Integral Test

MATH 211, *Calculus II*

J. Robert Buchanan

Department of Mathematics

Fall 2021

Introduction

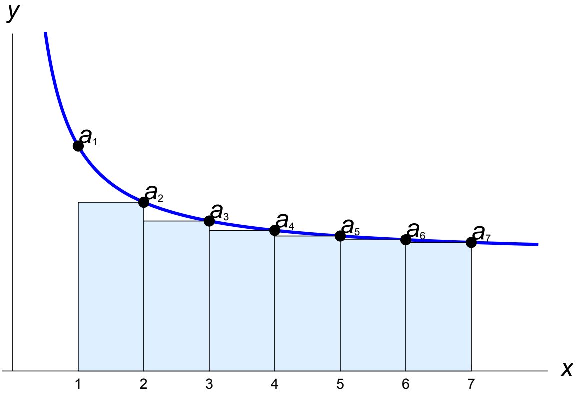
Remarks:

- Determining the **convergence** or **divergence** of a **series** from its **sequence of partial sums** is difficult for most series.
- We must develop some indirect techniques for determining if a series converges or diverges.
- Today we will work only with **positive term** series, *i.e.*, series

$$\sum_{k=1}^{\infty} a_k \quad \text{where} \quad a_k \geq 0 \quad \text{for all } k.$$

Area Under the Curve

Consider $\sum_{k=1}^{\infty} a_k$ for which there is a function $f(x) \geq 0$ for $x \geq 1$ and $f(k) = a_k$ for $k = 1, 2, \ldots$.



$$0 \leq \sum_{k=2}^{n} a_k = S_n - a_1 \leq \int_1^n f(x)\,dx$$

Boundedness

$$
\begin{aligned}
0 \leq S_n - a_1 &\leq \int_1^n f(x)\,dx \\
a_1 \leq S_n &\leq a_1 + \int_1^n f(x)\,dx \\
a_1 \leq S_n &\leq a_1 + \int_1^n f(x)\,dx \leq a_1 + \int_1^\infty f(x)\,dx \\
a_1 \leq S_n &\leq a_1 + \int_1^\infty f(x)\,dx
\end{aligned}
$$

Remark: the sequence of partial sums is **bounded** if $\int_1^\infty f(x)\,dx$ **converges.**

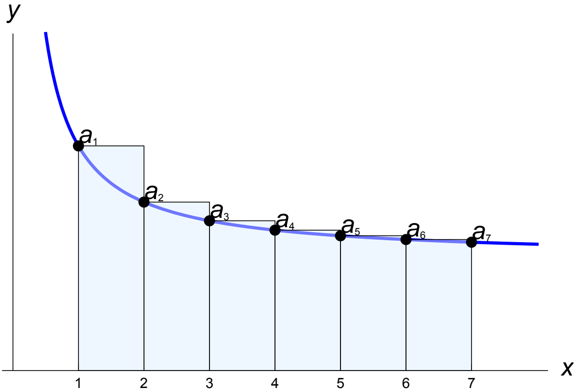
Monotonicity

$$\begin{aligned} S_n &\leq S_{n+1} \quad \text{for all } n = 1, 2, \ldots \qquad \textbf{Why?} \\ \sum_{k=1}^{n} a_k &\leq \sum_{k=1}^{n+1} a_k \\ \sum_{k=1}^{n} a_k &\leq a_{n+1} + \sum_{k=1}^{n} a_k \\ 0 &\leq a_{n+1} \end{aligned}$$

Conclusion: if $\int_1^\infty f(x)\,dx$ converges the $\{S_n\}_{n=1}^\infty$ is increasing and bounded and thus must converge. Hence $\sum_{k=1}^{\infty} a_k$ converges.

Area Under the Curve

Consider this scenario:



$$0 \leq \int_1^n f(x)\,dx \leq \sum_{k=1}^{n-1} a_k = S_{n-1}$$

Divergence

$$\begin{aligned}
\int_1^n f(x)\,dx &\leq S_{n-1} \\
\lim_{n\to\infty} \int_1^n f(x)\,dx &\leq \lim_{n\to\infty} S_{n-1} \\
\int_1^\infty f(x)\,dx &\leq \lim_{n\to\infty} S_{n-1}
\end{aligned}$$

Remark: if $\int_1^\infty f(x)\,dx$ **diverges**, then the sequence of partial sums diverges as well.

