Integration of Rational Functions

MATH 211, *Calculus II*

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Rational Functions

Definition

A **polynomial function in** x is a function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

where a_0, a_1, ..., a_n are called constants called **coefficients** and n is a non-negative integer. If $a_n \neq 0$ the **degree** of the polynomial, denoted $\deg p(x)$, is n.

Definition

If $f(x)$ is a function of the form $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, then $f(x)$ is said to be a **rational function.**

Today's discussion will center on a technique for evaluating integrals containing rational functions.

Example

$$\int \frac{1}{x^2-4}\,dx = \int \left(\frac{1}{4(x-2)} - \frac{1}{4(x+2)} \right) dx$$
$$= \frac{1}{4} \int \left(\frac{1}{x-2} - \frac{1}{x+2} \right) dx$$

The expression $\frac{1}{4(x-2)} - \frac{1}{4(x+2)}$ is the **partial fraction decomposition** of $\frac{1}{x^2-4}$.

Distinct Linear Factors

Suppose

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)}$$

and $\deg p(x) < \deg q(x)$ and the $a_ix + b_i$ for $i = 1, 2, \ldots, n$ are all distinct, then

$$\frac{p(x)}{q(x)} = \frac{c_1}{a_1x + b_1} + \frac{c_2}{a_2x + b_2} + \cdots + \frac{c_n}{a_nx + b_n}$$

where c_i is a constant for $i = 1, 2, \ldots, n$.

Examples

Find the partial fraction decompositions of the following rational functions.

- $\dfrac{1}{x^2 - 4}$
- $\dfrac{5x + 3}{x^3 - 2x^2 - 3x}$

Solution (1 of 2)

$$\begin{aligned}
\frac{1}{x^2-4} &= \frac{1}{(x-2)(x+2)} \\
\frac{1}{(x-2)(x+2)} &= \frac{A}{x-2} + \frac{B}{x+2} \\
\frac{1}{(x-2)(x+2)} &= \frac{A(x+2)}{(x-2)(x+2)} + \frac{B(x-2)}{(x-2)(x+2)} \\
1 &= A(x+2) + B(x-2)
\end{aligned}$$

If $x = 2$ then

$$1 = A(2+2) + B(2-2) = 4A \implies A = 1/4.$$

If $x = -2$ then

$$1 = A(-2+2) + B(-2-2) = -4B \implies B = -1/4.$$

Therefore

$$\frac{1}{x^2-4} = \frac{1/4}{x-2} - \frac{1/4}{x+2}.$$

Solution (2 of 2)

$$\begin{aligned}
\frac{5x+3}{x^3-2x^2-3x} &= \frac{5x+3}{x(x-3)(x+1)} \\
\frac{5x+3}{x(x-3)(x+1)} &= \frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+1} \\
\frac{5x+3}{x(x-3)(x+1)} &= \frac{A(x-3)(x+1)+Bx(x+1)+Cx(x-3)}{x(x-3)(x+1)} \\
5x+3 &= A(x-3)(x+1)+Bx(x+1)+Cx(x-3)
\end{aligned}$$

If $x = 0$ then

$$3 = A(0-3)(0+1)+B(0)(0+1)+C(0)(0-3) = -3A \implies A = -1.$$

If $x = 3$ then

$$18 = A(3-3)(3+1)+B(3)(3+1)+C(3)(3-3) = 12B \implies B = 3/2.$$

If $x = -1$ then

$$-2 = A(-1-3)(-1+1)+B(-1)(-1+1)+C(-1)(-1-3) = 4C \implies C = -1/2.$$

Therefore

$$\frac{5x+3}{x^3-2x^2-3x} = \frac{-1}{x}+\frac{3/2}{x-3}+\frac{-1/2}{x+1}.$$

Example

Evaluate the indefinite integral:

$$\begin{aligned}
\int \frac{5x+3}{x^3-2x^2-3x}\,dx &= \int \left(\frac{3/2}{x-3} - \frac{1}{x} - \frac{1/2}{x+1} \right) dx \\
&= \frac{3}{2}\int \frac{1}{x-3}\,dx - \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx \\
&= \frac{3}{2}\ln|x-3| - \ln|x| - \frac{1}{2}\ln|x+1| + C
\end{aligned}$$

Polynomial Long Division

If $f(x) = \frac{p(x)}{q(x)}$ and $\deg p(x) \geq \deg q(x)$ then we must perform **polynomial long division** in order to write

$$f(x) = \frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$$

where $s(x)$ is a polynomial called the **quotient** and $r(x)$ is polynomial for which $\deg r(x) < \deg q(x)$.

