second derivative is the derivative of the first derivative.

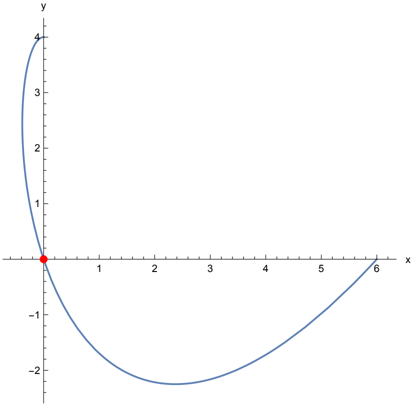
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Note: $\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$

Example

Find $\dfrac{d^2y}{dx^2}$ for the following parametric equations at $t = 1$.

$$\begin{aligned} x &= t^3 - t \\ y &= t^4 - 5t^2 + 4 \end{aligned}$$



Solution

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(\frac{4t^3-10t}{3t^2-1}\right)}{3t^2-1}$$

$$= \frac{\frac{(12t^2-10)(3t^2-1)-(4t^3-10t)(6t)}{(3t^2-1)^2}}{3t^2-1}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{\frac{(12-10)(3-1)-(4-10)(6)}{(3-1)^2}}{3-1} = 5$$

Finding Horizontal and Vertical Tangent Lines

Theorem

Suppose that $x'(t)$ and $y'(t)$ are continuous. Then for the curve defined by the parametric equations

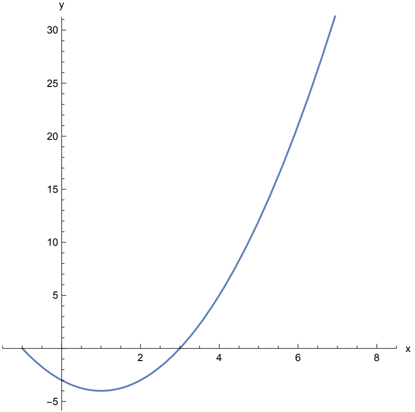
$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

1. *if $y'(c) = 0$ and $x'(c) \neq 0$, there is a horizontal tangent line at the point $(x(c), y(c))$.*
2. *if $x'(c) = 0$ and $y'(c) \neq 0$, there is a vertical tangent line at the point $(x(c), y(c))$.*

Example

Find the points at which the graph of the following parametric equations has horizontal or vertical tangent lines.

$$\begin{aligned} x &= t^2 - 1 \\ y &= t^4 - 4t^2 \end{aligned}$$



Solution

$$\begin{aligned}\frac{dx}{dt} &= 2t \\ \frac{dy}{dt} &= 4t^3 - 8t = 4t(t^2 - 2)\end{aligned}$$

Since when $y'(\pm\sqrt{2}) = 0$ and $x'(\pm\sqrt{2}) = \pm 2\sqrt{2} \neq 0$ then the graph has horizontal tangents when $t = \pm\sqrt{2}$.

$$\left(x(\pm\sqrt{2}), y(\pm\sqrt{2})\right) = (2 - 1, 4 - 8) = (1, -4)$$

Note that $x'(0) = y'(0) = 0$ so the slope of the tangent line when $t = 0$ is

$$\lim_{t \to 0} \frac{4t^3 - 8t}{2t} = \lim_{t \to 0} 2(t^2 - 2) = -4 \neq 0.$$

There are no vertical tangents.

Velocity and Speed

If the position of a moving object is given by the parametric equations

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

where $x(t)$ and $y(t)$ are differentiable we say

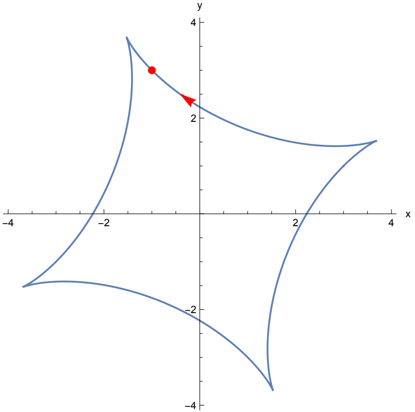
- the **horizontal component of velocity** is given by $x'(t)$,
- the **vertical component of velocity** is given by $y'(t)$, and
- the **speed** is given by $\sqrt{[x'(t)]^2 + [y'(t)]^2}$.

