Improper Integrals

MATH 211, *Calculus II*

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Definite Integrals

Theorem (Fundamental Theorem of Calculus (Part I))

If f is continuous on $[a, b]$ then

$$\int_a^b f(x)\,dx = [F(x)]_{x=a}^{x=b} = F(b) - F(a)$$

where F is any antiderivative of f on (a, b).

Question: Can we evaluate the definite integral $\int_{-1}^{1} \frac{1}{x^2}\,dx$?

Answer

We cannot use the Fundamental Theorem of Calculus to evaluate

$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$

since the integrand has a discontinuity at $x = 0$. If we try to evaluate it using the Fundamental Theorem of Calculus we get

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{x=-1}^{x=1} = -2$$

a result which is impossible since $1/x^2 > 0$ for $-1 \leq x < 0$ and $0 < x \leq 1$.

Improper Integrals

Extra care must be exercised when attempting to evaluate definite integrals for which

- the interval over which we integrate is of infinite length (Type 1),

and/or

- the integrand possesses isolated discontinuities within the integration interval (Type 2).

First Type (Type 1)

Definition

If f is continuous on $[a, \infty)$ we define the **improper integral**

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

If f is continuous on $(-\infty, a]$ we define the **improper integral**

$$\int_{-\infty}^a f(x)\,dx = \lim_{R\to-\infty} \int_R^a f(x)\,dx.$$

If the limit is L (finite) we say the improper integral **converges**, otherwise we say it **diverges.**

Examples

Determine if the following improper integrals converge or diverge.

- $\int_{1}^{\infty} e^{-x}\, dx$
- $\int_{5}^{\infty} \frac{1}{x}\, dx$
- $\int_{5}^{\infty} \frac{1}{x^2}\, dx$

Interval $(-\infty, \infty)$

Definition

If f is continuous on $(-\infty, \infty)$ then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$$

for any constant a. We say $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ converges if both $\displaystyle\int_{-\infty}^{a} f(x)\,dx$ and $\displaystyle\int_{a}^{\infty} f(x)\,dx$ converge, otherwise $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ diverges.

Example

Determine if the following improper integral converges.

$$\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\, dx$$

Solution (1 of 2)

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\,dx \\
&= \int_{-\infty}^{0} \frac{1}{e^x + e^{-x}}\,dx + \int_{0}^{\infty} \frac{1}{e^x + e^{-x}}\,dx \\
&= \lim_{R\to-\infty} \int_{R}^{0} \frac{1}{e^x + e^{-x}}\,dx + \lim_{S\to\infty} \int_{0}^{S} \frac{1}{e^x + e^{-x}}\,dx \\
&= \lim_{R\to-\infty} \int_{R}^{0} \frac{e^x}{e^{2x} + 1}\,dx + \lim_{S\to\infty} \int_{0}^{S} \frac{e^x}{e^{2x} + 1}\,dx
\end{aligned}$$

Use integration by substitution with $u = e^x$ and $du = e^x\,dx$.

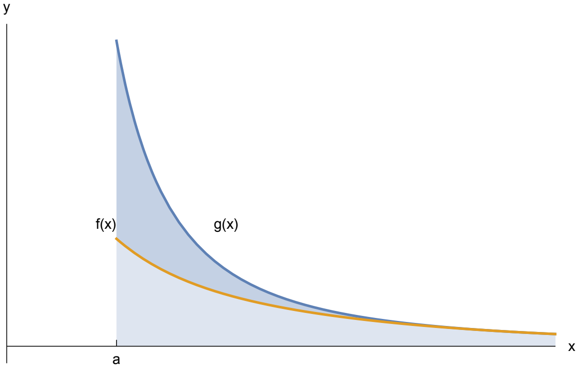
Solution (2 of 2)

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}}\,dx \\
&= \lim_{R\to-\infty} \int_{e^R}^{1} \frac{1}{u^2+1}\,du + \lim_{S\to\infty} \int_{1}^{e^S} \frac{1}{u^2+1}\,du \\
&= \lim_{R\to-\infty} \left[\tan^{-1} u\right]_{u=e^R}^{u=1} + \lim_{S\to\infty} \left[\tan^{-1} u\right]_{u=1}^{u=e^S} \\
&= \lim_{R\to-\infty} \left[\tan^{-1} 1 - \tan^{-1} e^R\right] + \lim_{S\to\infty} \left[\tan^{-1} e^S - \tan^{-1} 1\right] \\
&= \left[\frac{\pi}{4} - 0\right] + \left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{2}
\end{aligned}$$

This improper integral **converges.**

Graphical Approach

Suppose f and g are two continuous functions defined on $[a, \infty)$ and such that $0 \leq f(x) \leq g(x)$ for all $x \geq a$.



