Fourier Series

MATH 211, *Calculus II*

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Introduction

- Not all functions can be represented by Taylor series.
- A Taylor series $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ is exact at $x = c$ but as x moves away from c the error in the Taylor polynomial $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ grows quickly.
- Many natural phenomena are periodic and thus we may use periodic functions such as sine and cosine in infinite series.

Periodic Functions

Definition

A function $f(x)$ is periodic with period T if $f(x+T) = f(x)$ for all x.

Example

$\sin(k\,x)$ is periodic of period $\dfrac{2\pi}{k}$ since

$$\sin\left(k\left[x + \frac{2\pi}{k}\right]\right) = \sin\left(kx + 2\pi\right) = \sin(k\,x).$$

Likewise $\cos(k\,x)$ is periodic of period $\dfrac{2\pi}{k}$.

Fourier Series

Definition

A **Fourier Series** with period 2π is an infinite series of the form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\,x) + b_k \sin(k\,x))$$

where a_k for $k = 0, 1, \ldots$ and b_k for $k = 1, 2, \ldots$ are constants.

Questions:

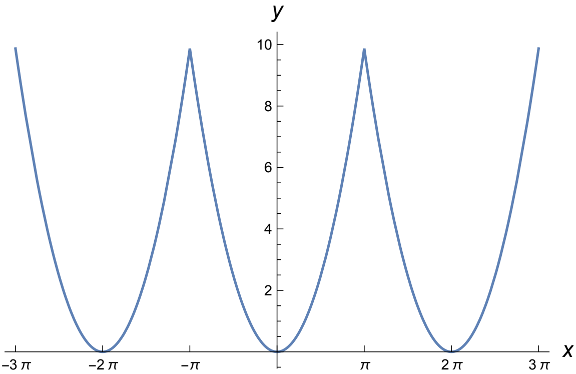
- Which functions can be represented by Fourier series?
- How do we calculate the coefficients (a_k and b_k) of the series?
- Does the Fourier series converge?
- If the series converges, to what function?

Answers

Suppose the Fourier series converges for $-\pi \leq x \leq \pi$ then

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx) \right)$$

and $f(x)$ must be 2π-periodic for all x.



Calculating the Coefficients (1 of 6)

Suppose $$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\,x) + b_k \sin(k\,x))$$

Since a_0 appears outside of the infinite series, we will find it first.

Integrate both sides of the equation on $[-\pi, \pi]$.

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx \\ &\quad + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(k\,x)\,dx + b_k \int_{-\pi}^{\pi} \sin(k\,x)\,dx \right)\end{aligned}$$

Calculating the Coefficients (2 of 6)

Note:

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{a_0}{2}\,dx &= a_0\pi \\
\int_{-\pi}^{\pi} \cos(k\,x)\,dx &= 0 \\
\int_{-\pi}^{\pi} \sin(k\,x)\,dx &= 0
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\,dx &= a_0\pi \\
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx
\end{aligned}$$

Calculating the Coefficients (3 of 6)

To calculate a_n for $n = 1, 2, \ldots$, multiply both sides of the Fourier series by $\cos(n\,x)$ and integrate over $[-\pi, \pi]$.

$$\int_{-\pi}^{\pi} f(x) \cos(n\,x)\,dx$$

$$= \int_{-\pi}^{\pi} \frac{a_0}{2} \cos(n\,x)\,dx + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(k\,x) \cos(n\,x)\,dx + b_k \int_{-\pi}^{\pi} \sin(k\,x) \cos(n\,x)\,dx \right)$$

Two new types of definite integral appear on the right hand side of the equation.

$$\int_{-\pi}^{\pi} \cos(k\,x) \cos(n\,x)\,dx \text{ and } \int_{-\pi}^{\pi} \sin(k\,x) \cos(n\,x)\,dx$$

We should evaluate them first.

