Power Series

MATH 211, *Calculus II*

J. Robert Buchanan

Department of Mathematics

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Introduction

- We have been studying series for which the individual terms are constant.
- Today we consider the case in which the terms of a series contain a variable.

Example

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

provided $|x| < 1$.

Remark: the series converges for $|x| < 1$ to a function, $f(x) = \frac{1}{1-x}$, but diverges for $|x| \geq 1$.

Power Series

Definition

If x is a variable, a **power series in x** is a series of the form

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

where the coefficients a_k are constants.

Definition

If x is a variable and c is a constant, a **power series in $x - c$** is a series of the form

$$\sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1 (x-c) + a_2 (x-c)^2 + \cdots$$

where the coefficients a_k are constants.

Where does a Power Series Converge?

Almost always the Ratio Test will answer this question.

Example

Determine the values of x for which the following series converge.

1. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{4^k}$
2. $\displaystyle\sum_{k=0}^{\infty} \frac{(x-1)^k}{k!}$
3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}(x-2)^k$

$$\sum_{k=0}^{\infty} \frac{x^k}{4^k}$$

Using the Ratio Test:

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{\frac{x^{k+1}}{4^{k+1}}}{\frac{x^k}{4^k}} \right| &= \lim_{k\to\infty} \left| \frac{4^k}{4^{k+1}} \frac{x^{k+1}}{x^k} \right| \\
&= \lim_{k\to\infty} \frac{1}{4}|x| \\
&= \frac{1}{4}|x|.
\end{aligned}$$

The series converges absolutely when

$$\frac{1}{4}|x| < 1 \iff |x| < 4 \iff -4 < x < 4.$$

$$\sum_{k=0}^{\infty} \frac{(x-1)^k}{k!}$$

Using the Ratio Test:

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{\frac{(x-1)^{k+1}}{(k+1)!}}{\frac{(x-1)^k}{k!}} \right| &= \lim_{k\to\infty} \left| \frac{k!}{(k+1)!} \frac{(x-1)^{k+1}}{(x-1)^k} \right| \\
&= \lim_{k\to\infty} \frac{1}{k+1} |x-1| \\
&= 0.
\end{aligned}$$

The series converges absolutely for all $-\infty < x < \infty$.

$$\sum_{k=1}^{\infty} \frac{1}{k}(x-2)^k$$

Using the Ratio Test:

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{\frac{1}{k+1}(x-2)^{k+1}}{\frac{1}{k}(x-2)^k} \right| &= \lim_{k\to\infty} \left| \frac{k}{k+1} \, \frac{(x-2)^{k+1}}{(x-2)^k} \right| \\
&= \lim_{k\to\infty} \frac{k}{k+1} |x-2| \\
&= |x-2|.
\end{aligned}$$

The series converges absolutely when

$$|x-2| < 1 \iff -1 < x-2 < 1 \iff 1 < x < 3.$$

Remarks

Consider the generic power series $\sum_{k=0}^{\infty} a_k(x-c)^k$.

Remarks:

- The series always converges for $x = c$.
- If the series converges for some $x \neq c$ there is a number $r > 0$ called the **radius of convergence** such that the power series converges absolutely for x in the interval $(c - r, c + r)$, the **interval of convergence**, and diverges when $|x - c| > r$.
- Further investigation on a case-by-case basis is needed to determine if the power series converges for $x = c \pm r$.

Result

Theorem

Given any power series $\sum_{k=0}^{\infty} a_k(x-c)^k$*, there are exactly three possibilities:*

1. *The series converges absolutely for all* x *in* $(-\infty, \infty)$ *and the radius of convergence is* $r = \infty$*.*
2. *The series converges only for* $x = c$ *and diverges for all* $x \neq c$ *and the radius of convergence is* $r = 0$*.*
3. *The series converges absolutely for* x *in* $(c - r, c + r)$ *and diverges for* $x < c - r$ *and* $x > c + r$ *for some number* r *with* $0 < r < \infty$*.*

Examples (1 of 2)

Find the interval and radius of convergence for the following power series.

