Conic Sections

MATH 211, *Calculus II*

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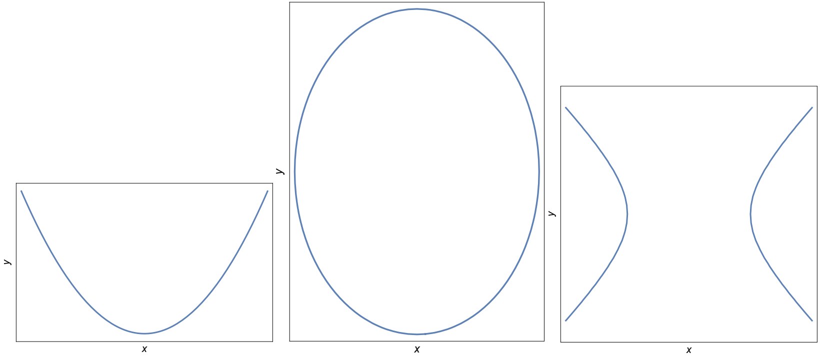
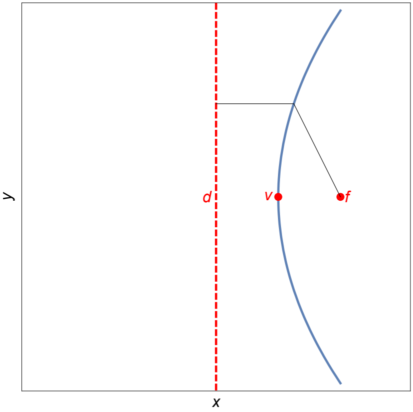
Introduction

The **conic sections** include the **parabola**, the **ellipse**, and the **hyperbola**.



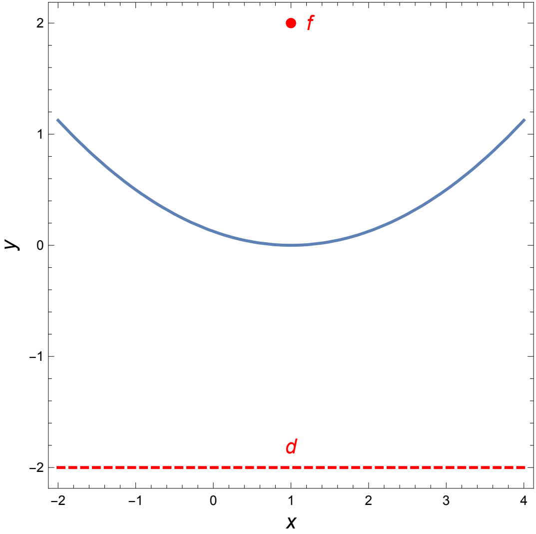
Parabola

A **parabola** is the set of all points in the plane that are equidistant from a fixed point called the **focus** and a fixed line called the **directrix**. The **vertex** is the point on the parabola at the intersection of the line perpendicular to the directrix and passing through the focus.



Example

Find the equation of the parabola with focus at $(1, 2)$ and directrix $y = -2$.



General Formula for the Parabola (1 of 2)

Theorem

The parabola with vertex at (b, c)*, focus at* $(b, c + \frac{1}{4a})$*, and directrix given by* $y = c - \frac{1}{4a}$ *has equation* $y = a(x - b)^2 + c$*.*

$$
\begin{aligned}
\left(y - c + \frac{1}{4a}\right)^2 &= (x - b)^2 + \left(y - c - \frac{1}{4a}\right)^2 \\
\left((y - c) + \frac{1}{4a}\right)^2 &= (x - b)^2 + \left((y - c) - \frac{1}{4a}\right)^2 \\
(y - c)^2 + \frac{1}{2a}(y - c) + \frac{1}{16a^2} &= (x - b)^2 + (y - c)^2 - \frac{1}{2a}(y - c) + \frac{1}{16a^2} \\
\frac{1}{a}(y - c) &= (x - b)^2 \\
y &= a(x - b)^2 + c
\end{aligned}
$$

General Formula for the Parabola (2 of 2)

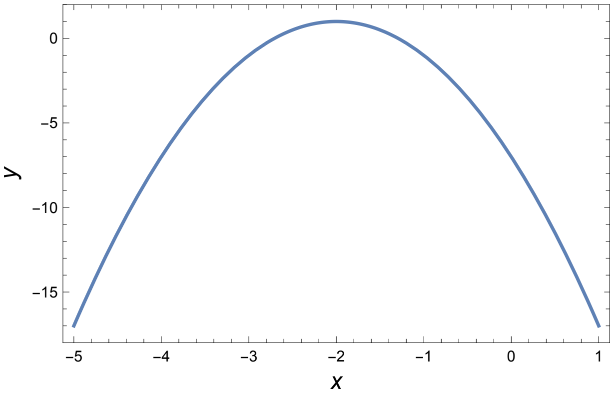
Remark: if we switch the roles of x and y we have the following result.