Product-to-Sum Formulas

We will need the trigonometric identities:

$$\begin{aligned}
\cos(k\,x)\cos(n\,x) &= \frac{1}{2}\left(\cos(n+k)x + \cos(n-k)x\right) \\
\sin(k\,x)\cos(n\,x) &= \frac{1}{2}\left(\sin(n+k)x - \sin(n-k)x\right)
\end{aligned}$$

Proof.

$$\begin{aligned}
\cos(n+k)x + \cos(n-k)x &= \cos(n\,x)\cos(k\,x) - \sin(n\,x)\sin(k\,x) \\
&\quad + \cos(n\,x)\cos(k\,x) + \sin(n\,x)\sin(k\,x) \\
&= 2\cos(n\,x)\cos(k\,x) \\
\sin(n+k)x - \sin(n-k)x &= \sin(n\,x)\cos(k\,x) + \cos(n\,x)\sin(k\,x) \\
&\quad - \sin(n\,x)\cos(k\,x) + \cos(n\,x)\sin(k\,x) \\
&= 2\cos(n\,x)\sin(k\,x)
\end{aligned}$$

□

$\int_{-\pi}^{\pi} \cos(k\,x)\cos(n\,x)\,dx$

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(k\,x)\cos(n\,x)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-k)x + \cos(n+k)x\,dx \\
&= \frac{1}{2}\left[\frac{\sin(n-k)x}{n-k} + \frac{\sin(n+k)x}{n+k}\right]_{x=-\pi}^{x=\pi} \\
&= 0 \text{ if } n \neq k. \\
\int_{-\pi}^{\pi} \cos(n\,x)\cos(n\,x)\,dx &= \int_{-\pi}^{\pi} \cos^2(n\,x)\,dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos(2n\,x))\,dx \\
&= \frac{1}{2}\left[x + \frac{1}{2n}\sin(2n\,x)\right]_{x=-\pi}^{x=\pi} \\
&= \pi \\
\int_{-\pi}^{\pi} \cos(k\,x)\cos(n\,x)\,dx &= \begin{cases} 0 & \text{if } n \neq k, \\ \pi & \text{if } n = k. \end{cases}
\end{aligned}$$

$\int_{-\pi}^{\pi} \sin(kx)\cos(nx)\,dx$

$$
\begin{aligned}
\int_{-\pi}^{\pi} \sin(k\,x)\cos(n\,x)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+k)x - \sin(n-k)x\,dx \\
&= \frac{1}{2}\left[\frac{-\cos(n+k)x}{n+k} + \frac{\cos(n-k)x}{n-k}\right]_{x=-\pi}^{x=\pi} \\
&= 0 \text{ if } n \neq k \\
\int_{-\pi}^{\pi} \sin(n\,x)\cos(n\,x)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} \sin(2n\,x)\,dx \\
&= \frac{1}{2}\left[\frac{-\cos(2n\,x)}{2n}\right]_{x=-\pi}^{x=\pi} \\
&= 0
\end{aligned}
$$

Calculating the Coefficients (4 of 6)

Now if

$$\int_{-\pi}^{\pi} f(x)\cos(n\,x)\,dx$$

$$= \int_{-\pi}^{\pi} \frac{a_0}{2}\cos(n\,x)\,dx + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos(k\,x)\cos(n\,x)\,dx + b_k\int_{-\pi}^{\pi}\sin(k\,x)\cos(n\,x)\,dx\right)$$

then

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\cos(n\,x)\,dx &= a_n\int_{-\pi}^{\pi}\cos^2(n\,x)\,dx \\ &= a_n\pi \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(n\,x)\,dx.\end{aligned}$$

Calculating the Coefficients (5 of 6)

To calculate b_n for $n = 1, 2, \ldots$ multiply both sides of the Fourier series by $\sin(nx)$ and integrate over $[-\pi, \pi]$.

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x) \sin(n\,x)\,dx \\
&= \int_{-\pi}^{\pi} \frac{a_0}{2} \sin(n\,x)\,dx \\
&\quad + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos(k\,x) \sin(n\,x)\,dx + b_k \int_{-\pi}^{\pi} \sin(k\,x) \sin(n\,x)\,dx \right)
\end{aligned}$$

One new type of definite integral appears on the right hand side of the equation.

$$\int_{-\pi}^{\pi} \sin(k\,x) \sin(n\,x)\,dx$$

We should evaluate it first.

