Polar Coordinates

MATH 211, *Calculus II*

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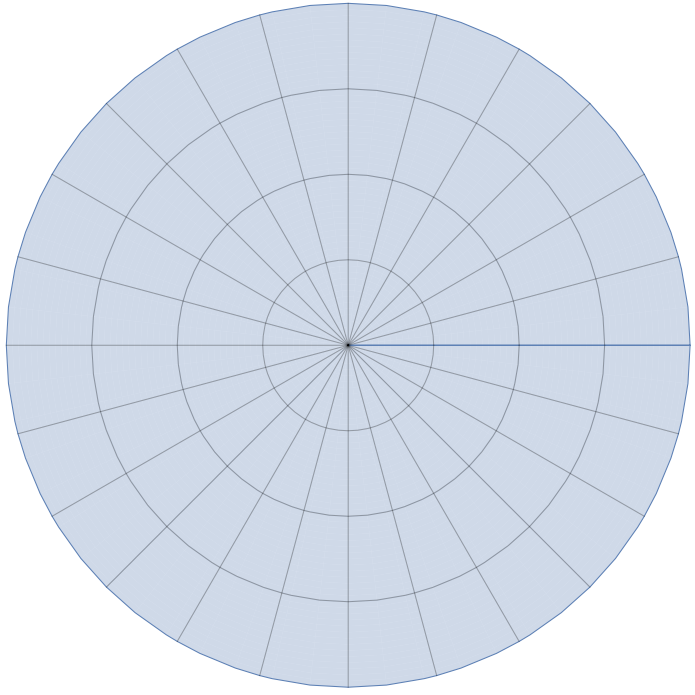
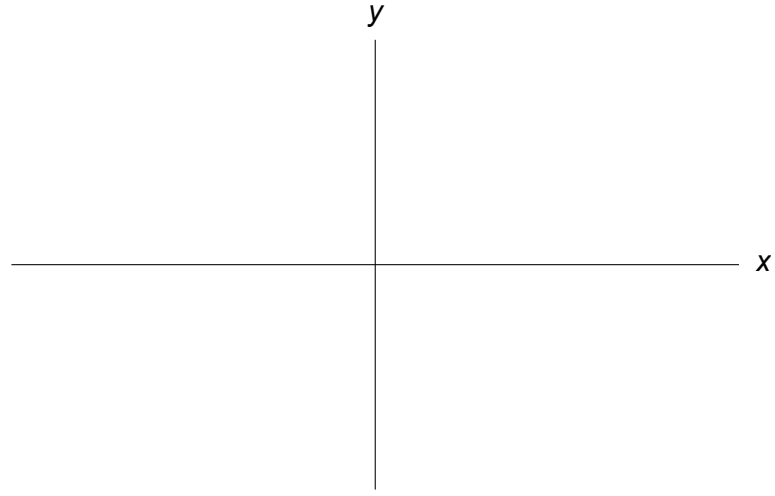
Fall 2021

Introduction

The **polar coordinate** system is an alternative to the **rectangular coordinate** system (sometimes called the **Cartesian coordinate** system).



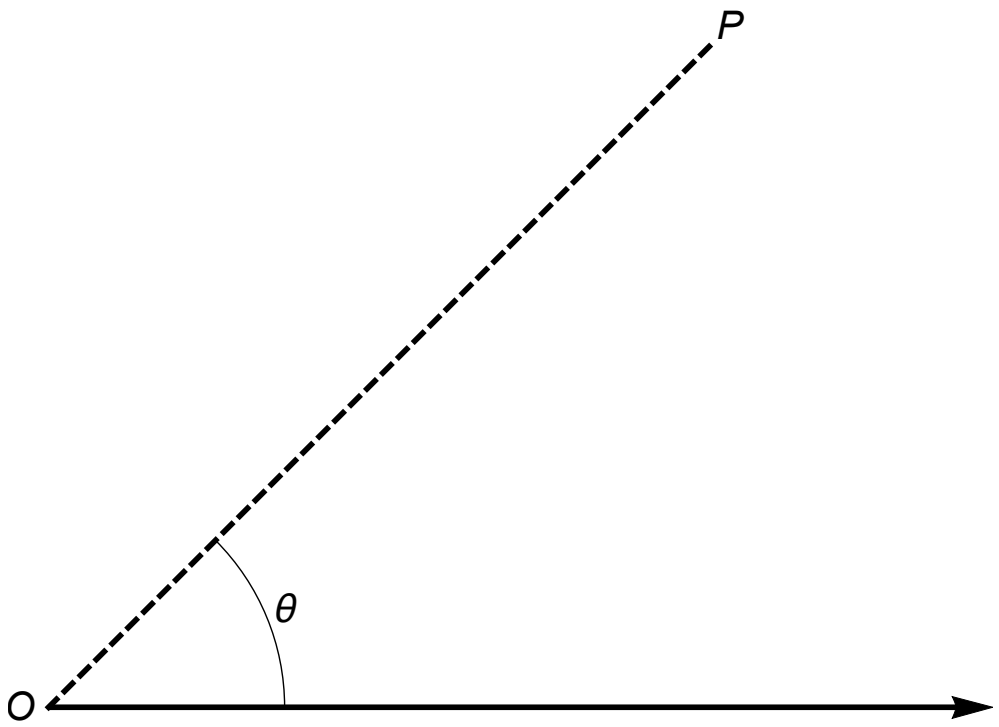


Polar Coordinate System

The polar coordinate system consists of

- an **origin point** O sometimes called the **pole**,
- a directed half-line called the **polar axis**.

