Plane Curves and Parametric Equations

MATH 211, *Calculus II*

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Fall 2021

Introduction

We typically think of a **graph** as a curve in the *xy*-plane generated by the set of all ordered pairs of the form $(x, y) = (x, f(x))$ for $a \leq x \leq b$.

Today we will generalize this notion.

Plane Curves

Definition

A **plane curve** is a set ordered pairs $(x(t), y(t))$ where x and y are now thought of a functions of a **parameter** t with $a \leq t \leq b$. The equations

$$\begin{aligned} x &= x(t) \\ y &= y(t) \end{aligned}$$

are called **parametric equations.**

Example

If $0 \leq t \leq 2\pi$ and

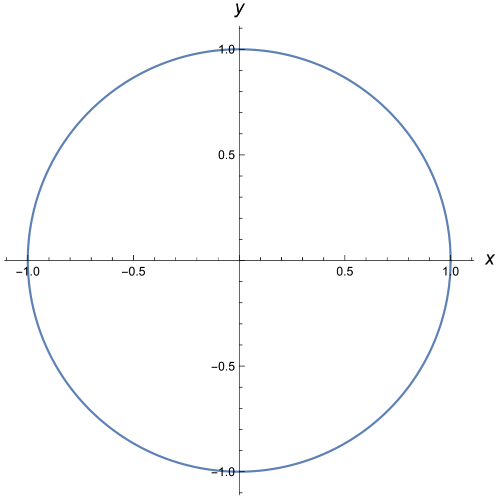
$$\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}$$

plot the plane curve.

If you wish to plot in a web browser, use the TI-83+/TI-84+ Graphing Calculator Emulator.

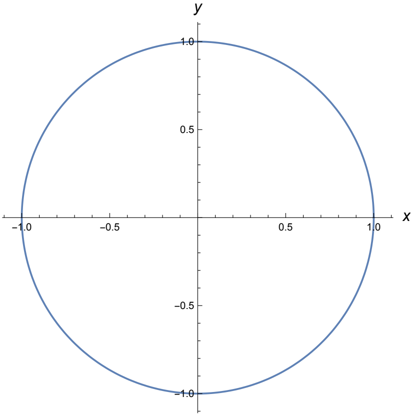
Example: Result

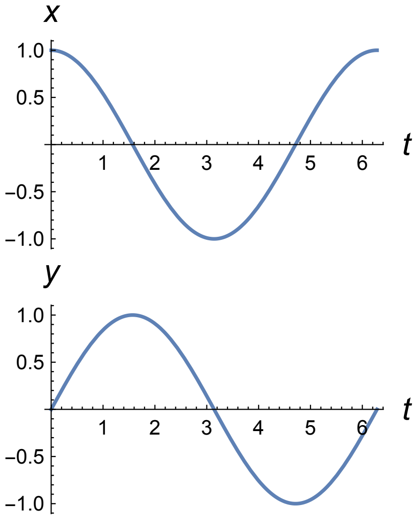
$$x = \cos t$$
$$y = \sin t$$



Example: Comparison

$$x = \cos t$$
$$y = \sin t$$





Points

We can define $P(t) = (x(t), y(t))$ to be a **point** on a plane curve. Suppose $a \leq t \leq b$.

- If $P(a) \neq P(b)$, then $P(a)$ and $P(b)$ are called the **endpoints** of the plane curve.
- If $P(a) = P(b)$, then the plane curve is called a **closed curve**.
- If $P(a) = P(b)$, and the plane curve does not intersect itself anywhere else, the plane curve is called a **simple closed curve**.