Theorem

The parabola with vertex at (c, b)*, focus at* $(c + \frac{1}{4a}, b)$*, and directrix given by* $x = c - \frac{1}{4a}$ *has equation* $x = a(y - b)^2 + c$*.*

Example

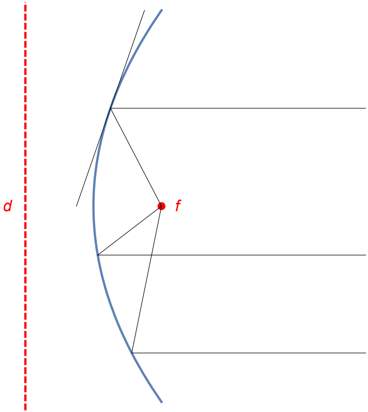
Given the equation for a parabola $y = -2(x+2)^2 + 1$, find the vertex, focus, and directrix.



vertex = $(-2, 1)$, focus = $(-2, 7/8)$, directrix: $y = 9/8$

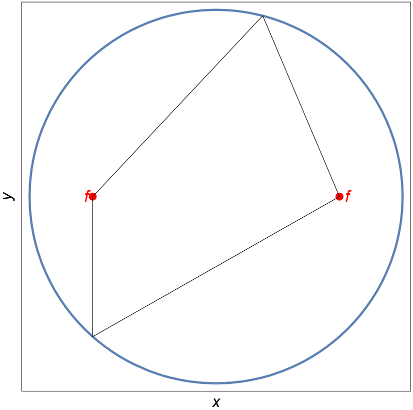
Reflective Property of Parabolas

If a parabola is thought of as a reflector (for example in a flashlight or satellite dish), all rays traveling perpendicular to the directrix and striking the parabola are reflected through the focus.



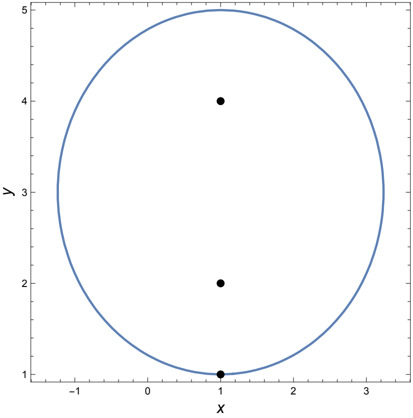
Ellipse

An **ellipse** is the set of all points in the plane for which the sum of the distances from two fixed points called **foci** is a constant.



Example

Suppose an ellipse has foci located at $(1, 2)$ and $(1, 4)$ and the point with coordinates $(1, 1)$ lies on the ellipse. Find the equation of the ellipse.



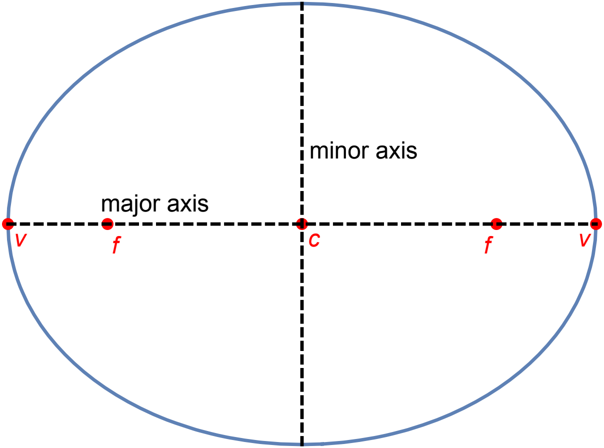
General Formula for Ellipse

Theorem

The equation $\dfrac{(x-x_0)^2}{a^2} + \dfrac{(y-y_0)^2}{b^2} = 1$ *with* $a > b > 0$ *describes an ellipse with foci at* $(x_0 - c, y_0)$ *and* $(x_0 + c, y_0)$ *where* $c = \sqrt{a^2 - b^2}$. *The* ***center*** *of the ellipse is at the point* (x_0, y_0) *and the* ***vertices*** *are located at* $(x_0 \pm a, y_0)$ *on the* ***major axis****. The endpoints of the* ***minor axis*** *are at* $(x_0, y_0 \pm b)$.