Product-to-Sum Formulas

We will need the trigonometric identity:

$$\sin(k\,x)\sin(n\,x) = \frac{1}{2}\left(\cos(n-k)x - \cos(n+k)x\right)$$

Proof.

$$\begin{aligned}
\cos(n-k)x - \cos(n+k)x &= \cos(n\,x)\cos(k\,x) + \sin(n\,x)\sin(k\,x) \\
&\quad - \cos(n\,x)\cos(k\,x) + \sin(n\,x)\sin(k\,x) \\
&= 2\sin(n\,x)\sin(k\,x)
\end{aligned}$$

□

$\int_{-\pi}^{\pi} \sin(kx)\sin(nx)\,dx$

$$
\begin{aligned}
\int_{-\pi}^{\pi} \sin(k\,x)\sin(n\,x)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-k)x - \cos(n+k)x\,dx \\
&= \frac{1}{2}\left[\frac{\sin(n-k)x}{n-k} - \frac{\sin(n+k)x}{n+k}\right]_{x=-\pi}^{x=\pi} \\
&= 0 \text{ if } n \neq k \\
\int_{-\pi}^{\pi} \sin(n\,x)\sin(n\,x)\,dx &= \int_{-\pi}^{\pi} \sin^2(n\,x)\,dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos(2n\,x))\,dx \\
&= \frac{1}{2}\left[x + \frac{1}{2n}\sin(2n\,x)\right]_{x=-\pi}^{x=\pi} \\
&= \pi \\
\int_{-\pi}^{\pi} \sin(k\,x)\sin(n\,x)\,dx &= \begin{cases} 0 & \text{if } n \neq k, \\ \pi & \text{if } n = k. \end{cases}
\end{aligned}
$$

Calculating the Coefficients (6 of 6)

Now if

$$\int_{-\pi}^{\pi} f(x)\sin(n\,x)\,dx$$

$$= \int_{-\pi}^{\pi} \frac{a_0}{2}\sin(n\,x)\,dx + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos(k\,x)\sin(n\,x)\,dx + b_k\int_{-\pi}^{\pi}\sin(k\,x)\sin(n\,x)\,dx\right)$$

then

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\sin(n\,x)\,dx &= b_n\int_{-\pi}^{\pi}\sin^2(n\,x)\,dx \\ &= b_n\pi \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(n\,x)\,dx.\end{aligned}$$

Euler-Fourier Formulas

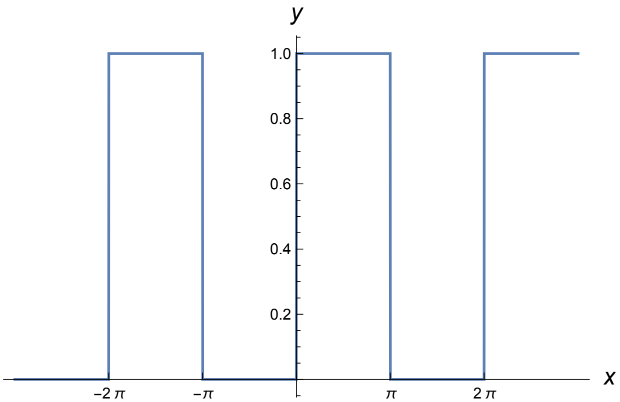
To summarize, the coefficients of the Fourier series for a 2π-periodic function $f(x)$ are:

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(k\,x)\,dx \quad (k = 1, 2, \ldots) \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(k\,x)\,dx \quad (k = 1, 2, \ldots)
\end{aligned}
$$

Example

Find the Fourier series representation of the 2π-periodic extension of the function

