Calculus and Polar Coordinates

MATH 211, *Calculus II*

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Motivation

Today we will focus on extending the ideas of slope, equation of the tangent line, arc length, and area to curves that are described as equations in polar coordinates.

Slope of the Tangent Line

The curve described by the polar equation $r = f(\theta)$ is equivalent to the parametric equations in rectangular coordinates:

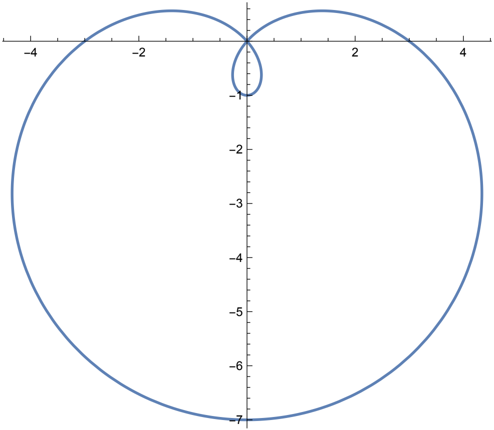
$$\begin{aligned} x &= f(\theta)\cos\theta \\ y &= f(\theta)\sin\theta. \end{aligned}$$

The slope of the tangent line at $\theta = \theta_0$ is

$$\begin{aligned} \left[\frac{dy}{dx}\right]_{\theta=\theta_0} &= \frac{\frac{dy}{d\theta}(\theta_0)}{\frac{dx}{d\theta}(\theta_0)} \\ &= \frac{f'(\theta_0)\sin\theta_0 + f(\theta_0)\cos\theta_0}{f'(\theta_0)\cos\theta_0 - f(\theta_0)\sin\theta_0} \end{aligned}$$

Example

Find a formula for the slope of the tangent line to the graph of $r = 3 - 4\sin\theta$.

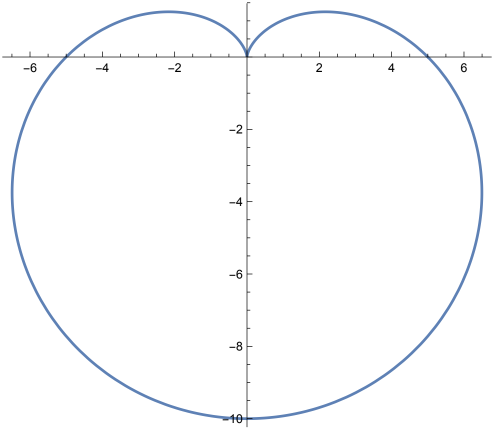


Solution

$$
\begin{aligned}
x(\theta) &= (3 - 4\sin\theta)\cos\theta \\
y(\theta) &= (3 - 4\sin\theta)\sin\theta \\
\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\
&= \frac{-4\cos\theta\sin\theta + (3 - 4\sin\theta)\cos\theta}{-4\cos\theta\cos\theta - (3 - 4\sin\theta)\sin\theta} \\
&= \frac{-8\cos\theta\sin\theta + 3\cos\theta}{4\sin^2\theta - 4\cos^2\theta - 3\sin\theta} \\
&= \frac{(3 - 8\sin\theta)\cos\theta}{8\sin^2\theta - 3\sin\theta - 4}
\end{aligned}
$$

Example

Find the points at which the graph of $r = 5 - 5\sin\theta$ has horizontal tangent lines.



Solution (1 of 2)

$$
\begin{aligned}
x(\theta) &= (5 - 5\sin\theta)\cos\theta \\
y(\theta) &= (5 - 5\sin\theta)\sin\theta \\
\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\
&= \frac{-5\cos\theta\sin\theta + (5 - 5\sin\theta)\cos\theta}{-5\cos\theta\cos\theta - (5 - 5\sin\theta)\sin\theta} \\
&= \frac{-2\cos\theta\sin\theta + \cos\theta}{\sin^2\theta - \cos^2\theta - \sin\theta} \\
&= \frac{(1 - 2\sin\theta)\cos\theta}{(2\sin\theta + 1)(\sin\theta - 1)}
\end{aligned}
$$

Solution (2 of 2)

$$\frac{dy}{dx} = \frac{(1 - 2\sin\theta)\cos\theta}{(2\sin\theta + 1)(\sin\theta - 1)} = 0$$

implies

$$\begin{aligned} 1 - 2\sin\theta &= 0 \implies \theta = \frac{\pi}{6} \quad \text{or} \quad \theta = \frac{5\pi}{6}, \\ \cos\theta &= 0 \implies \theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{3\pi}{2}. \end{aligned}$$

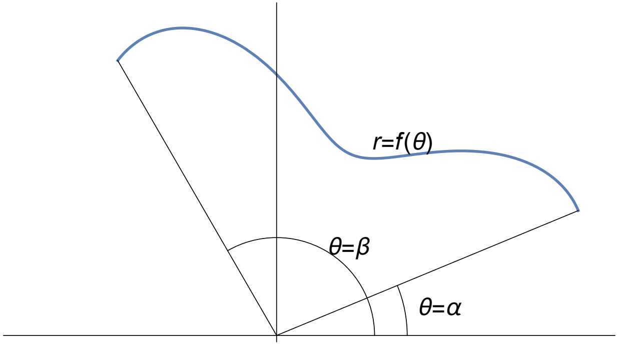
Note that if $\theta = \pi/2$ then dy/dx is indeterminate of form $0/0$.

$$\lim_{\theta\to\pi/2} \frac{dy}{dx} = \lim_{\theta\to\pi/2} \frac{(1 - 2\sin\theta)\cos\theta}{(2\sin\theta + 1)(\sin\theta - 1)}$$

does not exist.