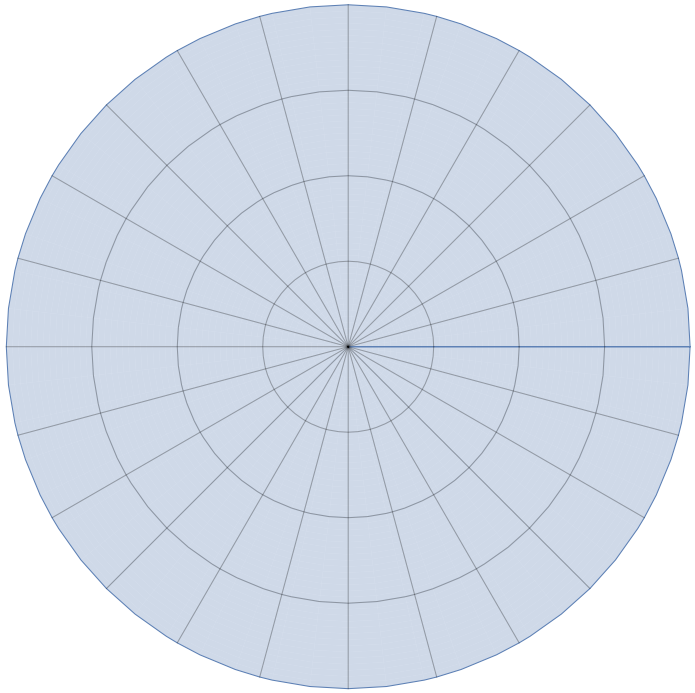
A point P in the plane can be located using its polar coordinates (r, θ) where r is the distance from the origin O and θ is the angle the line from O to P makes with the polar axis.

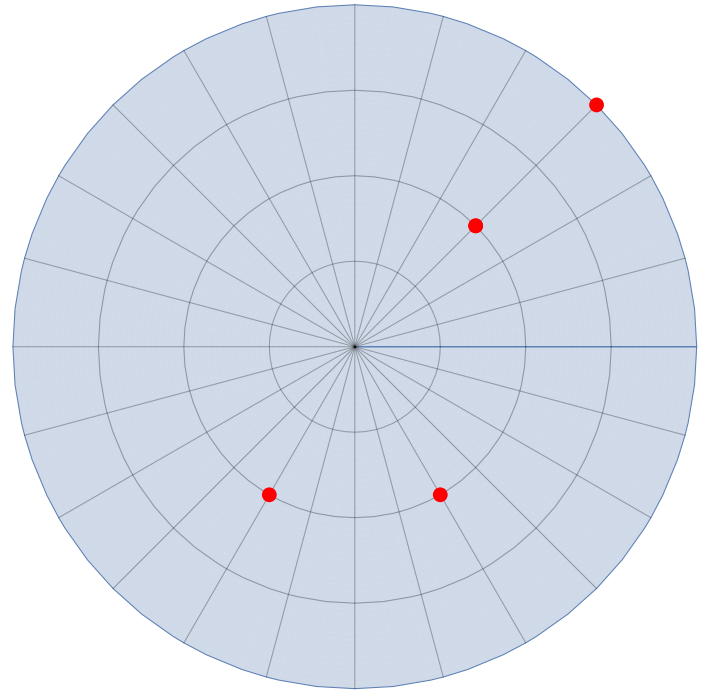


Example

Locate the points with polar coordinates:

$$(1, \pi/4), \quad (2, \pi/4), \quad (-1, \pi/3), \quad (1, -\pi/3), \quad (1, -7\pi/4)$$





Remarks

Remarks:

- The polar coordinates of a point are not unique.
- The origin O has coordinates $(0, \theta)$ for any θ.
- The point with coordinates (r, θ) can also be located with $(r, \theta + 2n\pi)$ for any integer n.

Converting Between Polar and Rectangular Coordinates

Given polar coordinates (r, θ), the corresponding rectangular coordinates are

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

Given rectangular coordinates (x, y), the corresponding polar coordinates are

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right). \end{aligned}$$

Remark: choose $r < 0$ when $x < 0$.

Examples (1 of 2)

Find the rectangular coordinates of the points whose polar coordinates are

- $(-1, \pi/3)$

$$(x, y) = (r\cos\theta, r\sin\theta) = \left(-1\cos\frac{\pi}{3}, -1\sin\frac{\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

- $(3, \pi/8)$

$$(x, y) = (r\cos\theta, r\sin\theta) = \left(3\cos\frac{\pi}{8}, 3\sin\frac{\pi}{8}\right)$$

- $(-3, 1)$

$$(x, y) = (r\cos\theta, r\sin\theta) = (-3\cos 1, -3\sin 1)$$

Examples (2 of 2)

Find the polar coordinates of the points whose rectangular coordinates are

- $(-1, 1)$

$$\begin{aligned} r &= -\sqrt{x^2 + y^2} = -\sqrt{(-1)^2 + (1)^2} = -\sqrt{2} \\ \theta &= \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{1}{-1} = -\frac{\pi}{4} \end{aligned}$$

- $(2, -1)$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (-1)^2} = \sqrt{5} \\ \theta &= \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{-1}{2} \approx -0.4636 \end{aligned}$$

- $(-3, 1)$

$$\begin{aligned} r &= -\sqrt{x^2 + y^2} = -\sqrt{(-3)^2 + (1)^2} = -\sqrt{10} \\ \theta &= \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{1}{-3} \approx -0.3218 \end{aligned}$$

