Volume: Slicing, Disks, and Washers

MATH 211, *Calculus II*

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Background

Today's discussion will center on using the definite integral to find the volume of a three-dimensional object.

Recall: for a rectangular box

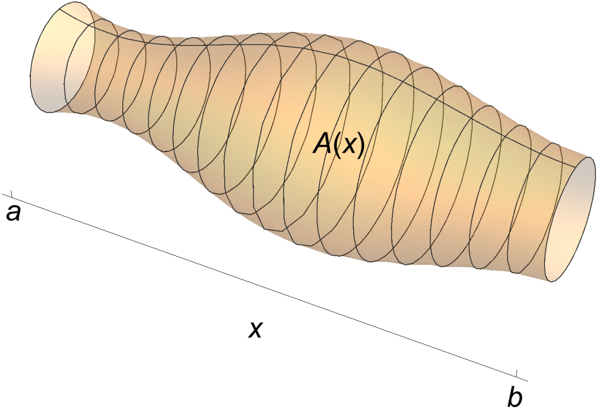
$$V = l\,w\,h = \underbrace{(\text{length} \times \text{width})}_{\text{cross-sectional area}} \times \text{height}$$

and for a cylinder

$$V = \pi r^2 h = \underbrace{\pi(\text{radius})^2}_{\text{cross-sectional area}} \times \text{height}$$

General Idea

Suppose a three-dimensional object lies along the x-axis covering the interval $[a, b]$ and the cross-sectional area is a continuous function of x, call it $A(x)$.



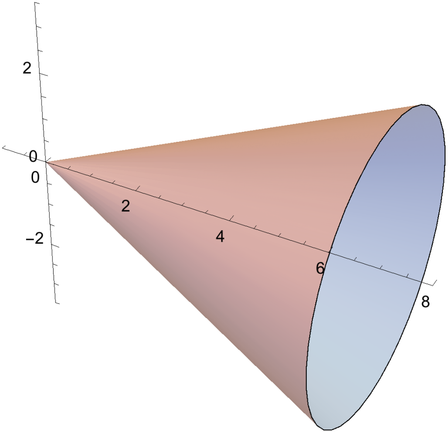
Riemann Sum Approach

Let $\Delta x = \dfrac{b-a}{n}$ for $n \in \mathbb{N}$ and define $x_i = a + i\Delta x$ for $i = 1, 2, \ldots, n$.

$$
\begin{aligned}
V &\approx \sum_{i=1}^{n} A(x_i)\Delta x \\
V &= \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x \\
&= \int_a^b A(x)\,dx
\end{aligned}
$$

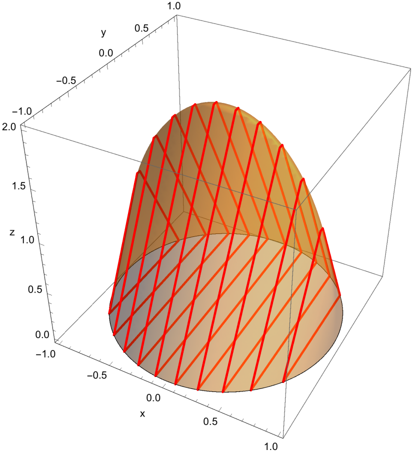
Example

Find the volume of a right circular cone whose base radius is 3 and whose height is 7.



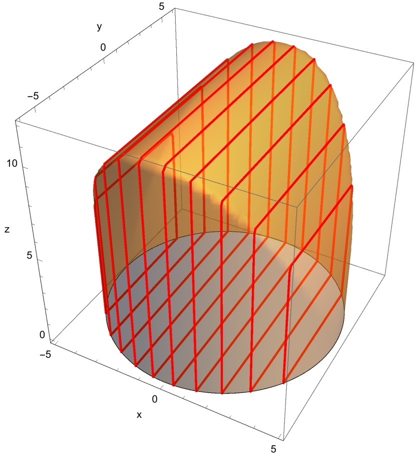
Example

A solid object has as its base the circular region defined by $x^2 + y^2 = 1$. Every cross section of the object perpendicular to the x-axis is a triangle whose base vertices are on the circle and whose height equals the length of the base. Find the volume of this object.