Remark: the roles of the major and minor axes are reversed when $b > a > 0$.

Anatomy of an Ellipse



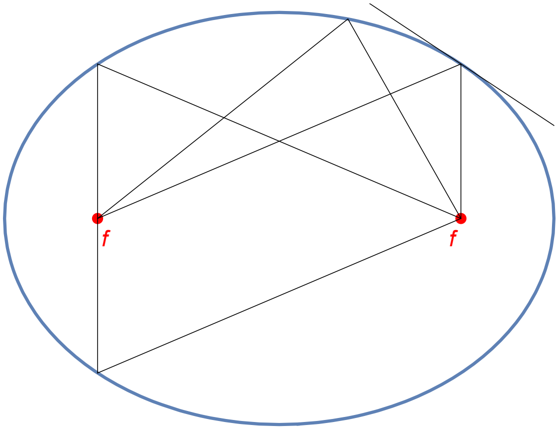
Example

Identify the following features of the ellipse
$$\frac{(x+1)^2}{9} + \frac{(y-3)^2}{4} = 1.$$

1. Center $(-1, 3)$
2. Foci $(-1 \pm \sqrt{5}, 3)$
3. Vertices $(-4, 3)$ and $(2, 3)$
4. Endpoints of minor axis $(-1, 1)$ and $(-1, 5)$

Reflective Property of Ellipses

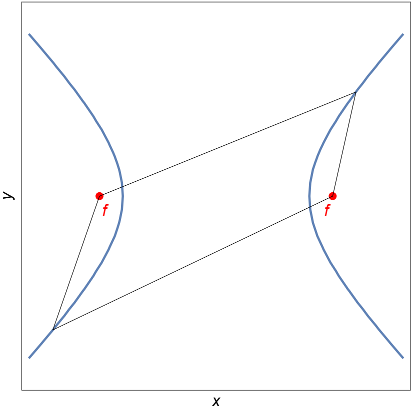
A ray emanating from one focus will always reflect off the ellipse and pass through the other focus.



Hyperbola

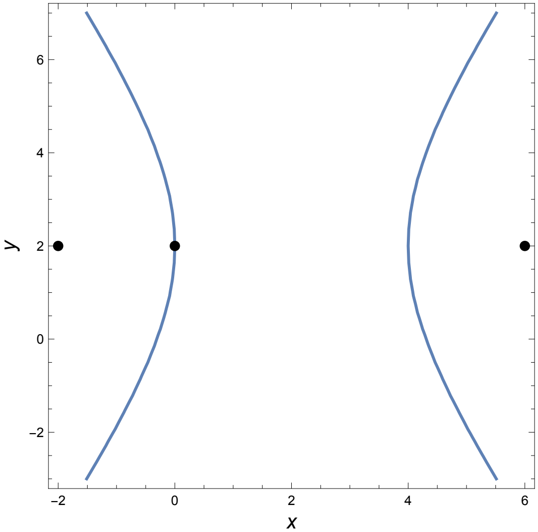
Definition

A **hyperbola** is the set of all points in the plane for which the difference of the distances from two fixed points called **foci** is a constant.



Example

Suppose a hyperbola has foci located at $(-2, 2)$ and $(6, 2)$ and the point with coordinates $(0, 2)$ lies on the hyperbola. Find the equation of the hyperbola.