Example

Find the partial fraction decomposition of

$$\frac{x^4 + 8x^2 + 8}{x^3 - 4x}$$

and use it to evaluate

$$\int \frac{x^4 + 8x^2 + 8}{x^3 - 4x}\, dx.$$

Solution

$$
\begin{aligned}
\frac{x^4 + 8x^2 + 8}{x^3 - 4x} &= x + \frac{12x^2 + 8}{x^3 - 4x} \\
&= x - \frac{2}{x} + \frac{7}{x-2} + \frac{7}{x+2} \\
\int \frac{x^4 + 8x^2 + 8}{x^3 - 4x}\,dx &= \int \left(x - \frac{2}{x} + \frac{7}{x-2} + \frac{7}{x+2} \right) dx \\
&= \frac{x^2}{2} - 2\ln|x| + 7\ln|x-2| + 7\ln|x+2| + C
\end{aligned}
$$

Repeated Linear Factors

Question: what if the linear factors of $q(x)$ are not all distinct, *i.e.*, one or more of them is repeated?

Answer: if $f(x) = \dfrac{p(x)}{(ax+b)^n}$ then the partial fraction decomposition will have the form

$$\frac{p(x)}{(ax+b)^n} = \frac{c_1}{ax+b} + \frac{c_2}{(ax+b)^2} + \cdots + \frac{c_n}{(ax+b)^n}$$

where c_i is a constant for $i = 1, 2, \ldots, n$.

Example

Find the partial fraction decomposition of

$$\frac{x}{x^2 - 6x + 9}$$

and use it to evaluate

$$\int \frac{x}{x^2 - 6x + 9}\, dx.$$

Solution

$$\begin{aligned}
\frac{x}{x^2-6x+9} &= \frac{x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} \\
&= \frac{A(x-3)+B}{(x-3)^2} \\
x &= A(x-3)+B \\
\text{Let x = 3: } 3 &= B \\
\text{Let x = 0: } 0 &= -3A+B = -3A+3 \implies A = 1 \\
\frac{x}{x^2-6x+9} &= \frac{1}{x-3} + \frac{3}{(x-3)^2}
\end{aligned}$$

Solution

$$\begin{aligned}\frac{x}{x^2-6x+9} &= \frac{1}{x-3}+\frac{3}{(x-3)^2}\\ \int \frac{x}{x^2-6x+9}\,dx &= \int\left(\frac{1}{x-3}+\frac{3}{(x-3)^2}\right)\,dx\end{aligned}$$

Make the substitutions $u = x - 3$ and $du = dx$.

$$\begin{aligned}\int \frac{x}{x^2-6x+9}\,dx &= \int\left(\frac{1}{u}+\frac{3}{u^2}\right)\,du\\ &= \ln|u| - \frac{3}{u} + C\\ &= \ln|x-3| - \frac{3}{x-3} + C\end{aligned}$$

Irreducible Quadratic Factors (1 of 2)

Sometimes the factorization of $q(x)$ will contain a quadratic factor which cannot be factored as the product of two linear functions with real number coefficients. Such a factor is called an **irreducible quadratic factor.**

Example

$$\frac{1}{x^3 - x^2 + x - 1} = \frac{1}{(x-1)(x^2+1)}$$

Irreducible Quadratic Factors (2 of 2)

Suppose

$$\frac{p(x)}{q(x)} = \frac{p(x)}{(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2) \cdots (a_nx^2 + b_nx + c_n)}$$

and $\deg p(x) < \deg q(x)$ and the $a_ix^2 + b_ix + c_i$ for $i = 1, 2, \ldots, n$ are all distinct, then

$$\frac{p(x)}{q(x)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_nx + B_n}{a_nx^2 + b_nx + c_n}$$

where A_i and B_i are constants for $i = 1, 2, \ldots, n$.

Examples

Find the partial fraction decompositions of the following rational functions.