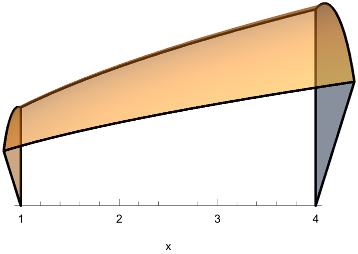
Example

A solid object has as its base the ellipse defined by $\frac{x^2}{25} + \frac{y^2}{36} = 1$. Every cross section of the object perpendicular to the x-axis is a square whose base vertices are on the ellipse. Find the volume of this object.



Solid of Revolution

If a region in the plane is revolved around a line in the plane the resulting three-dimensional object is called a **solid of revolution**.



Method of Disks

Suppose $f(x) \geq 0$ and continuous on $[a, b]$ and the region bounded beneath the graph of $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ is revolved around the x-axis.

Every cross section is a circular disk of radius $r = f(x)$, thus the volume of the solid of revolution is

$$V = \int_a^b \underbrace{\pi[f(x)]^2}_{\text{cross-sectional area}} \, dx.$$

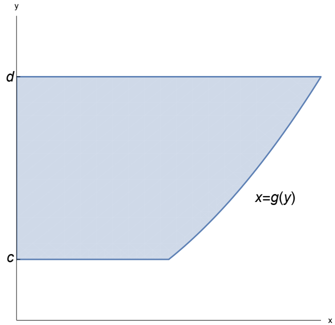
Example

Find the volume of the solid of revolution generated when $y = 2 - \frac{1}{2}x$ is revolved around the x-axis while $0 \leq x \leq 4$.

y as Independent Variable

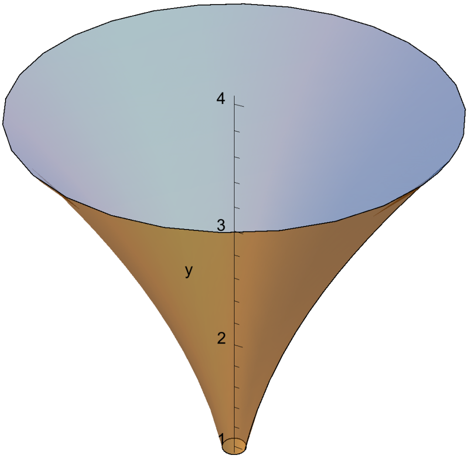
If $x = g(y)$ and $g(y) \geq 0$ for $c \leq y \leq d$ then the volume of the solid of revolution generated by revolving the region bounded between the graph of $x = g(y)$, the y-axis, and the lines $y = c$ and $y = d$ is given by

$$V = \int_c^d \underbrace{\pi[g(y)]^2}_{\text{cross-sectional area}} dy.$$



Example

Find the volume of the solid of revolution generated when $y = \sqrt{x}$ for $1 \leq y \leq 4$ is revolved around the y-axis.



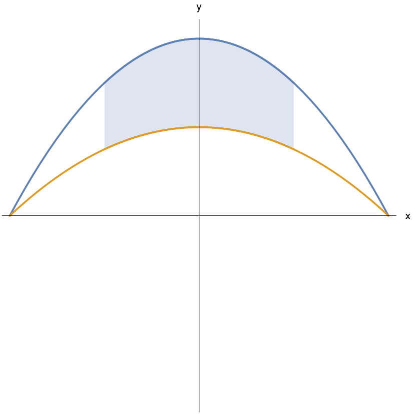
Solution

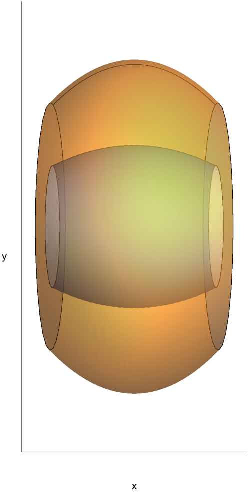
Since $y = \sqrt{x}$ then $x = y^2$ and the volume is calculated as follows.

$$
\begin{aligned}
V &= \pi \int_1^4 (y^2)^2 \, dy \\
&= \pi \int_1^4 y^4 \, dy \\
&= \left[\frac{\pi}{5} y^5 \right]_{y=1}^{y=4} \\
&= \frac{\pi (4)^5}{5} - \frac{\pi (1)^5}{5} \\
&= \frac{1024\pi}{5} - \frac{\pi}{5} = \frac{1023\pi}{5}
\end{aligned}
$$

Method of Washers

Sometimes a solid region has a "hole" inside it.