Path of a Projectile

If a projectile is fired into the air at an angle θ to the horizontal with

- initial position (x_0, y_0), and
- initial speed s_0,

then the parametric equations describing the path of the projectile are

$$\begin{aligned} x(t) &= x_0 + (s_0 \cos\theta)t \\ y(t) &= y_0 + (s_0 \sin\theta)t - \frac{1}{2}gt^2 \end{aligned}$$

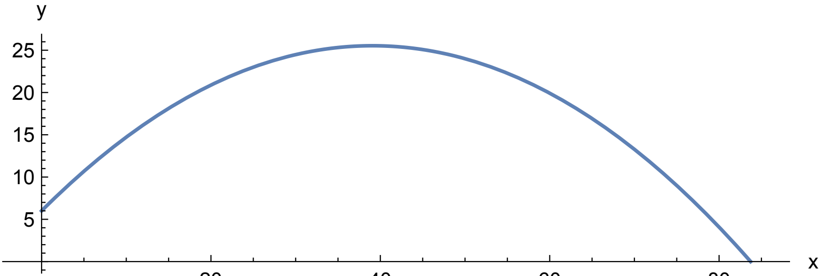
where g is the gravitational acceleration constant.

Example

Suppose a shot-putter releases the shot 6 feet above the ground at an angle of 45° to the horizontal with an initial speed of 50 feet per second. What is the distance of the throw?

Solution (1 of 2)

$$\begin{aligned} x(t) &= (50\cos 45^\circ)t \\ y(t) &= 6 + (50\sin 45^\circ)t - 16t^2 \end{aligned}$$



Solution (2 of 2)

The shot strikes the ground when $y(t) = 0$.

$$
\begin{aligned}
0 &= 6 + (50 \sin 45^\circ)t - 16t^2 \\
t &= \frac{-50 \sin 45^\circ - \sqrt{(50 \sin 45^\circ)^2 - 4(-16)(6)}}{2(-16)} \\
t &\approx 2.36807
\end{aligned}
$$

The distance of the effort is

$$x(2.36807) \approx 83.7238 \text{ feet.}$$

Collisions

Suppose the position of one object for $0 \leq t \leq 2\pi$ is given by the parametric curves:

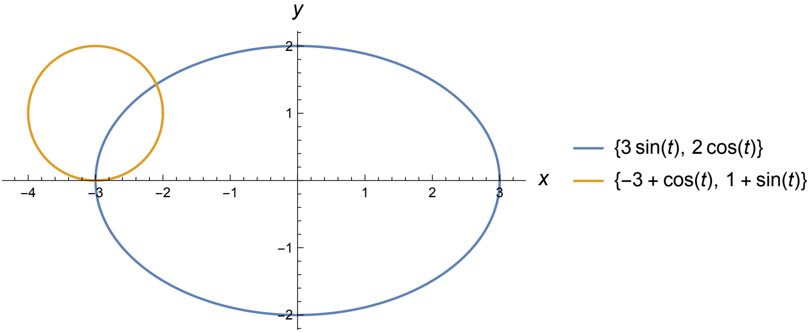
$$\begin{aligned} x_1(t) &= 3\sin t \\ y_1(t) &= 2\cos t. \end{aligned}$$

Suppose the position of a second object is given by the parametric curves:

$$\begin{aligned} x_2(t) &= -3 + \cos t \\ y_2(t) &= 1 + \sin t. \end{aligned}$$

1. How many points of intersection in the paths of the objects are there?
2. Are any the intersection points **collision** points (objects at the same place at the same time)?

Paths



There are two points of intersection of the paths.

Any Collisions?

If $x_1(t) = x_2(t)$ then $3\sin t = -3 + \cos t$ which implies $\cos t = 3 + 3\sin t$.

If $y_1(t) = y_2(t)$ then $2\cos t = 1 + \sin t$ or equivalently

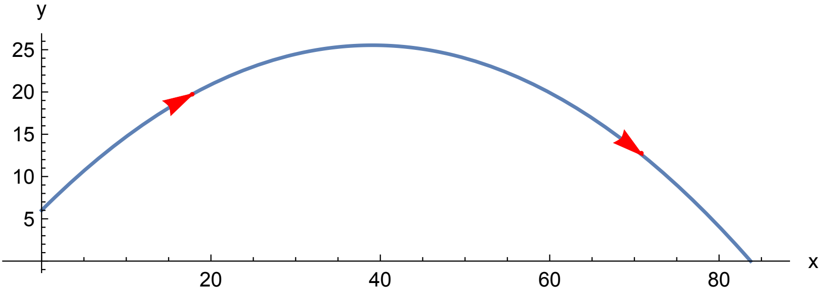
$$\begin{aligned} 2(3 + 3\sin t) &= 1 + \sin t \\ 6 + 6\sin t &= 1 + \sin t \\ \sin t &= -1 \\ t &= \frac{3\pi}{2}. \end{aligned}$$