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi. \end{cases}$$

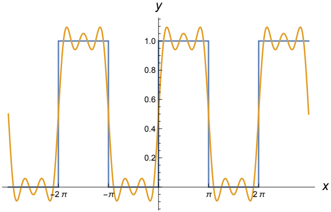


Solution

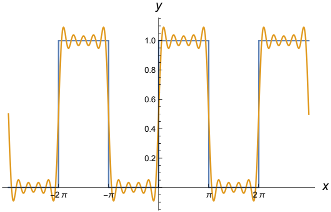
$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\,dx = 1 \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(k\,x)\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\cos(k\,x)\,dx = 0 \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(k\,x)\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\sin(k\,x)\,dx \\
&= \frac{1}{\pi}\left[\frac{-\cos(k\,x)}{k}\right]_{x=0}^{x=\pi} = \frac{1}{k\,\pi}(1-\cos(k\pi)) = \frac{(1-(-1)^k)}{k\,\pi}
\end{aligned}$$

Partial Sums

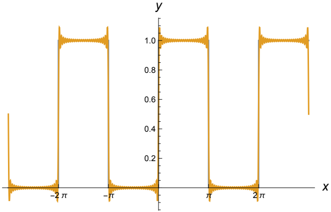
$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(1-(-1)^k)}{\pi k} \sin(k\,x)$$



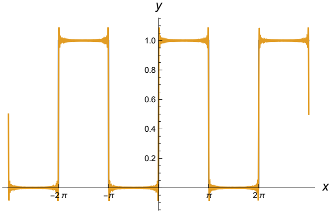
$n = 5$



$n = 10$



$n = 50$

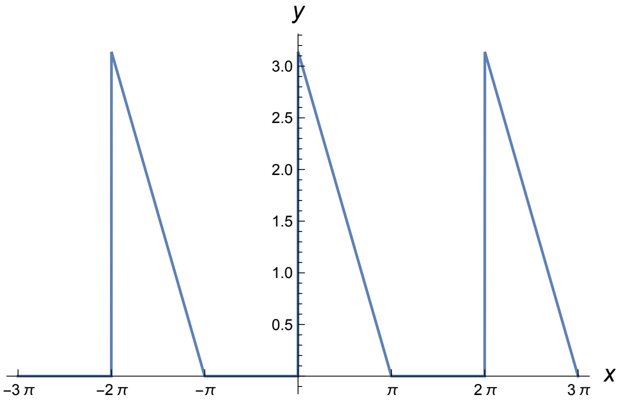


$n = 100$

Example

Find the Fourier series representation of the 2π-periodic extension of the function

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi. \end{cases}$$

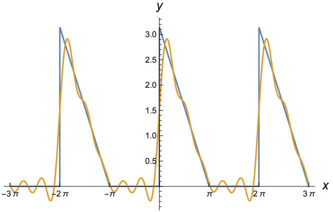


Solution

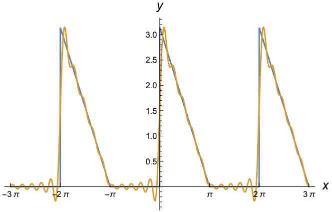
$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_{x=0}^{x=\pi} = \frac{\pi}{2} \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(k\,x)\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos(k\,x)\,dx \\
&= \frac{1}{k\pi}\int_0^{\pi}\sin(k\,x)\,dx = \frac{1}{k^2\pi}\left[-\cos(k\,x)\right]_{x=0}^{x=\pi} = \frac{(1-(-1)^k)}{k^2\pi} \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(k\,x)\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin(k\,x)\,dx \\
&= \frac{1}{\pi}\left(\left[\frac{-(\pi - x)\cos(k\,x)}{k}\right]_{x=0}^{x=\pi} + \frac{1}{k}\int_0^{\pi}\cos(k\,x)\,dx\right) = \frac{1}{k}
\end{aligned}
$$

Solution

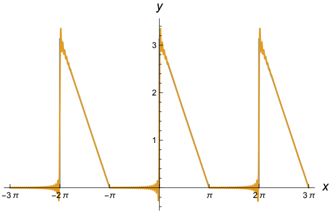
$$f(x) = \frac{\pi}{4} + \sum_{k=1}^{\infty} \left(\frac{(1-(-1)^k)}{\pi k^2} \cos(k\,x) + \frac{1}{k} \sin(k\,x) \right)$$