General Formula for the Hyperbola

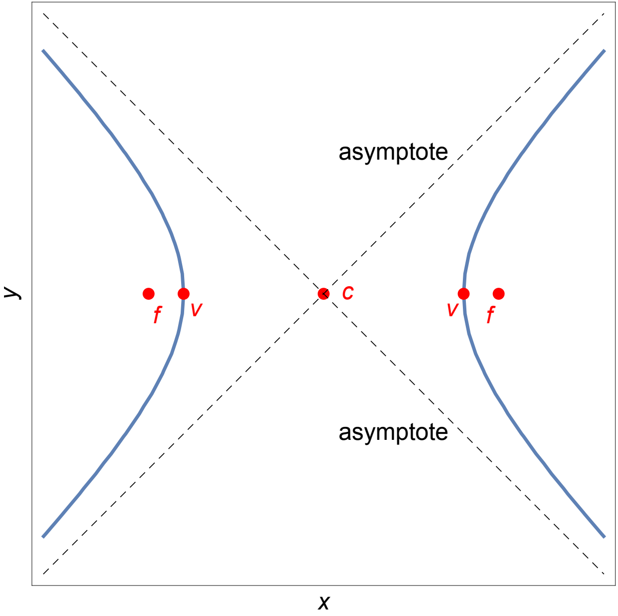
Theorem

The equation $\dfrac{(x-x_0)^2}{a^2} - \dfrac{(y-y_0)^2}{b^2} = 1$ *describes a hyperbola with foci at* $(x_0 - c, y_0)$ *and* $(x_0 + c, y_0)$ *where* $c = \sqrt{a^2 + b^2}$*. The* ***center*** *of the hyperbola is at the point* (x_0, y_0) *and the* ***vertices*** *are located at* $(x_0 \pm a, y_0)$*. The* ***asymptotes*** *are the lines* $y = \pm\dfrac{b}{a}(x - x_0) + y_0$*.*

Theorem

The equation $\dfrac{(y-y_0)^2}{a^2} - \dfrac{(x-x_0)^2}{b^2} = 1$ *describes a hyperbola with foci at* $(x_0, y_0 - c)$ *and* $(x_0, y_0 + c)$ *where* $c = \sqrt{a^2 + b^2}$*. The* ***center*** *of the hyperbola is at the point* (x_0, y_0) *and the* ***vertices*** *are located at* $(x_0, y_0 \pm a)$*. The* ***asymptotes*** *are the lines* $y = \pm\dfrac{a}{b}(x - x_0) + y_0$*.*

Anatomy of a Hyperbola



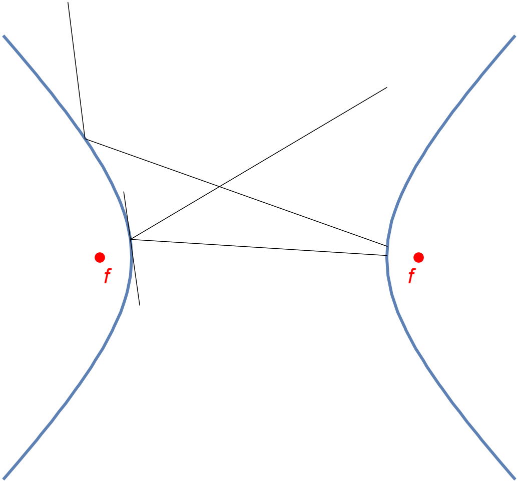
Example

Identify the following features of the hyperbola
$\frac{x^2}{4} - \frac{(y-1)^2}{16} = 1$.

1. Center $(0, 1)$
2. Foci $(\pm 2\sqrt{5}, 1)$
3. Vertices $(\pm 2, 1)$
4. Asymptotes $y = \pm 2x + 1$

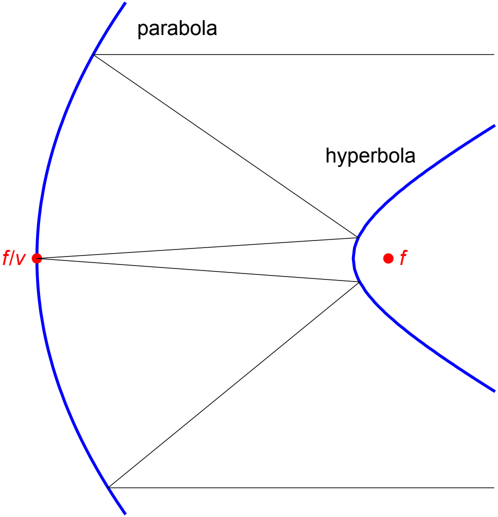
Reflective Property of Hyperbolas

A ray directed toward one focus will reflect off the hyperbola and travel toward the other focus.



Combination of Reflecting Properties

A reflecting parabola and reflecting hyperbola can be used together to focus incoming rays on a single point.



Homework

- Read Section 10.5
- Exercises: WebAssign/D2L