Integration by Parts

MATH 211, *Calculus II*

J. Robert Buchanan

Department of Mathematics

Fall 2021

Introduction

- If necessary refresh your skills with basic integration including **integration by substitution** by reviewing Section 5.5 of the eBook (or Section 4.6 of *Calculus*, 4th edition by Smith & Minton) and working the exercises at the end of the section.
- Today's discussion will focus on the second major technique of integration, **integration by parts.**
- We will see that integration by parts is related to the product rule for derivatives.

Product Rule for Derivatives

Suppose that u and v are functions of x then

$$
\begin{aligned}
\frac{d}{dx}[u\,v] &= v\,\frac{du}{dx} + u\,\frac{dv}{dx} \\
d(uv) &= v\,du + u\,dv \\
\int d(u\,v) &= \int (v\,du + u\,dv) \\
u\,v &= \int v\,du + \int u\,dv \\
\int u\,dv &= u\,v - \int v\,du
\end{aligned}
$$

Integration by Parts Formula

Theorem

If $u = f(x)$ and $v = g(x)$ and $f'(x)$ and $g'(x)$ are continuous then

$$\int u\,dv = u\,v - \int v\,du$$

or alternatively

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

Remark: the derivative of $g(x)$ in the integral on the left-hand side has moved over to $f(x)$ in the integral on the right-hand side.

Further Remarks

- When trying to apply integration by parts we must designate part of the integrand to be dv and the rest u.
- Usually dv is the most complicated part of the integrand that we can integrate using an elementary technique.

Examples

Use integration by parts to evaluate the following indefinite integrals.

- $\int x \sin x \, dx$
- $\int \ln x \, dx$
- $\int \cos^{-1} x \, dx$

$\int x \sin x \, dx$

Integrate by parts choosing

$$\begin{aligned} u &= x & v &= -\cos x \\ du &= dx & dv &= \sin x \, dx \end{aligned}$$

then

$$\begin{aligned} \int u \, dv &= u\, v - \int v \, du \\ \int x \sin x \, dx &= x\,(-\cos x) - \int (-\cos x)\, dx \\ &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + C. \end{aligned}$$

$\int \ln x \, dx$

Integrate by parts choosing

$$\begin{aligned} u &= \ln x & \qquad v &= x \\ du &= \tfrac{1}{x}\, dx & \qquad dv &= dx \end{aligned}$$

then

$$\begin{aligned} \int u \, dv &= u\, v - \int v \, du \\ \int \ln x \, dx &= x \ln x - \int x \left(\frac{1}{x} \right) dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + C. \end{aligned}$$

$\int \cos^{-1} x \, dx$ (1 of 2)

Integrate by parts choosing

$$\begin{aligned} u &= \cos^{-1} x & \qquad v &= x \\ du &= \frac{-1}{\sqrt{1-x^2}} \, dx & \qquad dv &= dx \end{aligned}$$

then

$$\begin{aligned} \int u \, dv &= u \, v - \int v \, du \\ \int \cos^{-1} x \, dx &= x \, \cos^{-1} x - \int x \left(\frac{-1}{\sqrt{1-x^2}} \right) \, dx \\ &= x \, \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} \, dx. \end{aligned}$$

$\int \cos^{-1} x \, dx$ (2 of 2)

Now integrate by substitution letting

$$\begin{aligned} w &= 1 - x^2 \\ -\frac{1}{2}\, dw &= x\, dx. \end{aligned}$$

$$\begin{aligned} \int \cos^{-1} x \, dx &= x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}\, dx \\ &= x \cos^{-1} x - \frac{1}{2} \int w^{-1/2}\, dw \\ &= x \cos^{-1} x - w^{1/2} + C \\ &= x \cos^{-1} x - \sqrt{1-x^2} + C \end{aligned}$$

Repeated Integration by Parts

Remark: Sometimes integration by parts must be used more than once.

Example

Use integration by parts to evaluate the following indefinite integrals.

- $\int x^2 \cos x \, dx$
- $\int e^x \sin x \, dx$

$\int x^2 \cos x \, dx$ (1 of 2)

$$\begin{aligned} u &= x^2 & v &= \sin x \\ du &= 2x\,dx & dv &= \cos x\,dx \end{aligned}$$

$$\int x^2 \cos x\,dx = x^2 \sin x - \int 2x \sin x\,dx$$

$\int x^2 \cos x \, dx$ (2 of 2)

$$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$$

$$\begin{aligned} u &= 2x & \qquad v &= -\cos x \\ du &= 2\,dx & \qquad dv &= \sin x \, dx \end{aligned}$$

$$\begin{aligned} \int x^2 \cos x \, dx &= x^2 \sin x - \int 2x \sin x \, dx \\ &= x^2 \sin x - \left(-2x \cos x - \int 2(-\cos x) \, dx \right) \\ &= x^2 \sin x + 2x \cos x - \int 2 \cos x \, dx \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C \end{aligned}$$

Tabular Integration

Consider $\int x^2 \cos x \, dx$ and suppose we arrange our work in the following table:

u	dv	sign
	$\cos x$	
x^2	$\sin x$	$+$
$2x$	$-\cos x$	$-$
2	$-\sin x$	$+$
0	$\cos x$	$-$

If we multiply across the completed rows and add these products we get

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

Note: differentiate down the first column, antidifferentiate down the second column, alternate signs (starting with $+$) in the third column.

Example

Use tabular integration (or repeated integration by parts) to evaluate $\displaystyle\int x^3 e^x \, dx$.