Integral Test

Theorem (Integral Test)

*If $f(k) = a_k$ for all $k = 1, 2, \ldots$, f is continuous and decreasing, and $f(x) \geq 0$ for $x \geq 1$, then the improper integral $\int_1^\infty f(x)\,dx$ and the infinite series $\sum_{k=1}^{\infty} a_k$ either **both** converge or **both** diverge.*

Remark: when the integral and the series both converge they do not necessarily converge to the same value.

Examples

Determine, using the Integral Test, whether the following infinite series converge or diverge.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$
2. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

Let $f(x) = 1/x$, then $f(k) = 1/k = a_k$ for all $k \in \mathbb{N}$.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x}\,dx &= \lim_{R\to\infty} \int_1^{R} \frac{1}{x}\,dx \\
&= \lim_{R\to\infty} [\ln x]_{x=1}^{x=R} \\
&= \lim_{R\to\infty} (\ln R - \ln 1) = \infty \text{ (diverges)}
\end{aligned}$$

Hence the Integral Test shows that the Harmonic Series diverges.

$\sum_{k=1}^{\infty} \frac{1}{k^2}$

Let $f(x) = 1/x^2$, then $f(k) = 1/k^2 = a_k$ for all $k \in \mathbb{N}$.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2}\,dx &= \lim_{R\to\infty} \int_1^{R} \frac{1}{x^2}\,dx \\
&= \lim_{R\to\infty} \left[\frac{-1}{x}\right]_{x=1}^{x=R} \\
&= \lim_{R\to\infty} \left(\frac{-1}{R} + 1\right) = 1 \text{ (converges)}
\end{aligned}$$

Thus by the Integral Test, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

p-Series

Definition

An infinite series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a ***p*-series.**

Theorem

The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

Proof (1 of 2)

Suppose $p = 1$.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{R\to\infty}\int_1^R \frac{1}{x}\,dx \\ &= \lim_{R\to\infty}\left[\ln x\right]_{x=1}^{x=R} \\ &= \lim_{R\to\infty}\ln R = \infty\end{aligned}$$

In this case the improper integral diverges and by the Integral Test, the series $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges.

Proof (2 of 2)

Suppose $p \neq 1$.

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{R\to\infty} \int_1^R x^{-p}\,dx \\
&= \lim_{R\to\infty} \left[\frac{1}{1-p} x^{1-p}\right]_{x=1}^{x=R} \\
&= \lim_{R\to\infty} \left(\frac{1}{1-p}\,\frac{1}{R^{p-1}} - \frac{1}{1-p}\right) \\
&= \begin{cases} \infty & \text{if } p < 1, \\ 1/(p-1) & \text{if } p > 1. \end{cases}
\end{aligned}$$

By the Integral Test, the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$.

Examples

Which of the following series converge and which diverge?

1. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ $\quad p = \dfrac{3}{2} > 1$ converges
2. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{1/4}}$ $\quad p = \dfrac{1}{4} \leq 1$ diverges
3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{1.001}}$ $\quad p = 1.001 > 1$ converges
4. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{-5/4}}$ $\quad p = -\dfrac{5}{4} \leq 1$ diverges

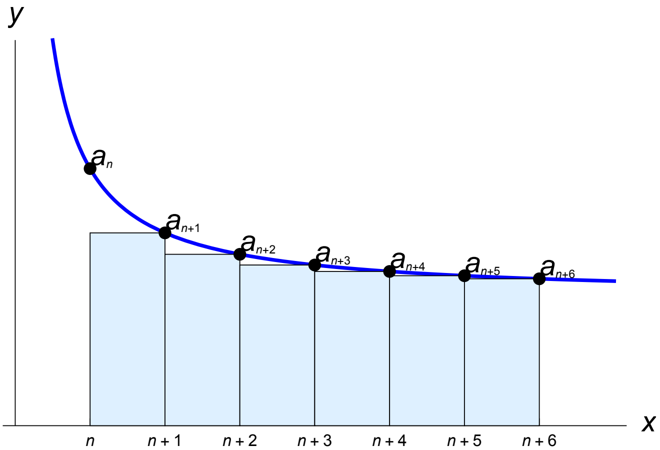
Series Remainder

Earlier we said that in general $\sum_{k=1}^{\infty} a_k \neq \int_1^{\infty} f(x)\,dx$, but an improper integral can be used to help **estimate** the sum of the series.