- If $\int_a^\infty f(x)\,dx$ diverges, what about $\int_a^\infty g(x)\,dx$?
- If $\int_a^\infty g(x)\,dx$ converges, what about $\int_a^\infty f(x)\,dx$?

Comparison Test

Theorem (Comparison Test)

Suppose that f *and* g *are continuous on* $[a, \infty)$ *and*
$0 \leq f(x) \leq g(x)$ *for all* $x \geq a$*.*

- *If* $\displaystyle\int_a^{\infty} g(x)\,dx$ *converges, then* $\displaystyle\int_a^{\infty} f(x)\,dx$ *converges.*
- *If* $\displaystyle\int_a^{\infty} f(x)\,dx$ *diverges, then* $\displaystyle\int_a^{\infty} g(x)\,dx$ *diverges.*

Examples

Determine if the following improper integrals converge or diverge.

- $\displaystyle\int_1^\infty \frac{x^2-2}{x^4+3}\,dx$
- $\displaystyle\int_1^\infty \frac{1+\sec^2 x}{x}\,dx$
- $\displaystyle\int_0^\infty e^{x+1}\,dx$

Gabriel's Horn

Rotate the curve $y = 1/x$ for $x \geq 1$ about the x-axis.

- Find the volume of the solid of revolution.
- Find the surface area of the solid of revolution.

Volume

Using the Method of Disks,

$$\begin{aligned} V &= \pi \int_1^\infty \left(\frac{1}{x}\right)^2 \, dx = \lim_{R\to\infty} \pi \int_1^R \frac{1}{x^2} \, dx \\ &= \lim_{R\to\infty} \pi \left[\frac{-1}{x}\right]_{x=1}^{x=R} \\ &= \lim_{R\to\infty} \pi \left(\frac{-1}{R} - \frac{-1}{1}\right) \\ &= \pi. \end{aligned}$$

The improper integral converges.

Surface Area

$$\begin{aligned} S &= 2\pi \int_1^{\infty} \left(\frac{1}{x}\right) \sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\, dx = 2\pi \int_1^{\infty} \left(\frac{1}{x}\right) \sqrt{1 + \frac{1}{x^4}}\, dx \\ &= 2\pi \int_1^{\infty} \left(\frac{1}{x}\right) \sqrt{\frac{x^4+1}{x^4}}\, dx = 2\pi \int_1^{\infty} \frac{1}{x^3} \sqrt{x^4+1}\, dx \\ &> 2\pi \int_1^{\infty} \frac{1}{x^3} \sqrt{x^4}\, dx = 2\pi \int_1^{\infty} \frac{1}{x}\, dx = \infty \end{aligned}$$

This improper integral diverges. Thus Gabriel’s Horn has a finite volume, but an infinite surface area.

Second Type (Type 2)

Definition

If f is continuous on the interval $[a, b)$ and $|f(x)| \to \infty$ as $x \to b^-$, the **improper integral** of f on $[a, b]$ is

$$\int_a^b f(x)\,dx = \lim_{R \to b^-} \int_a^R f(x)\,dx.$$

If f is continuous on the interval $(a, b]$ and $|f(x)| \to \infty$ as $x \to a^+$, the **improper integral** of f on $[a, b]$ is

$$\int_a^b f(x)\,dx = \lim_{R \to a^+} \int_R^b f(x)\,dx.$$

If the limit is L (finite), we say the improper integral **converges**, otherwise we say it **diverges.**

Examples

Determine if the following improper integrals converge or diverge.

- $\displaystyle\int_1^4 \frac{2x}{x^2-1}\,dx$
- $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$
- $\displaystyle\int_0^{\pi/2} \tan x\,dx$

Discontinuity in (a, b)

Definition

Suppose f is continuous on $[a, b]$ except at some $c \in (a, b)$ and $|f(x)| \to \infty$ as $x \to c$. The improper integral is

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

If $\int_a^c f(x)\,dx = L_1$ and $\int_c^b f(x)\,dx = L_2$ the improper integral $\int_a^b f(x)\,dx$ converges to $L_1 + L_2$. If either of the improper integrals $\int_a^c f(x)\,dx$ or $\int_c^b f(x)\,dx$ diverges then $\int_a^b f(x)\,dx$ diverges as well.

Examples

Determine if the following improper integrals converge or diverge.

- $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\, dx$
- $\displaystyle\int_{0}^{4} \frac{2x}{x^2 - 1}\, dx$

Homework

- Read Section 3.7
- Exercises: 347, 351, 355, . . . , 387, 395, 399/handout