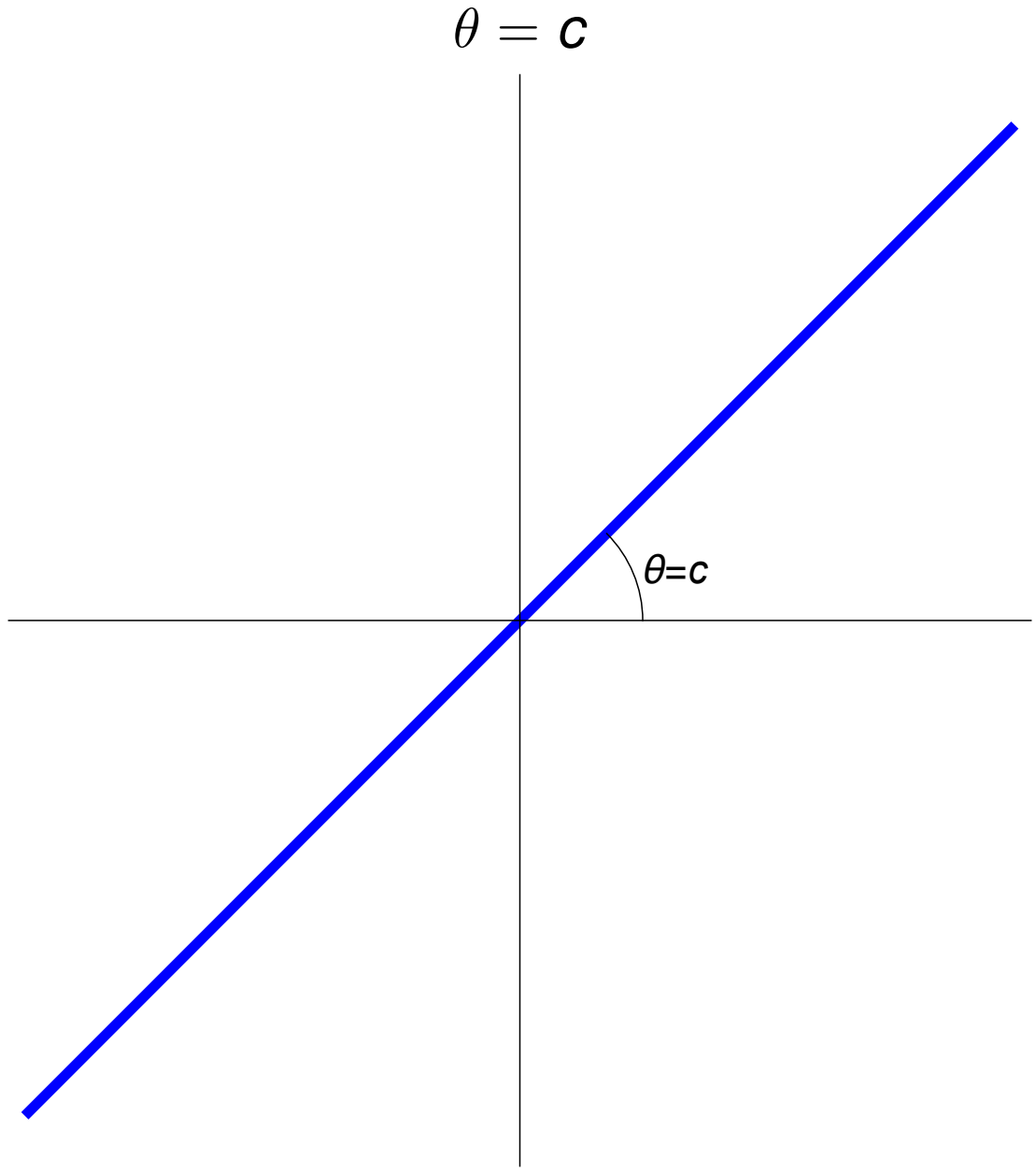
Graphs

The **graph** of the polar equation $r = f(\theta)$ is the set of all points in the plane whose polar coordinates are $(f(\theta), \theta)$.