Let $S = \sum_{k=1}^{\infty} a_k$ and define the **remainder** R_n as

$$R_n = S - S_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k$$

Error Estimate (1 of 2)



$$R_n = \sum_{k=n+1}^{\infty} a_k \leq \int_n^{\infty} f(x)\, dx$$

Error Estimate (2 of 2)

Theorem

Suppose that $f(k) = a_k$ *for all* $k = 1, 2, \ldots,$ *where* f *is continuous and decreasing and* $f(x) \geq 0$ *for all* $x \geq 1$. *Further suppose that* $\int_1^\infty f(x)\,dx$ *converges. The remainder* R_n *satisfies the inequality*

$$0 \leq R_n = \sum_{k=n+1}^{\infty} a_k \leq \int_n^\infty f(x)\,dx.$$

Example

Estimate the error in using S_{100} to approximate

$$\sum_{k=1}^{\infty} \frac{4}{1+k^2}.$$

$$\begin{aligned} S - S_{100} &\leq \int_{100}^{\infty} \frac{4}{1+x^2}\,dx = \lim_{R\to\infty} \int_{100}^{R} \frac{4}{1+x^2}\,dx \\ &= \lim_{R\to\infty} \left[4\tan^{-1} x\right]_{x=100}^{x=R} = 4 \lim_{R\to\infty} (\tan^{-1} R - \tan^{-1} 100) \\ &\approx 0.04 \end{aligned}$$

Note: $S_{100} = \sum_{k=1}^{100} \frac{4}{1+k^2} \approx 4.2669$

Unusual Sums

Determine if the following infinite series converge or diverge.

- $\displaystyle\sum_{k=1}^{\infty} \int_{k}^{k+1} \frac{1}{x^{4/3}}\,dx$
- $\displaystyle\sum_{k=1}^{\infty} \int_{k}^{k+1} x^{1/3}\,dx$

$$\sum_{k=1}^{\infty} \int_{k}^{k+1} \frac{1}{x^{4/3}}\, dx$$

$$\begin{aligned}
\sum_{k=1}^{\infty} \int_{k}^{k+1} \frac{1}{x^{4/3}}\, dx &= \sum_{k=1}^{\infty} \int_{k}^{k+1} x^{-4/3}\, dx \\
&= \sum_{k=1}^{\infty} \left[-3x^{-1/3}\right]_{x=k}^{x=k+1} \\
&= 3\sum_{k=1}^{\infty} \left(\frac{1}{k^{1/3}} - \frac{1}{(k+1)^{1/3}}\right) \quad \text{(telescoping sum)} \\
&= 3
\end{aligned}$$

$$\sum_{k=1}^{\infty} \int_{k}^{k+1} x^{1/3}\, dx$$

$$\begin{aligned} \sum_{k=1}^{\infty} \int_{k}^{k+1} x^{1/3}\, dx &= \sum_{k=1}^{\infty} \left[\frac{3}{4} x^{4/3}\right]_{x=k}^{x=k+1} \\ &= \frac{3}{4} \sum_{k=1}^{\infty} \left((k+1)^{4/3} - k^{4/3}\right) \end{aligned}$$

While this sum telescopes, the Nth partial sum is

$$S_N = \frac{3}{4} \sum_{k=1}^{N} \left((k+1)^{4/3} - k^{4/3}\right) = \frac{3}{4} \left((N+1)^{4/3} - 1\right)$$

which diverges as $N \to \infty$.

Homework

- Read Section 5.3
- Exercises: 153, 157, 161, 165, . . . , 177, 181/handout