1. $$\sum_{k=0}^{\infty} \frac{x^k}{k+1}$$
2. $$\sum_{k=0}^{\infty} \frac{(x-2)^k}{3^k}$$
3. $$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{\sqrt{k}}$$
4. $$\sum_{k=1}^{\infty} k! x^k$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k+1}$$

Using the Ratio Test:

$$\lim_{k\to\infty} \left| \frac{\frac{x^{k+1}}{k+2}}{\frac{x^k}{k+1}} \right| = \lim_{k\to\infty} \left| \frac{k+1}{k+2} \frac{x^{k+1}}{x^k} \right| = \lim_{k\to\infty} \frac{k+1}{k+2}|x| = |x| < 1.$$

Thus the power series converges absolutely for $-1 < x < 1$.

- At $x = -1$ the series is $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ which converges by the Alternating Series Test.
- At $x = 1$ the series is $\sum_{k=0}^{\infty} \frac{1}{k+1}$ which diverges by the Integral Test.

The power series converges for $-1 \leq x < 1$.

$$\sum_{k=0}^{\infty} \frac{(x-2)^k}{3^k}$$

Using the Ratio Test:

$$\lim_{k\to\infty} \left| \frac{\frac{(x-2)^{k+1}}{3^{k+1}}}{\frac{(x-2)^k}{3^k}} \right| = \lim_{k\to\infty} \left| \frac{3^k}{3^{k+1}} \frac{(x-2)^{k+1}}{(x-2)^k} \right| = \lim_{k\to\infty} \frac{1}{3}|x-2| = \frac{1}{3}|x-2| <$$

Thus the power series converges absolutely for $-1 < x < 5$.

- At $x = -1$ the series is $\sum_{k=0}^{\infty} \frac{(-3)^k}{3^k} = \sum_{k=0}^{\infty} (-1)^k$ which diverges by the kth Term Test.
- At $x = 5$ the series is $\sum_{k=0}^{\infty} \frac{3^k}{3^k} = \sum_{k=0}^{\infty} 1$ which diverges by the kth Term Test.

The power series converges for $-1 < x < 5$.

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{\sqrt{k}}$$

Using the Ratio Test:

$$\lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1} x^{k+1}}{\sqrt{k+1}}}{\frac{(-1)^k x^k}{\sqrt{k}}} \right| = \lim_{k\to\infty} \left| \frac{\sqrt{k}}{\sqrt{k+1}} \frac{(-1)^{k+1}}{(-1)^k} \frac{x^{k+1}}{x^k} \right| = \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k+1}} |x| =$$

Thus the power series converges absolutely for $-1 < x < 1$.

- At $x = -1$ the series is $\sum_{k=1}^{\infty} \frac{(-1)^k(-1)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ which diverges by the p-Series Test.
- At $x = 1$ the series is $\sum_{k=1}^{\infty} \frac{(-1)^k(1)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/2}}$ which converges by the Alternating Series Test.

The power series converges for $-1 < x \leq 1$.

$$\sum_{k=1}^{\infty} k! x^k$$

Using the Ratio Test:

$$\lim_{k\to\infty} \left| \frac{(k+1)! x^{k+1}}{k! x^k} \right| = \lim_{k\to\infty} (k+1)|x| = \infty > 1 \text{ if } x \neq 0.$$

Thus the power series converges only for $x = 0$. The radius of convergence is $r = 0$.

Questions

1. If the interval of convergence of a power series is $(2, 4)$, what is the radius of convergence?
2. If the power series $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence $r = 5$, what do we know about the convergence of
   - $\sum_{k=0}^{\infty} a_k 3^k$?
   - $\sum_{k=0}^{\infty} a_k (-6)^k$?
   - $\sum_{k=0}^{\infty} a_k 5^k$?

Examples (2 of 2)

Find power series whose sums are the following functions. State the radius and interval of convergence of those series.

1. $f(x) = \dfrac{3}{1-x^2} = \displaystyle\sum_{k=0}^{\infty} 3x^{2k}$ if $-1 < x < 1$, radius $r = 1$.
2. $g(x) = \dfrac{4}{3-5x} = \displaystyle\sum_{k=0}^{\infty} 4\frac{(5x)^k}{3^{k+1}}$ if $-\frac{3}{5} < x < \frac{3}{5}$, radius $r = \frac{3}{5}$.