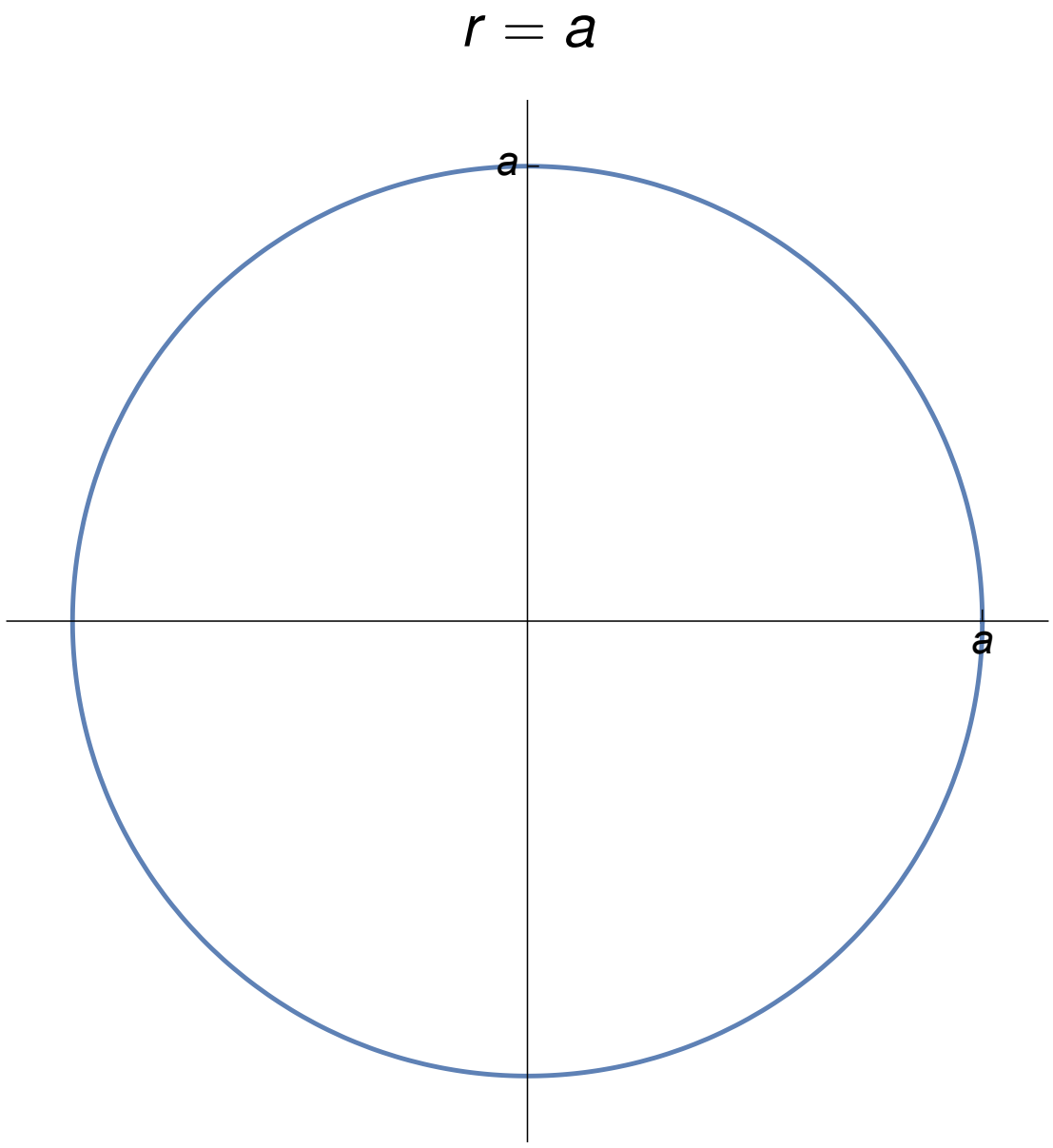
Equivalently, the **graph** of the polar equation $r = f(\theta)$ is the set of all points in the plane whose rectangular coordinates are given by the parametric equations

$$\begin{aligned} x &= f(\theta)\cos\theta \\ y &= f(\theta)\sin\theta. \end{aligned}$$