Thus we see there is a collision when $t = \frac{3\pi}{2}$ at

$$x_1\left(\frac{3\pi}{2}\right) = x_2\left(\frac{3\pi}{2}\right) = -3 \text{ with } y_1\left(\frac{3\pi}{2}\right) = y_2\left(\frac{3\pi}{2}\right) = 0.$$

Orientation

We sometimes add arrows to the plane curve to indicate the direction in which $P(t)$ moves as t increases. The arrows indicate the **orientation** of the curve.



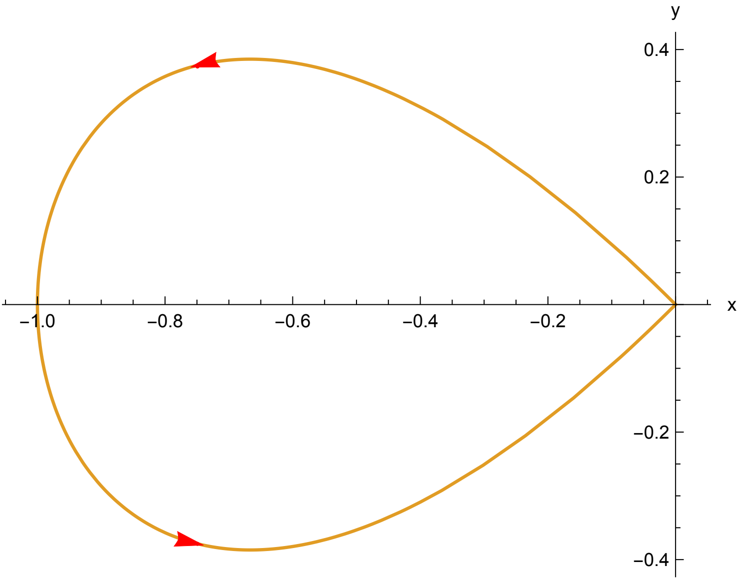
Example

For $-1 \leq t \leq 1$ sketch the plane curve given by

$$\begin{aligned} x(t) &= t^2 - 1 \\ y(t) &= t^3 - t \end{aligned}$$

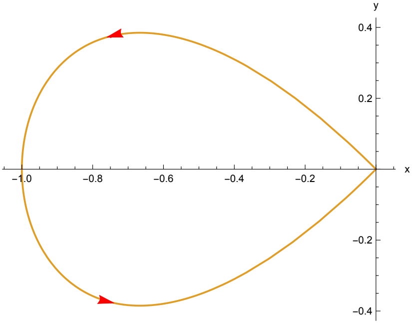
and indicate its orientation.

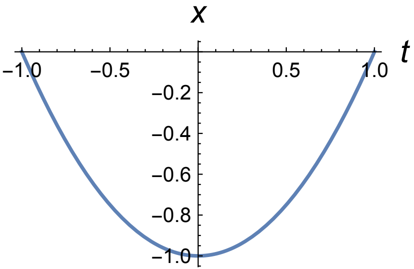
Example: Result

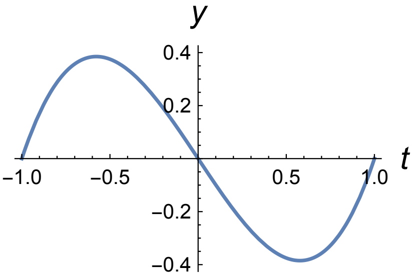


Example: Comparison

$$x = t^2 - 1$$
$$y = t^3 - t$$







Functional *vs.* Parametric Form

Given $y = f(x)$ for $a \leq x \leq b$ then the equivalent parametric equations are

$$\begin{aligned} x(t) &= t \\ y(t) &= f(t) \end{aligned}$$

for $a \leq t \leq b$.