- $\dfrac{1}{x^3 + x}$
- $\dfrac{3}{x^4 + 5x^2 + 4}$

Solution (1 of 2)

$$
\begin{aligned}
\frac{1}{x^3+x} &= \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \\
\frac{1}{x(x^2+1)} &= \frac{A(x^2+1)}{x(x^2+1)} + \frac{(Bx+C)x}{x(x^2+1)} \\
1 &= A(x^2+1) + (Bx+C)x
\end{aligned}
$$

If $x = 0$ then $A = 1$ and we can replace A in the equation above.

$$
\begin{aligned}
1 &= x^2 + 1 + (Bx+C)x \\
0 &= x^2(1+B) + Cx
\end{aligned}
$$

Matching coefficients of x on both sides of the equation implies $B = -1$ and $C = 0$. Thus

$$
\frac{1}{x^3+x} = \frac{1}{x} + \frac{-x}{x^2+1}
$$

Solution (2 of 2)

$$\begin{aligned}
\frac{3}{x^4+5x^2+4} &= \frac{3}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} \\
\frac{3}{(x^2+1)(x^2+4)} &= \frac{(Ax+B)(x^2+4)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+4)} \\
3 &= (Ax+B)(x^2+4)+(Cx+D)(x^2+1) \\
&= (A+C)x^3+(B+D)x^2+(4A+C)x+(4B+D)
\end{aligned}$$

Matching coefficients of x^3 and x on both sides of the equation implies $A = C = 0$. Matching coefficients of x^2 and the constants on both sides of the equation implies $B = 1$ and $D = -1$. Thus

$$\frac{3}{x^4+5x^2+4} = \frac{1}{x^2+1} + \frac{-1}{x^2+4}.$$

Example

Evaluate the indefinite integral:

$$\begin{aligned}\int \frac{3}{x^4+5x^2+4}\,dx &= \int \left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right) dx \\ &= \tan^{-1} x - \frac{1}{2}\tan^{-1}\frac{x}{2} + C\end{aligned}$$

Completing the Square (1 of 3)

Evaluate the indefinite integral below.

$$\int \frac{1}{x^3 + x^2 - 2}\,dx$$

$$\begin{aligned}
&= \int \frac{1}{(x-1)(x^2+2x+2)}\,dx \\
&= \int \left(\frac{1}{5(x-1)} - \frac{x+3}{5(x^2+2x+2)} \right)\,dx \\
&= \frac{1}{5}\int \frac{1}{x-1}\,dx - \frac{1}{5}\int \frac{x+3}{x^2+2x+2}\,dx \\
&= \frac{1}{5}\ln|x-1| - \frac{1}{5}\int \frac{x+1+2}{(x^2+2x+1)+1}\,dx \\
&= \frac{1}{5}\ln|x-1| - \frac{1}{5}\int \frac{x+1}{(x+1)^2+1}\,dx - \frac{2}{5}\int \frac{1}{(x+1)^2+1}\,dx
\end{aligned}$$

Completing the Square (2 of 3)

$$\begin{aligned}\int \frac{1}{x^3+x^2-2}\,dx \\ = \quad & \frac{1}{5}\ln|x-1| - \frac{1}{5}\int \frac{x+1}{(x+1)^2+1}\,dx - \frac{2}{5}\int \frac{1}{(x+1)^2+1}\,dx\end{aligned}$$

For the next integral make the substitution $u = (x+1)^2$ and $\frac{1}{2}\,du = (x+1)\,dx$.

$$\begin{aligned}\int \frac{1}{x^3+x^2-2}\,dx \\ = \quad & \frac{1}{5}\ln|x-1| - \frac{1}{5}\int \frac{1/2}{u+1}\,du - \frac{2}{5}\int \frac{1}{(x+1)^2+1}\,dx \\ = \quad & \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln|u+1| - \frac{2}{5}\int \frac{1}{(x+1)^2+1}\,dx \\ = \quad & \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln((x+1)^2+1) - \frac{2}{5}\int \frac{1}{(x+1)^2+1}\,dx\end{aligned}$$

Completing the Square (3 of 3)

$$\int \frac{1}{x^3 + x^2 - 2}\,dx$$
$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln((x+1)^2+1) - \frac{2}{5}\int \frac{1}{(x+1)^2+1}\,dx$$

For the final integration step make the trigonometric substitution $x + 1 = \tan\theta$ and $dx = \sec^2\theta\,d\theta$.

$$\int \frac{1}{x^3 + x^2 - 2}\,dx$$
$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln((x+1)^2+1) - \frac{2}{5}\int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta$$
$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln((x+1)^2+1) - \frac{2}{5}\int (1)\,d\theta$$
$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln((x+1)^2+1) - \frac{2}{5}\theta + C$$
$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln((x+1)^2+1) - \frac{2}{5}\tan^{-1}(x+1) + C$$

Repeated Quadratic Factors

Question: what if an irreducible quadratic factor of $q(x)$ is repeated?