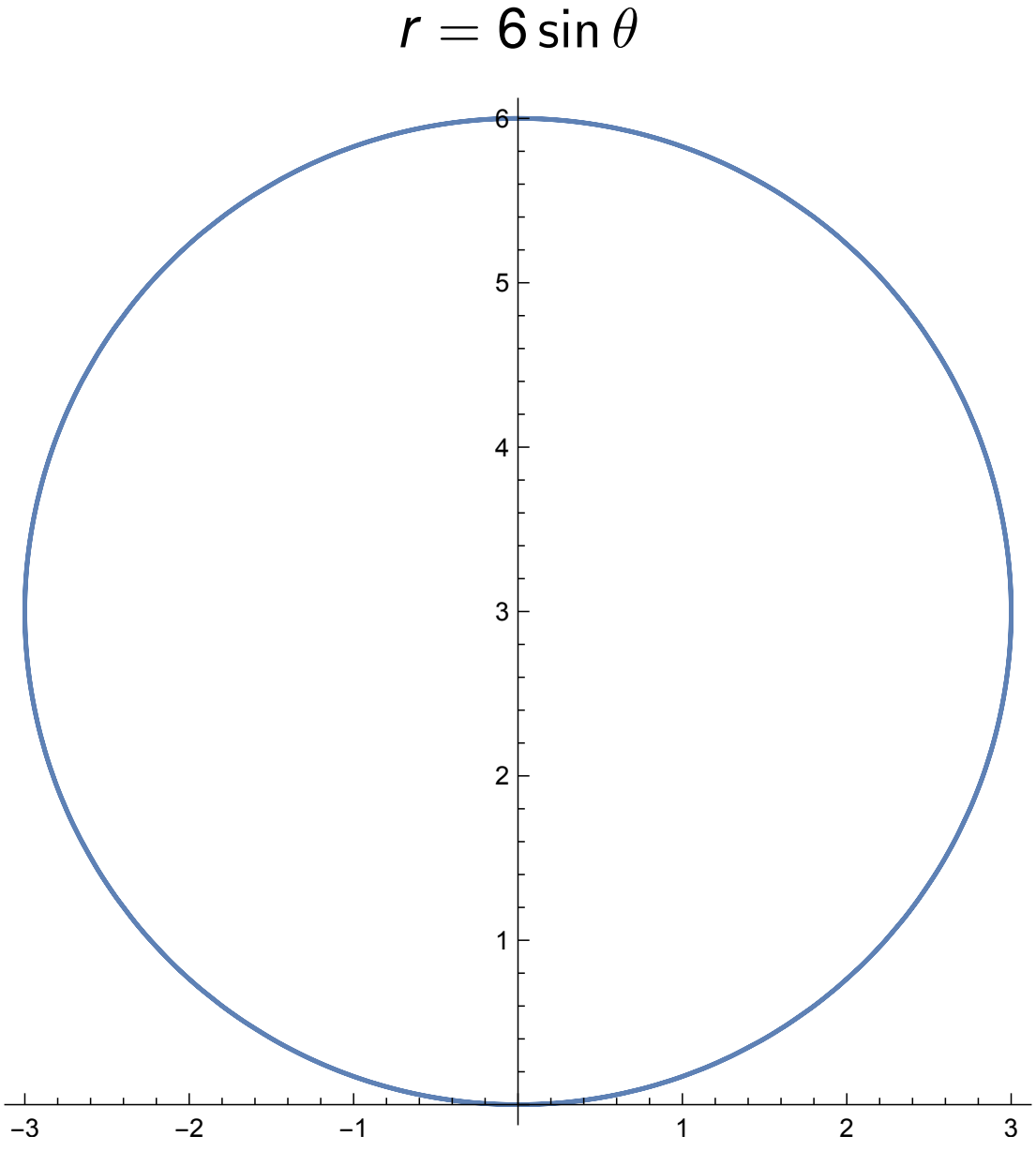
Graph: $\theta = c$, a constant



Graph: $r = a > 0$, a constant



Circle



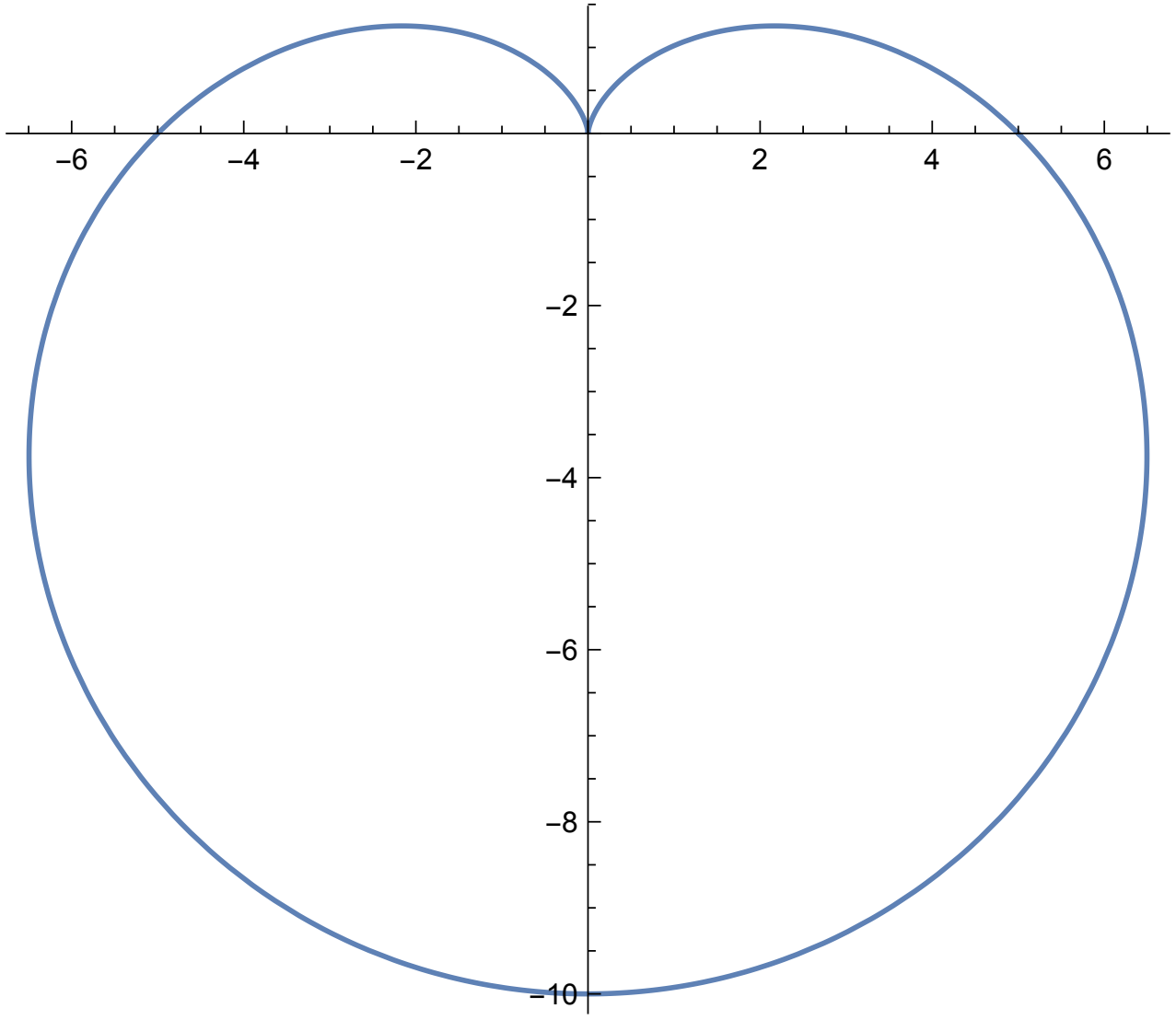
Cardioid

If $a \neq 0$ then any of the following equations graphs a **cardioid**.

$$\begin{aligned} r &= a(1 + \cos\theta) \\ r &= a(1 - \cos\theta) \\ r &= a(1 + \sin\theta) \\ r &= a(1 - \sin\theta) \end{aligned}$$