Integrals in Polar Coordinates

Consider the region in the plane bounded by the lines $\theta = \alpha$, $\theta = \beta$, the origin, and the graph of the equation $r = f(\theta) \geq 0$.

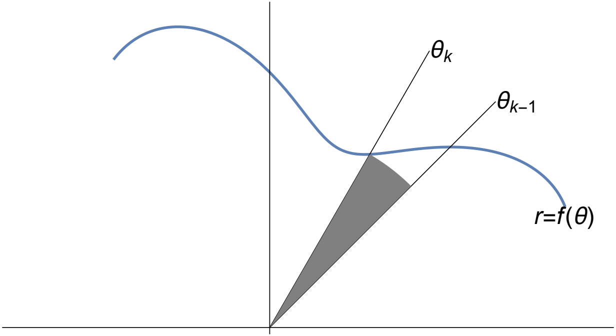


We would like to find the area of this region.

Area of a Circular Sector

Partition the interval $[\alpha, \beta]$ into n equal subintervals where $\Delta\theta = \dfrac{\beta - \alpha}{n}$ and $\theta_k = \alpha + k\Delta\theta$ for $k = 0, 1, \ldots, n$.

The area of the region in the subinterval $[\theta_{k-1}, \theta_k]$ can be approximated by the area of a circular sector.



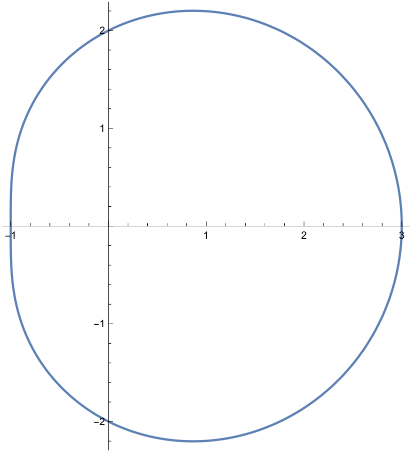
Riemann Sum

Area of a circular sector: $\Delta A_k = \frac{1}{2} r^2 \Delta\theta \approx \frac{1}{2} \left[f(\theta_k)\right]^2 \Delta\theta$.

$$\begin{aligned} A &\approx \sum_{k=1}^{n} \frac{1}{2} \left[f(\theta_k)\right]^2 \Delta\theta \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2} \left[f(\theta_k)\right]^2 \Delta\theta \\ &= \frac{1}{2} \int_{\alpha}^{\beta} \left[f(\theta)\right]^2 \, d\theta \end{aligned}$$

Example

Find the area enclosed by the limaçon: $r = 2 + \cos\theta$.

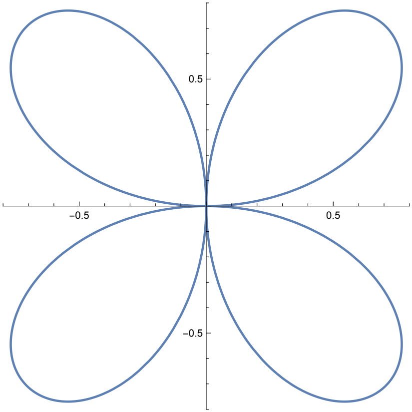


Solution

$$
\begin{aligned}
A &= \int_0^{2\pi} \frac{1}{2}(2+\cos\theta)^2\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi}(4+4\cos\theta+\cos^2\theta)\,d\theta \\
&= \int_0^{2\pi} 2\,d\theta + \int_0^{2\pi} 2\cos\theta\,d\theta + \frac{1}{2}\int_0^{2\pi}\cos^2\theta\,d\theta \\
&= 4\pi + 0 + \frac{1}{4}\int_0^{2\pi}(1+\cos 2\theta)\,d\theta \\
&= 4\pi + \frac{\pi}{2} = \frac{9\pi}{2}
\end{aligned}
$$

Example

Find the area enclosed by the rose: $r = \sin 2\theta$.