Answer: if $f(x) = \dfrac{p(x)}{(ax^2+bx+c)^n}$ then the partial fraction decomposition will have the form

$$\begin{aligned}\frac{p(x)}{(ax^2+bx+c)^n} &= \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots \\ &\quad + \frac{A_nx+B_n}{(ax^2+bx+c)^n}\end{aligned}$$

where A_i and B_i are constants for $i = 1, 2, \ldots, n$.

Example

Evaluate the indefinite integral below.

$$\int \frac{1}{x(1+x^2)^2}\,dx$$

First find the partial fraction decomposition of the integrand. Since the denominator is already factored,

$$\begin{aligned}
\frac{1}{x(1+x^2)^2} &= \frac{A}{x} + \frac{Bx+C}{1+x^2} + \frac{Dx+E}{(1+x^2)^2} \\
&= \frac{A(1+x^2)^2 + (Bx+C)x(1+x^2) + (Dx+E)x}{x(1+x^2)^2} \\
1 &= A(1+x^2)^2 + (Bx+C)x(1+x^2) + (Dx+E)x
\end{aligned}$$

Partial Fraction Decomposition (1 of 2)

$$
\begin{aligned}
1 &= A(1+x^2)^2 + (Bx+C)x(1+x^2) + (Dx+E)x \\
\text{Let } x = 0\text{: } 1 &= A \text{ (replace } A \text{ with 1 in the equation above)} \\
1 &= (1+x^2)^2 + (Bx+C)x(1+x^2) + (Dx+E)x \\
0 &= 2x^2 + x^4 + (Bx+C)x(1+x^2) + (Dx+E)x \\
0 &= 2x + x^3 + (Bx+C)(1+x^2) + (Dx+E) \\
\text{Let } x = 0\text{: } 0 &= C + E \text{ (replace } E \text{ with } -C \text{ in the equation above)} \\
0 &= 2x + x^3 + (Bx+C)(1+x^2) + (Dx-C) \\
0 &= 2x + x^3 + Bx^3 + Cx^2 + Bx + Dx \\
0 &= 2 + x^2 + Bx^2 + Cx + B + D \\
\text{Let } x = 0\text{: } 0 &= 2 + B + D \text{ (replace } D \text{ with } -2-B \text{ above)} \\
0 &= 2 + x^2 + Bx^2 + Cx + B - 2 - B \\
0 &= x^2 + Bx^2 + Cx \\
0 &= x + Bx + C
\end{aligned}
$$

Partial Fraction Decomposition (2 of 2)

$$\begin{aligned} 0 &= x + Bx + C \\ \text{Let } x = 0\text{: } 0 &= C \implies E = 0 \\ 0 &= x + Bx = (1 + B)x \implies B = -1 \implies D = -1 \end{aligned}$$

Finally we have the partial fraction decomposition:

$$\frac{1}{x(1+x^2)^2} = \frac{1}{x} - \frac{x}{1+x^2} - \frac{x}{(1+x^2)^2}$$

Integration

Evaluate the indefinite integral below.

$$\begin{aligned}\int \frac{1}{x(1+x^2)^2}\,dx &= \int \left(\frac{1}{x} - \frac{x}{1+x^2} - \frac{x}{(1+x^2)^2}\right)\,dx \\ &= \int \frac{1}{x}\,dx - \int \frac{x}{1+x^2}\,dx - \int \frac{x}{(1+x^2)^2}\,dx\end{aligned}$$

For the remaining two integrals make the substitution $u = 1 + x^2$ and $\frac{1}{2}\,du = x\,dx$.

$$\begin{aligned}\int \frac{1}{x(1+x^2)^2}\,dx &= \ln|x| - \frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{2}\int \frac{1}{u^2}\,du \\ &= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + C \\ &= \ln|x| - \frac{1}{2}\ln(1+x^2) + \frac{1}{2(1+x^2)} + C\end{aligned}$$

Summary

Given $f(x) = \dfrac{p(x)}{q(x)}$:

1. If $\deg p(x) \geq \deg q(x)$ use long division to rewrite $f(x) = s(x) + \frac{r(x)}{q(x)}$ where $\deg r(x) < \deg q(x)$.
2. Assuming $\deg p(x) < \deg q(x)$, factor $q(x)$ into linear and irreducible quadratic factors.
3. For each factor $(ax+b)^n$, the partial fraction decomposition contains terms of the form

$$\frac{c_1}{ax+b} + \frac{c_2}{(ax+b)^2} + \cdots + \frac{c_n}{(ax+b)^n}.$$

   For each irreducible factor $(ax^2+bx+c)^n$, the partial fraction decomposition contains terms of the form

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

Homework

- Read Section 3.4
- Exercises: 183, 187, 191, . . . , 227/handout