Example: Cardioid

$$r = 5(1 - \sin\theta)$$



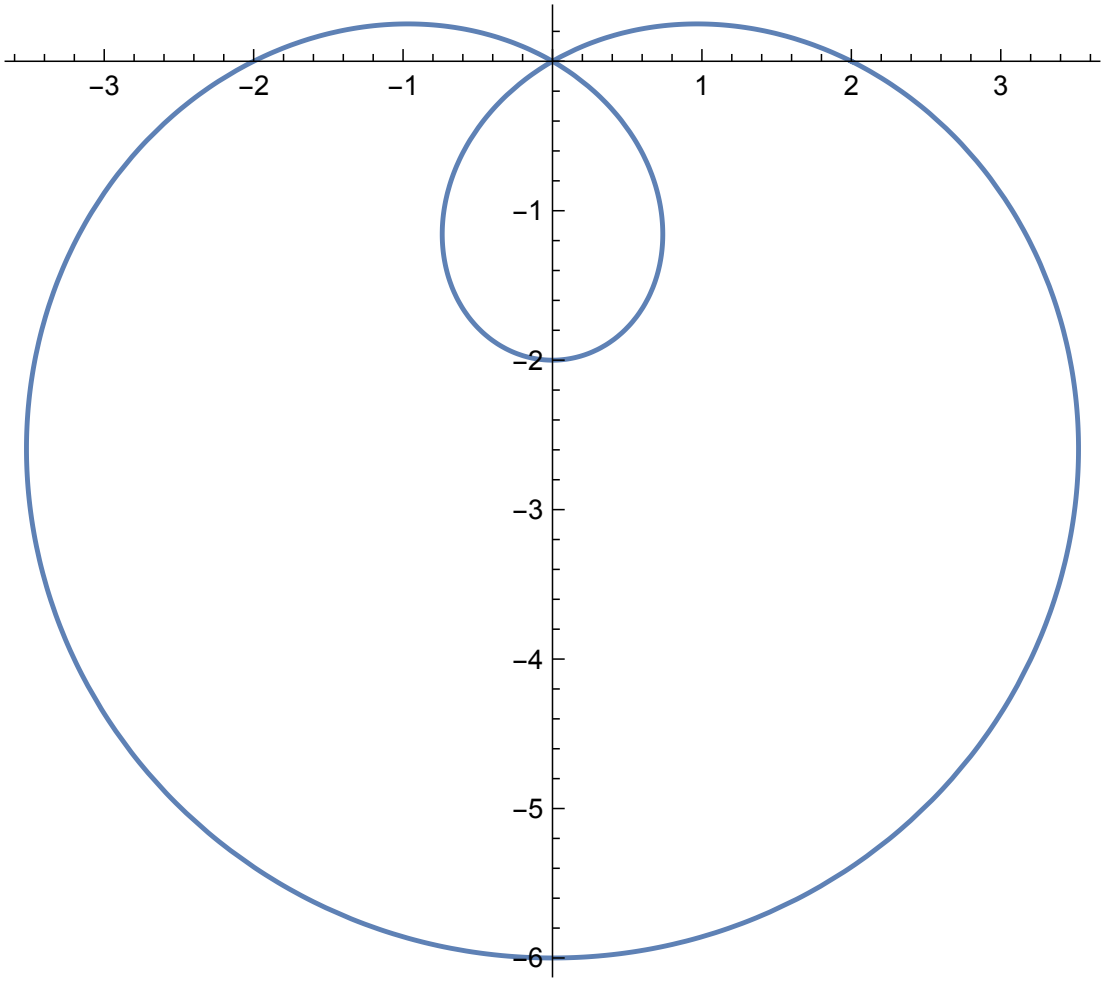
Limaçon

If $a \neq 0$ and $a \neq b \neq 0$ then either of the following equations graphs a **limaçon**.

$$\begin{aligned} r &= a + b\cos\theta \\ r &= a + b\sin\theta \end{aligned}$$

Example: Limaçon

$$r = 2 - 4\sin\theta$$



If $|a| = |b|$ the limaçon is a cardioid.