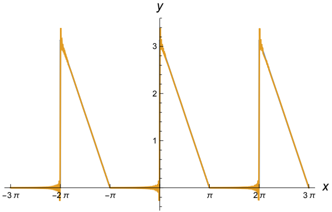
$n = 5$



$n = 10$



$n = 50$



$n = 100$

Functions of Period $T \neq 2\pi$

If f is a function of period $T = 2L > 0$ then the Fourier series for f is

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right]$$

where

$$\begin{aligned}
a_0 &= \frac{1}{L} \int_{-L}^{L} f(x)\, dx \\
a_k &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi x}{L}\right) dx \quad (k = 1, 2, \ldots) \\
b_k &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{k\pi x}{L}\right) dx \quad (k = 1, 2, \ldots)
\end{aligned}$$

Convergence of Fourier Series

Theorem

If f is $2L$-periodic and f and f' are continuous on $[-L, L]$ except for at most a finite number of jump discontinuities, then f has a convergent Fourier series expansion.

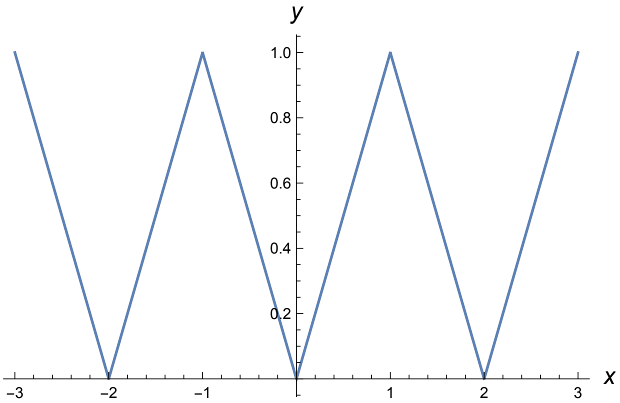
$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right) \right]$$

The series converges to $f(c)$ if f is continuous at $x = c$. If f has a jump discontinuity at $x = c$ then the series converges to

$$\frac{1}{2}\left(\lim_{x \to c^-} f(x) + \lim_{x \to c^+} f(x) \right).$$

Example

Find the Fourier series representation of the 2-periodic extension of the function $f(x) = |x|$.

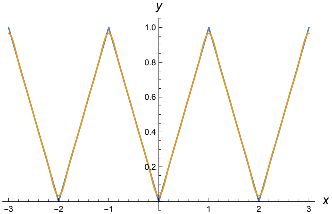


Solution

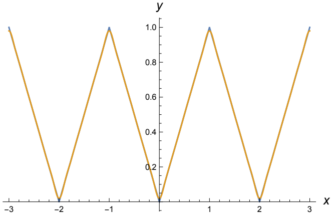
$$
\begin{aligned}
a_0 &= \int_{-1}^{1} f(x)\,dx = 2\int_0^1 x\,dx = 2\left[\frac{x^2}{2}\right]_{x=0}^{x=1} = 1\\
a_k &= \int_{-1}^{1} f(x)\cos(k\pi x)\,dx = 2\int_0^1 x\cos(k\pi x)\,dx\\
&= \frac{-2}{k\,\pi}\int_0^1 \sin(k\pi x)\,dx = \frac{2}{k^2\pi^2}\left[\cos(k\pi x)\right]_{x=0}^{x=1} = \frac{2((-1)^k-1)}{k^2\pi^2}\\
b_k &= \int_{-1}^{1} f(x)\sin(k\pi x)\,dx = 0
\end{aligned}
$$

Solution

$$f(x) = |x| = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2((-1)^k - 1)}{\pi^2 k^2} \cos(k\pi x)$$