Question: how can we find the volume of the solid of revolution with a cavity?

Method of Washers Formula

If $f(x)$ and $g(x)$ are continuous on $[a, b]$ and $0 \leq f(x) \leq g(x)$ on $[a, b]$ the volume of the solid of revolution generated by revolving the region bounded between the graphs of $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ is given by

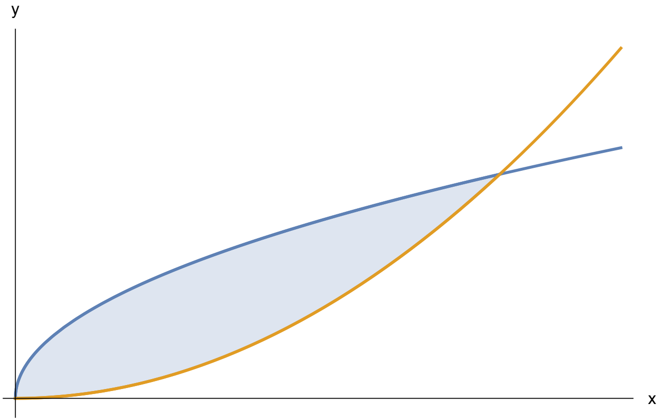
$$V = \int_a^b \pi \Big(\underbrace{[g(x)]^2 - [f(x)]^2}_{(\text{outer radius})^2 - (\text{inner radius})^2} \Big) \, dx$$

Example

Find the volume of the solid of revolution generated by revolving the region bounded by $y = 2 - x^2/2$, $y = 1 - x^2/4$, $x = -1$, and $x = 1$ around the x-axis.

Example

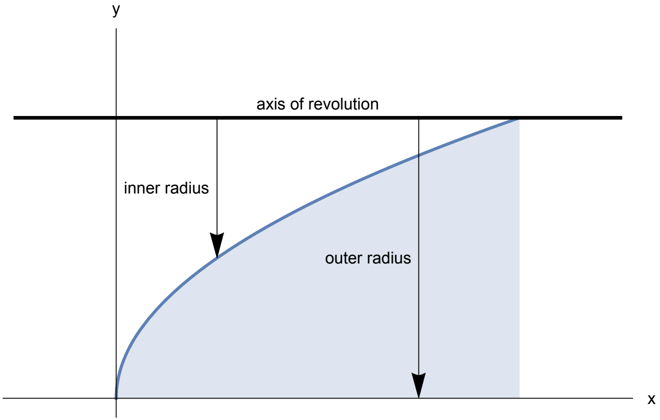
Find the volume of the solid of revolution generated when the region bounded by the curves $y = \sqrt{x}$ and $y = x^2$ is revolved around the y-axis.



Revolving Around Different Lines

We can modify our previous formulas to handle the cases in which we revolve a region around horizontal or vertical line not intersecting the region.

The key is to think about the radius of revolution in each case.

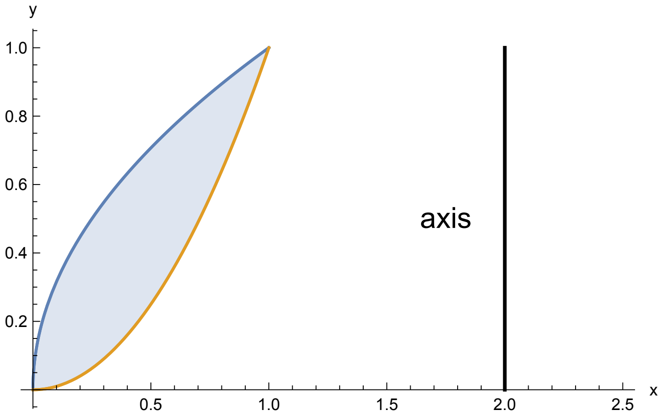


Example

Find the volume of the solid of revolution generated when the region bounded between $y = \sqrt{x}$, $y = 0$ and $x = 4$ is revolved around the line $y = 2$.

Example

Find the volume of the solid of revolution generated when the region bounded between $y = \sqrt{x}$ and $y = x^2$ is revolved around the line $x = 2$.



Homework

- Read Section 2.2
- Exercises: 69–89 odd, 99, 101/handout