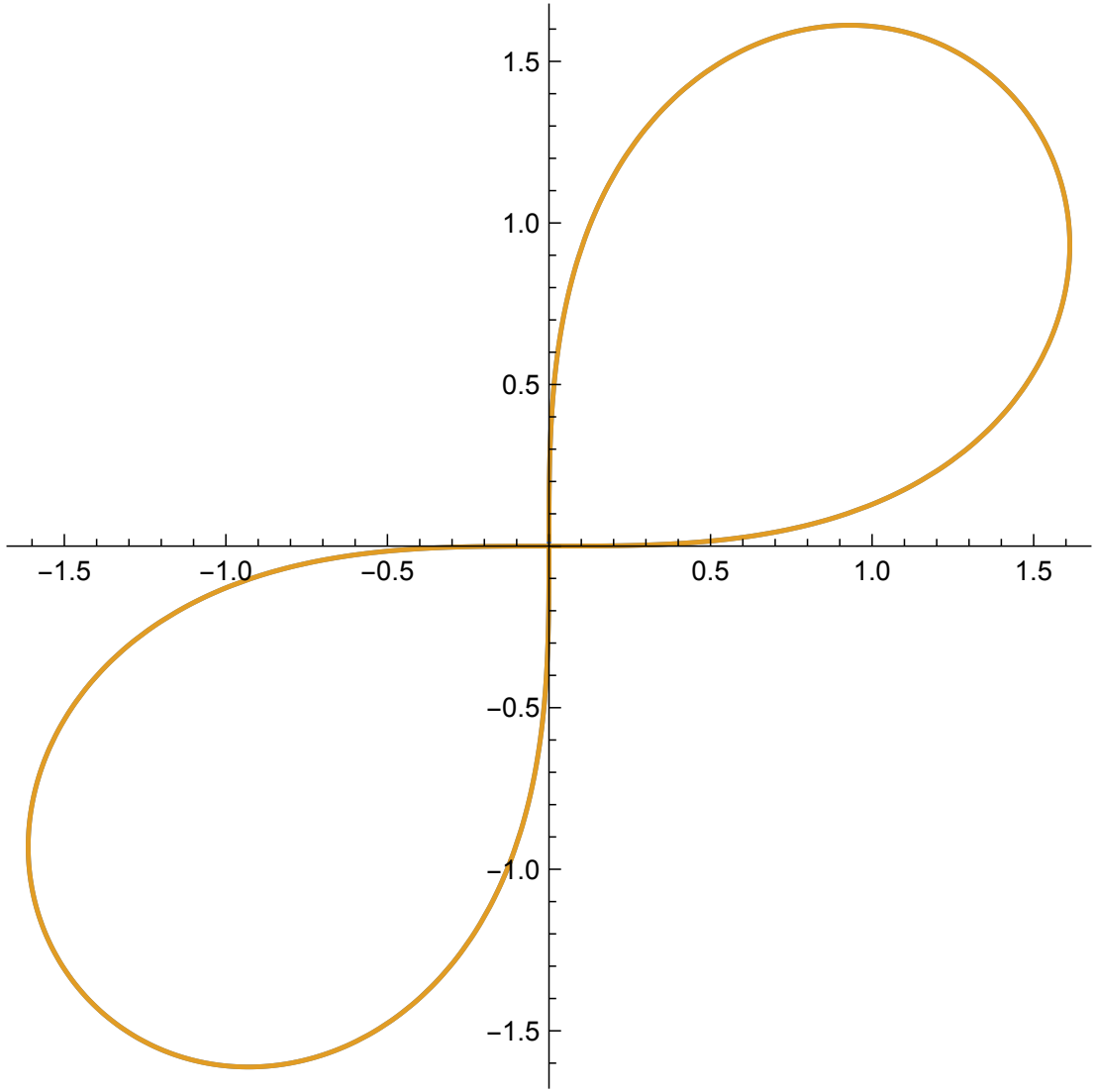
Lemniscate

If $a \neq 0$ and $b \neq 0$ then either of the following equations graphs a **lemniscate**.

$$\begin{aligned} r^2 &= a \cos b\theta \\ r^2 &= a \sin b\theta \end{aligned}$$

Example: Lemniscate

$$r^2 = 4 \sin 2\theta$$



Roses

If $a \neq 0$ and n is a positive whole number then equations of the form

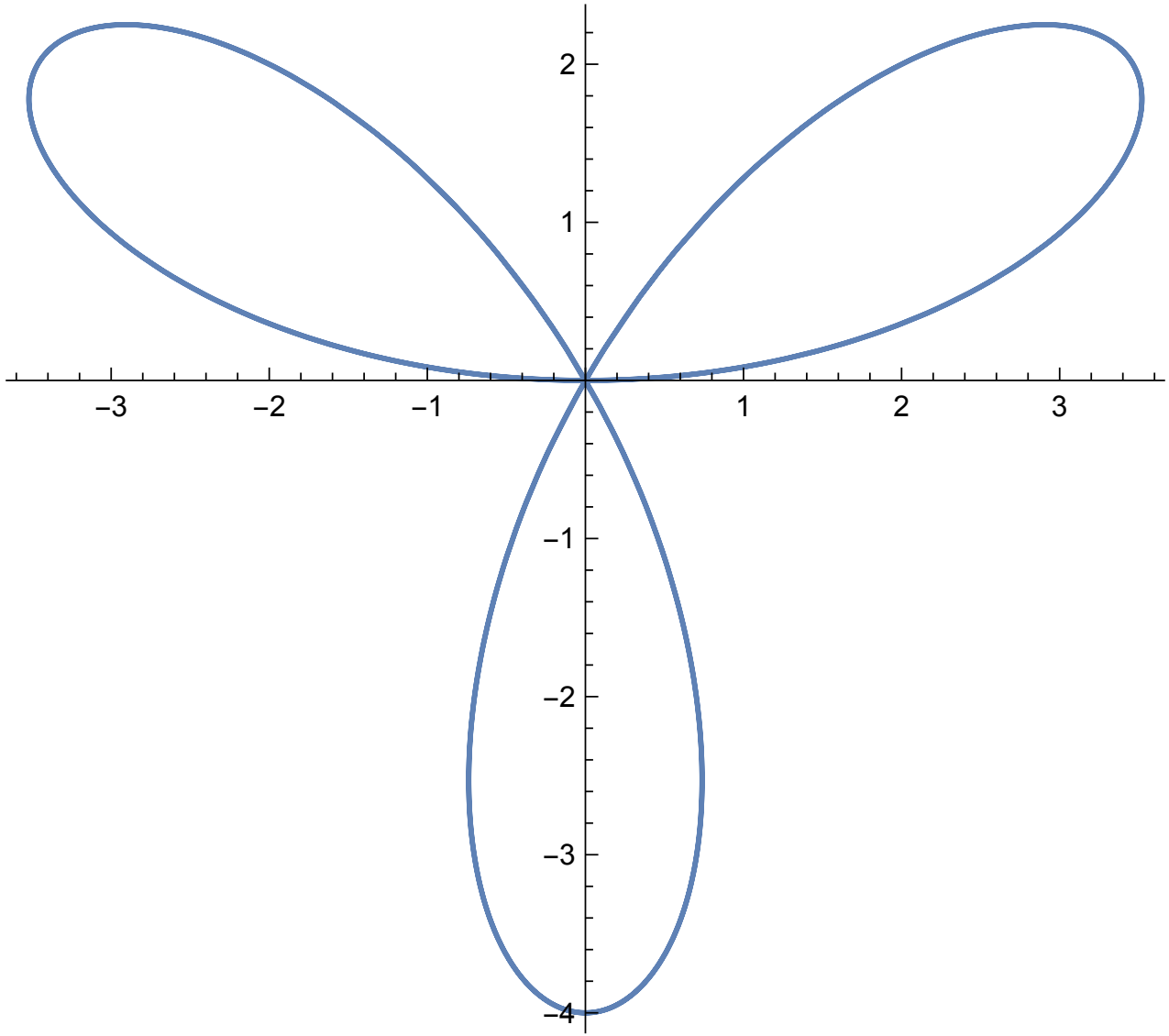
$$r = a\sin(n\theta) \qquad \text{or} \qquad r = a\cos(n\theta)$$

produce a curve called a **rose**.