Calculus and Power Series

Remark: inside of the radius of convergence of a power series, the series converges to a function of x, *i.e.*,

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

for x in $(c-r, c+r)$.

We can differentiate and integrate these convergent power series term-by-term.

Differentiating a Power Series

If $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$, then

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\sum_{k=0}^{\infty} a_k (x-c)^k\right] \\
&= \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k (x-c)^k\right] \\
&= \sum_{k=0}^{\infty} k\, a_k (x-c)^{k-1} \\
&= \sum_{k=1}^{\infty} k\, a_k (x-c)^{k-1}
\end{aligned}
$$

Example

Use the power series representation of $f(x) = \frac{1}{1-x}$ to find the power series representation of $\frac{1}{(1-x)^2}$.

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k \quad (\text{if } |x| < 1) \\
\frac{d}{dx}\left[\frac{1}{1-x}\right] &= \frac{d}{dx}\left[\sum_{k=0}^{\infty} x^k\right] \quad (\text{if } |x| < 1) \\
\frac{1}{(1-x)^2} &= \sum_{k=1}^{\infty} kx^{k-1} \quad (\text{if } |x| < 1)
\end{aligned}$$

Integrating a Power Series

If $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$, then

$$\begin{aligned}
\int_c^x f(t)\,dt &= \int_c^x \left(\sum_{k=0}^{\infty} a_k (t-c)^k \right) dt \\
&= \sum_{k=0}^{\infty} a_k \left(\int_c^x (t-c)^k\,dt \right) \\
&= \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-c)^{k+1}
\end{aligned}$$

Examples (1 of 2)

Use the power series representation of $f(x) = \dfrac{1}{1-x}$ to find the power series representation of $\ln(1-x)$.

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{k=0}^{\infty} x^k \text{ (if } |x| < 1) \\
\int_0^x \frac{1}{1-t}\,dt &= \int_0^x \left[\sum_{k=0}^{\infty} t^k\right] dt \text{ (if } |x| < 1) \\
\left[-\ln(1-t)\right]_{t=0}^{t=x} &= \sum_{k=0}^{\infty} \int_0^x t^k\,dt \text{ (if } |x| < 1) \\
-\ln(1-x) + \ln 1 &= \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} \text{ (if } |x| < 1) \\
\ln(1-x) &= \sum_{k=0}^{\infty} \frac{-1}{k+1} x^{k+1} \text{ (if } |x| < 1)
\end{aligned}$$

Examples (2 of 2)

Find a power series representation for $\tan^{-1} x$ using the fact that $\dfrac{d}{dx}\left[\tan^{-1} x\right] = \dfrac{1}{1+x^2}$.

$$
\begin{aligned}
\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} &= \sum_{k=0}^{\infty}(-x^2)^k \text{ (if } |x| < 1) \\
\int_0^x \frac{1}{1+t^2}\,dt &= \int_0^x \left[\sum_{k=0}^{\infty}(-1)^k t^{2k}\right] dt \text{ (if } |x| < 1) \\
\left[\tan^{-1} t\right]_{t=0}^{t=x} &= \sum_{k=0}^{\infty}(-1)^k \int_0^x t^{2k}\,dt \text{ (if } |x| < 1) \\
\tan^{-1} x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \text{ (if } |x| < 1)
\end{aligned}
$$

Other Series Containing Variables

There are other types of infinite series which contain variables, but which are not power series.

For example, the infinite series of the form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\,x) + b_k \sin(k\,x))$$

is called a **Fourier series.**

Differentiating a Non-Power Series

Remark: Differentiating a non-power series does not necessarily produce a convergent derivative series.

Example

Suppose $f(x) = \sum_{k=1}^{\infty} \frac{\cos(k^3 x)}{k^2}$. Show that the original series converges but that its derivative does not when $x \neq 0$.

By the Comparison Test, the absolute value series converges for all x, thus the given series converges absolutely for $-\infty < x < \infty$.

$$f'(x) = \sum_{k=1}^{\infty} (-k)\sin(k^3 x)$$

diverges by the kth Term Test except for $x = 0$.

Homework

- Read Section 6.1
- Exercises: 1, 5, 9, . . . , 41/handout