Given the parametric equations

$$\begin{aligned} x(t) &= t \\ y(t) &= f(t) \end{aligned}$$

we must eliminate the parameter t to obtain the functional form of the curve.

Example

Given the parametric equations:

$$\begin{aligned} x(t) &= \sqrt{1+t} \\ y(t) &= \sqrt{1-t} \end{aligned}$$

eliminate the parameter to find a functional Cartesian form of the curve.

Solution

Square both equations.

$$\begin{aligned} x^2 &= 1+t \\ y^2 &= 1-t \end{aligned}$$

Add the equations.

$$\begin{aligned} x^2+y^2 &= 2 \\ y &= \sqrt{2-x^2} \end{aligned}$$

Example

The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ generates the graph of an ellipse. Find a set of parametric equations to graph an ellipse.

Solution

Recall the trigonometric identity $\cos^2 t + \sin^2 t = 1$*. If* $\cos t = \frac{x}{a}$ *and* $\sin t = \frac{y}{b}$ *then*

$$\begin{aligned} x(t) &= a\cos t \\ y(t) &= b\sin t \end{aligned}$$

for $0 \leq t \leq 2\pi$ *will graph an ellipse.*

Example: Lissajous Figure

Parametric equations:

$$\begin{aligned} x(t) &= a \sin \omega_1 t \\ y(t) &= b \cos \omega_2 t \end{aligned}$$

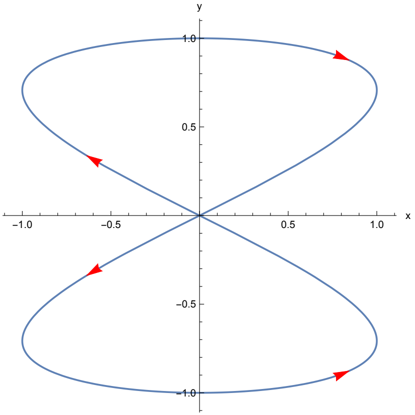
Example

$$\begin{aligned} x(t) &= \sin 2\pi t \\ y(t) &= \cos \pi t \end{aligned}$$

- Sketch the graph of this Lissajous figure and note its orientation.
- Find a formula for this Lissajous figure in functional Cartesian form.

Example: Lissajous Figure

$$\begin{aligned} x(t) &= \sin 2\pi t \\ y(t) &= \cos \pi t \end{aligned}$$



Example: Lissajous Figure

Given

$$\begin{aligned} x &= \sin 2\pi t \\ y &= \cos \pi t \end{aligned}$$

we must use some trigonometric identities to eliminate t.

$$\begin{aligned} x &= 2\cos \pi t \, \sin \pi t \quad \text{(double angle formula)} \\ &= \pm 2(\cos \pi t)\sqrt{1 - \cos^2 \pi t} \quad \text{(fundamental trig. identity)} \\ &= \pm 2y\sqrt{1 - y^2} \\ x^2 &= 4y^2(1 - y^2) \end{aligned}$$

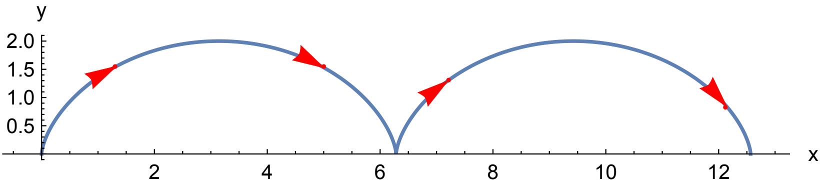
Example: Brachistochrone

Parametric equations:

$$\begin{aligned} x(t) &= a(t - \sin t) \\ y(t) &= a(1 - \cos t) \end{aligned}$$

Example

When $a = 1$



Homework

- Read Section 7.1
- Exercises: 1, 5, 9, . . . , 53, 57/handout