Remark: if n is odd there are n loops. If n is even there are $2n$ loops.

Example: Roses

$$r = 4\sin(3\theta)$$

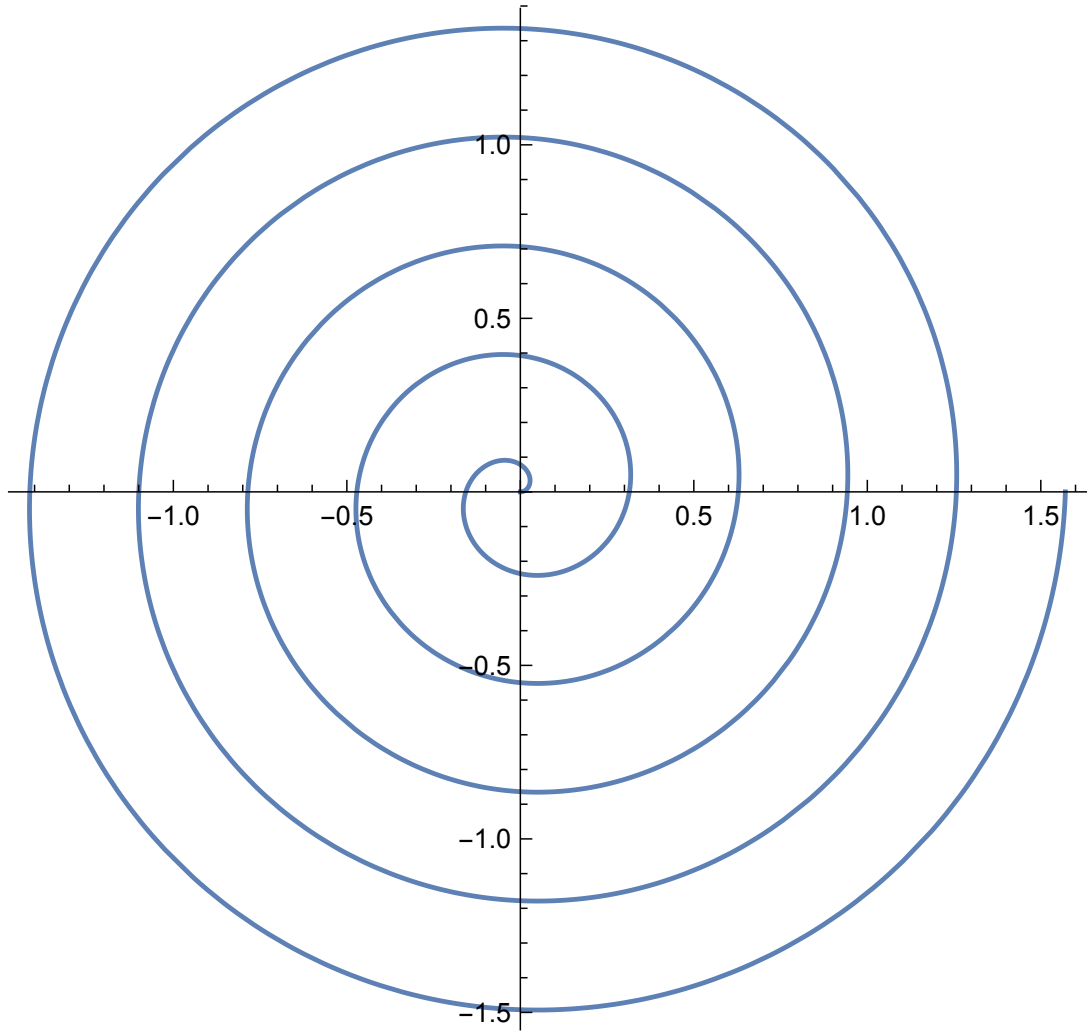


Spiral of Archimedes

If $a \neq 0$ the equation $r = a\theta$ has a graph called an **Archimedean spiral**.

Example

$$r = \frac{1}{20}\theta$$



Polar Equations *vs.* Rectangular Equations (1 of 2)

Convert the following polar coordinate equations into rectangular coordinate equations.

1. $r \sin\theta - 4 = 0$

$$\begin{aligned} r\sin\theta &= 4 \\ y &= 4 \end{aligned}$$

2. $r = 6\cos\theta$

$$\begin{aligned} r^2 &= 6r\cos\theta \\ x^2 + y^2 &= 6x \\ x^2 - 6x + y^2 &= 0 \\ (x-3)^2 + y^2 &= 9 \end{aligned}$$

Polar Equations *vs.* Rectangular Equations (2 of 2)

Convert the following rectangular coordinate equations into polar coordinate equations.

1. $x - y = 0$

$$\begin{aligned} r\cos\theta - r\sin\theta &= 0 \\ r\cos\theta &= r\sin\theta \\ 1 &= \tan\theta \\ \theta &= \frac{\pi}{4} \end{aligned}$$

2. $y^2 = 4x$

$$\begin{aligned} r^2\sin^2\theta &= 4r\cos\theta \\ r &= \frac{4\cos\theta}{\sin^2\theta} \\ r &= 4\cot\theta\,\csc\theta \end{aligned}$$

Homework

- Read Section 7.3
- Exercises: 125, 129, 133, . . . , 145, 155, 159, . . . , 175/handout