Example

Find the components of velocity and the speed of an object moving according to the parametric equations

$$\begin{aligned} x &= 3\cos t + \sin 3t \\ y &= 3\sin t + \cos 3t \end{aligned}$$

at $t = \pi/2$.



Solution

$$\begin{aligned} x'(t) &= -3\sin t + 3\cos 3t \\ y'(t) &= 3\cos t - 3\sin 3t \\ x'(\pi/2) &= -3 \\ y'(\pi/2) &= 3 \\ s(\pi/2) &= \sqrt{(-3)^2 + (3)^2} = 3\sqrt{2} \end{aligned}$$

Area Enclosed by a Curve (1 of 2)

Recall: if $y = f(x) \geq 0$ for $a \leq x \leq b$ then the area under the curve, above the x-axis and between $x = a$ and $x = b$ is given by

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx.$$

If the region is enclosed by parametrically defined curves

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

with $c \leq t \leq d$ then

$$A = \int_a^b \underbrace{y}_{y(t)} \underbrace{dx}_{x'(t)\,dt} = \int_c^d y(t)x'(t)\,dt.$$

Area Enclosed by a Curve (2 of 2)

Theorem

Suppose that the parametric equations $x = x(t)$ and $y = y(t)$ with $c \leq t \leq d$ describe a curve that is traced out clockwise exactly once as t increases from c to d and where the curve does not intersect itself, except that the initial and terminal points are the same, i.e., $x(c) = x(d)$ and $y(c) = y(d)$. Then the enclosed area is given by

$$A = \int_c^d y(t)x'(t)\,dt = -\int_c^d x(t)y'(t)\,dt.$$

If the curve is traced out counterclockwise, then the enclosed curve is given by

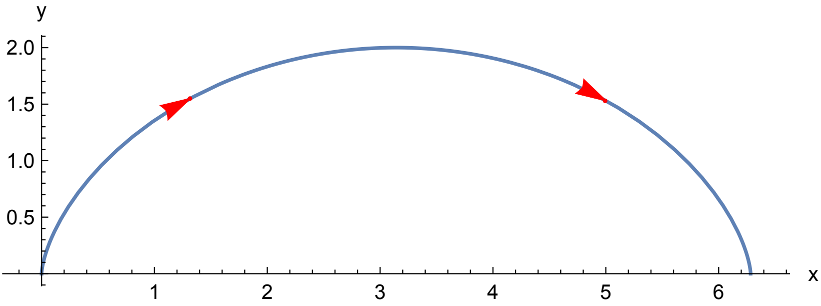
$$A = -\int_c^d y(t)x'(t)\,dt = \int_c^d x(t)y'(t)\,dt.$$

Example

Find the area enclosed by the graph of the parametric curve described by

$$\begin{aligned} x &= t - \sin t \\ y &= 1 - \cos t \end{aligned}$$

for $0 \leq t \leq 2\pi$.



Solution

$$
\begin{aligned}
A &= \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt \\
&= \int_0^{2\pi} (1-2\cos t+\cos^2 t)\,dt \\
&= \int_0^{2\pi} \left(1-2\cos t+\frac{1}{2}(1+\cos 2t)\right)\,dt \\
&= \int_0^{2\pi} \left(\frac{3}{2}-2\cos t+\frac{1}{2}\cos 2t\right)\,dt \\
&= 3\pi
\end{aligned}
$$

Example

The ellipse whose general formula is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for a, $b > 0$ is described parametrically by

$$\begin{aligned} x &= a\cos t \\ y &= b\sin t \end{aligned}$$

for $0 \leq t \leq 2\pi$. Use the parametric equations to find a formula for the area of an ellipse.

Solution

$$
\begin{aligned}
A &= -\int_0^{2\pi} (b\sin t)(-a\sin t)\,dt \\
&= ab\int_0^{2\pi} \sin^2 t\,dt \\
&= \frac{ab}{2}\int_0^{2\pi} (1-\cos 2t)\,dt \\
&= a\,b\,\pi
\end{aligned}
$$

Homework

- Read Section 7.2
- Exercises: 63, 67, 71, . . . , 103, 107/handout