$n = 5$



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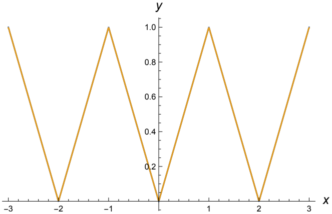
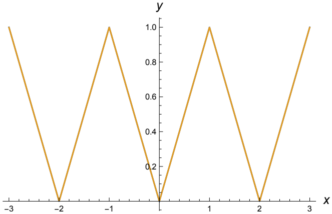


$n = 50$

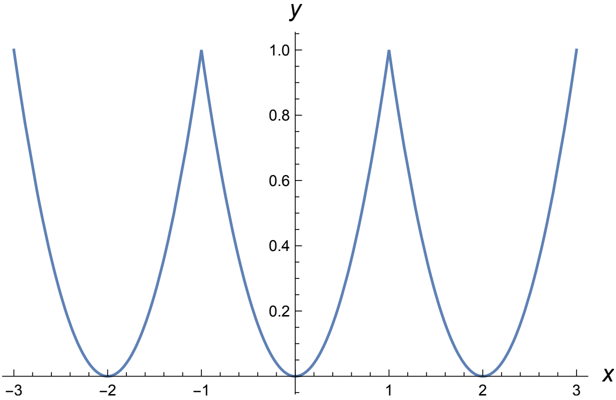


$n = 100$

Example

Find the Fourier series representation of the 2-periodic extension of the function $f(x) = x^2$. Use the series and the Fourier Convergence Theorem to find the sums:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ and } \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}.$$



Solution (1 of 3)

$$
\begin{aligned}
a_0 &= \int_{-1}^{1} x^2\,dx = 2\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{2}{3} \\
a_k &= \int_{-1}^{1} x^2 \cos(k\pi x)\,dx = 2\int_0^1 x^2\cos(k\pi x)\,dx \\
&= \frac{-4}{k\pi}\int_0^1 x\sin(k\pi x)\,dx \\
&= \frac{-4}{k\pi}\left(\left[\frac{-x}{k\pi}\cos(k\pi x)\right]_{x=0}^{x=1} + \int_0^1 \frac{1}{k\pi}\cos(k\pi x)\,dx\right) \\
&= \frac{4}{k^2\pi^2}\cos(k\pi) = \frac{4(-1)^k}{k^2\pi^2} \\
b_k &= \int_{-1}^{1} x^2\sin(k\pi x)\,dx = 0
\end{aligned}
$$

Solution (2 of 3)

$$x^2 = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2\pi^2} \cos(k\pi x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(k\pi x)$$

Let $x = 0$, then

$$\begin{aligned}
(0)^2 &= \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(k\pi(0)) \\
0 &= \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \\
\frac{-\pi^2}{12} &= \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}.
\end{aligned}$$

Solution (3 of 3)

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(k\pi x)$$

Let $x = 1$, then

$$\begin{aligned}
(1)^2 &= \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(k\pi(1)) \\
1 &= \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} (-1)^k \\
\frac{\pi^2}{6} &= \sum_{k=1}^{\infty} \frac{1}{k^2}.
\end{aligned}$$

Other Sums (1 of 2)

$$
\begin{aligned}
|x| &= \frac{1}{2} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{((-1)^k - 1)}{k^2} \cos(k\pi x) \\
|0| &= \frac{1}{2} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{((-1)^k - 1)}{k^2} \cos(k\pi(0)) \\
0 &= \frac{1}{2} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{((-1)^k - 1)}{k^2} \\
0 &= \frac{1}{2} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{-2}{(2k-1)^2} \\
\frac{\pi^2}{8} &= \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}
\end{aligned}
$$

Other Sums (2 of 2)

$$\begin{aligned}
f(x) &= \begin{cases} 0 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi. \end{cases} \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{(1 - (-1)^k)}{k\pi} \sin(k\,x) \\
&= \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin((2k-1)x) \\
f(\pi/2) &= \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin\left(\frac{(2k-1)\pi}{2}\right) \\
1 &= \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \\
\frac{\pi}{4} &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1}
\end{aligned}$$