Solution: $\int x^3 e^x \, dx$

u	dv	sign
	e^x	
x^3	e^x	$+$
$3x^2$	e^x	$-$
$6x$	e^x	$+$
6	e^x	$-$
0	e^x	$+$

$$\int x^3 e^x \, dx = e^x(x^3 - 3x^2 + 6x - 6) + C$$

$\int e^x \sin x \, dx$ (1 of 2)

$$\begin{aligned} u &= e^x & v &= -\cos x \\ du &= e^x \, dx & dv &= \sin x \, dx \end{aligned}$$

$$\begin{aligned} \int e^x \sin x \, dx &= -e^x \cos x - \int e^x (-\cos x) \, dx \\ &= -e^x \cos x + \int e^x \cos x \, dx \end{aligned}$$

$\int e^x \sin x \, dx$ (2 of 2)

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

$$\begin{aligned} u &= e^x & v &= \sin x \\ du &= e^x \, dx & dv &= \cos x \, dx \end{aligned}$$

$$\begin{aligned} \int e^x \sin x \, dx &= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx \\ 2 \int e^x \sin x \, dx &= -e^x \cos x + e^x \sin x \\ \int e^x \sin x \, dx &= \frac{1}{2}\left(-e^x \cos x + e^x \sin x\right) + C \end{aligned}$$

Reduction Formulas

Often common integration formulas are expressed recursively. For example:

$$\int \cos^n x \, dx = \int \cos^{n-1} x \cos x \, dx$$

using integration by parts with

$$\begin{aligned} u &= \cos^{n-1} x & v &= \sin x \\ du &= -(n-1) \sin x \cos^{n-2} x \, dx & dv &= \cos x \, dx \end{aligned}$$

we obtain

$$\int \cos^n x \, dx = \sin x \cos^{n-1} x + \int (n-1) \sin^2 x \cos^{n-2} x \, dx$$

Simplification

$$\begin{aligned}
\int \cos^n x\,dx &= \sin x \cos^{n-1} x + \int (n-1)\sin^2 x \cos^{n-2} x\,dx \\
&= \sin x \cos^{n-1} x \\
&\quad + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx \\
&= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx \\
&\quad - (n-1)\int \cos^n x\,dx \\
n\int \cos^n x\,dx &= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx \\
\int \cos^n x\,dx &= \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx
\end{aligned}$$

Definite Integrals

Integration by parts is easily adapted to definite integrals:

$$\int_{x=a}^{x=b} u\,dv = [u\,v]_{x=a}^{x=b} - \int_{x=a}^{x=b} v\,du$$

Example

Use integration by parts to evaluate the following definite integrals.

- $\displaystyle\int_0^{\pi/2} x \sin x \, dx$
- $\displaystyle\int_{-1}^{0} 3x\, e^{2x} \, dx$
- $\displaystyle\int_1^{e} \sqrt{x} \ln x \, dx$

Solution: $\int_0^{\pi/2} x \sin x \, dx$

$$\begin{aligned} u &= x & v &= -\cos x \\ du &= dx & dv &= \sin x \, dx \end{aligned}$$

then

$$\begin{aligned} \int_0^{\pi/2} x \sin x \, dx &= [-x \cos x]_{x=0}^{x=\pi/2} + \int_0^{\pi/2} \cos x \, dx \\ &= 0 + \int_0^{\pi/2} \cos x \, dx \\ &= [\sin x]_{x=0}^{x=\pi/2} = 1. \end{aligned}$$

Solution: $\int_{-1}^{0} 3x\, e^{2x}\, dx$

$$\begin{aligned} u &= 3x & \qquad v &= (1/2)e^{2x} \\ du &= 3\, dx & \qquad dv &= e^{2x}\, dx \end{aligned}$$

then

$$\begin{aligned} \int_{-1}^{0} 3x\, e^{2x}\, dx &= \left[\frac{3x}{2} e^{2x}\right]_{x=-1}^{x=0} - \int_{-1}^{0} \frac{3}{2} e^{2x}\, dx \\ &= \frac{3}{2} e^{-2} - \frac{3}{2} \int_{-1}^{0} e^{2x}\, dx \\ &= \frac{3}{2} e^{-2} - \left[\frac{3}{4} e^{2x}\right]_{x=-1}^{x=0} \\ &= \frac{3}{2} e^{-2} - \frac{3}{4} + \frac{3}{4} e^{-2} = \frac{9}{4} e^{-2} - \frac{3}{4} \end{aligned}$$

Solution: $\int_1^e \sqrt{x} \ln x \, dx$

$$\begin{aligned} u &= \ln x & \qquad v &= (2/3)x^{3/2} \\ du &= x^{-1}\,dx & \qquad dv &= x^{1/2}\,dx \end{aligned}$$

then

$$\begin{aligned} \int_1^e \sqrt{x} \ln x \, dx &= \left[\frac{2}{3}x^{3/2}\ln x\right]_{x=1}^{x=e} - \int_1^e \frac{2}{3}x^{3/2}x^{-1}\,dx \\ &= \frac{2}{3}e^{3/2} - \frac{2}{3}\int_1^e x^{1/2}\,dx \\ &= \frac{2}{3}e^{3/2} - \left[\frac{4}{9}x^{3/2}\right]_{x=1}^{x=e} \\ &= \frac{2}{3}e^{3/2} - \frac{4}{9}e^{3/2} + \frac{4}{9} = \frac{4}{9} + \frac{2}{9}e^{3/2}, \end{aligned}$$

Homework

- Read Section 3.1
- Exercises: 7, 11, 15, 19, . . . , 67/handout