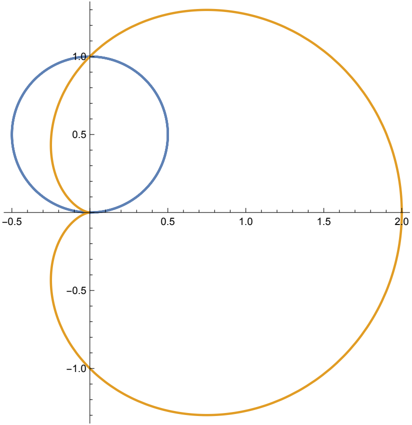


Solution

$$
\begin{aligned}
A &= 4\int_0^{\pi/2} \frac{1}{2}(\sin 2\theta)^2 \, d\theta \\
&= 2\int_0^{\pi/2} \sin^2 2\theta \, d\theta \\
&= \int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta \\
&= \frac{\pi}{2}
\end{aligned}
$$

Example

Find the area inside the circle $r = \sin\theta$ and outside the cardioid $r = 1 + \cos\theta$.



Solution

$$
\begin{aligned}
A &= \frac{1}{2}\int_{\pi/2}^{\pi} (\sin\theta)^2 - (1+\cos\theta)^2\, d\theta \\
&= \frac{1}{2}\int_{\pi/2}^{\pi} (\sin^2\theta - 1 - 2\cos\theta - \cos^2\theta)\, d\theta \\
&= \frac{1}{2}\int_{\pi/2}^{\pi} (-1 - 2\cos\theta - \cos 2\theta)\, d\theta \\
&= \left[\frac{1}{2}\left(-\theta - 2\sin\theta - \frac{1}{2}\sin 2\theta\right)\right]_{\theta=\pi/2}^{\theta=\pi} \\
&= -\frac{\pi}{2} - \frac{1}{2}\left(-\frac{\pi}{2} - 2\right) \\
&= 1 - \frac{\pi}{4}
\end{aligned}
$$

Arc Length

The curve described by the polar equation $r = f(\theta)$ is equivalent to the parametric equations in rectangular coordinates:

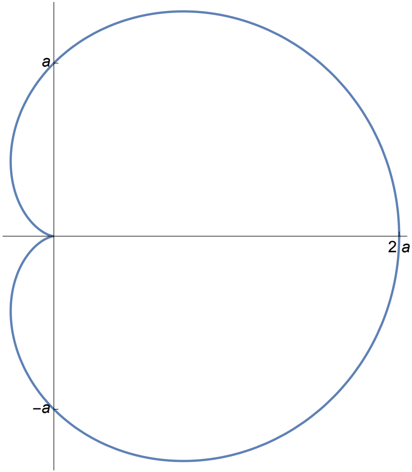
$$\begin{aligned} x &= f(\theta)\cos\theta \\ y &= f(\theta)\sin\theta. \end{aligned}$$

We already have a definite integral for the arc length of a parametrically defined curve.

$$\begin{aligned} s &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \end{aligned}$$

Example

Find the perimeter of the cardioid $r = a(1 + \cos\theta)$ where $a > 0$.

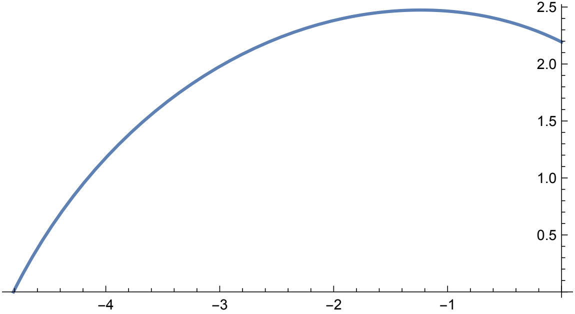


Solution

$$
\begin{aligned}
s &= \int_0^{2\pi} \sqrt{[a(1+\cos\theta)]^2 + [-a\sin\theta]^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{a^2(1+2\cos\theta+\cos^2\theta) + a^2\sin^2\theta}\, d\theta \\
&= a\int_0^{2\pi} \sqrt{2+2\cos\theta}\, d\theta = a\sqrt{2}\int_0^{2\pi} \sqrt{1+\cos\theta}\, d\theta \\
&= a\sqrt{2}\int_0^{2\pi} \sqrt{2\cos^2\frac{\theta}{2}}\, d\theta = 2a\int_0^{2\pi} \sqrt{\cos^2\frac{\theta}{2}}\, d\theta \\
&= 4a\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = \left[8a\sin\frac{\theta}{2}\right]_{\theta=0}^{\theta=\pi} \\
&= 8a
\end{aligned}
$$

Example

Find the arc length of the exponential spiral $r = e^{\theta/2}$ for $\pi/2 \leq \theta \leq \pi$.



Solution

$$
\begin{aligned}
s &= \int_{\pi/2}^{\pi} \sqrt{[e^{\theta/2}]^2 + \left[\frac{1}{2}e^{\theta/2}\right]^2}\, d\theta \\
&= \int_{\pi/2}^{\pi} \sqrt{e^{\theta} + \frac{1}{4}e^{\theta}}\, d\theta \\
&= \frac{\sqrt{5}}{2} \int_{\pi/2}^{\pi} e^{\theta/2}\, d\theta \\
&= \left[\sqrt{5}e^{\theta/2}\right]_{\theta=\pi/2}^{\theta=\pi} \\
&= \sqrt{5}(e^{\pi/2} - e^{\pi/4})
\end{aligned}
$$

Homework

- Read Section 7.4
- Exercises: 189, 193, 197, . . . , 251/handout