Absolute Convergence and the Ratio Test

MATH 211, *Calculus II*

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Background

Remark: All previously covered tests for convergence/divergence apply only to positive term series (except for the Alternating Series Test).

Question: what about series which do not consist exclusively of positive terms, but which are not alternating series?

Example

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\cos k}{k^2} &= \cos 1 + \frac{\cos 2}{4} + \frac{\cos 3}{9} + \cdots \\ &\approx 0.540302 - 0.104037 - 0.109999 + \cdots \end{aligned}$$

Absolute Convergence

Definition

An infinite series $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** if the series

$$\sum_{k=1}^{\infty} |a_k| = |a_1| + |a_2| + |a_3| + \cdots$$

converges.

Remark: The series $\sum_{k=1}^{\infty} |a_k|$ is a positive term series.

Conditional Convergence

Definition

An infinite series $\sum_{k=1}^{\infty} a_k$ is **conditionally convergent** if the series converges but the series $\sum_{k=1}^{\infty} |a_k|$ diverges.

For an arbitrary series $\sum_{k=1}^{\infty} a_k$, the series may be classified in only one of the following ways:

- absolutely convergent
- conditionally convergent
- divergent

Examples

Determine which of the following infinite series are absolutely convergent, conditionally convergent, or divergent.

1. $$\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$$
2. $$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$
3. $$\sum_{k=1}^{\infty} \frac{(-1)^k \tan^{-1} k}{k^3}$$
4. $$\sum_{k=1}^{\infty} \frac{\sin(k\pi/6)}{k\sqrt{k}}$$

$\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$

- $\left|\frac{\cos k}{k^2}\right| \leq \frac{1}{k^2}$ for all $k = 1, 2, \ldots$.
- The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (p-Series Test) which implies

$$\sum_{k=1}^{\infty} \left|\frac{\cos k}{k^2}\right| \text{ converges.}$$

- Therefore, $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ **converges absolutely**.

$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$

- $\left|(-1)^{k+1}\frac{1}{k}\right| \leq \frac{1}{k}$ for all $k = 1, 2, \ldots$.
- The series $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges (harmonic series).
- The series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ converges by the Alternating Series Test.
- Therefore, $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ **converges conditionally.**

$$\sum_{k=1}^{\infty} \frac{(-1)^k \tan^{-1} k}{k^3}$$

- $\left| \frac{(-1)^k \tan^{-1} k}{k^3} \right| \leq \frac{\pi/2}{k^3}$ for all $k = 1, 2, \ldots$.
- The series $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges (*p*-Series Test) which implies

$$\sum_{k=1}^{\infty} \frac{\pi/2}{k^3} \text{ converges.}$$

- By the Comparison Test $\sum_{k=1}^{\infty} \left| \frac{(-1)^k \tan^{-1} k}{k^3} \right|$ converges.
- Therefore, $\sum_{k=1}^{\infty} \frac{(-1)^k \tan^{-1} k}{k^3}$ **converges absolutely**.

$$\sum_{k=1}^{\infty} \frac{\sin(k\pi/6)}{k\sqrt{k}}$$

- $\left|\frac{\sin(k\pi/6)}{k\sqrt{k}}\right| \leq \frac{1}{k^{3/2}}$ for all $k = 1, 2, \ldots$.
- The series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges (p-Series Test) which implies

$$\sum_{k=1}^{\infty} \left|\frac{\sin(k\pi/6)}{k\sqrt{k}}\right| \text{ converges.}$$

- Therefore, $\sum_{k=1}^{\infty} \frac{\sin(k\pi/6)}{k\sqrt{k}}$ **converges absolutely.**

Absolute Convergence Implies Convergence

Theorem

If $\sum_{k=1}^{\infty} |a_k|$ *converges then* $\sum_{k=1}^{\infty} a_k$ *converges.*

Proof.

$$\begin{aligned} -|a_k| &\leq a_k \leq |a_k| \\ 0 &\leq a_k + |a_k| \leq 2|a_k| \end{aligned}$$

Therefore

- $\sum_{k=1}^{\infty}(a_k + |a_k|)$ converges by the Comparison Test.
- $\sum_{k=1}^{\infty}(a_k + |a_k| - |a_k|) = \sum_{k=1}^{\infty} a_k$ converges.

□

Ratio Test

Theorem (Ratio Test)

Given $\sum_{k=1}^{\infty} a_k$, *with* $a_k \neq 0$ *for all* k, *suppose that*

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L.$$

Then

1. *if* $L < 1$, *the series converges absolutely,*
2. *if* $L > 1$, *the series diverges,*
3. *if* $L = 1$, *there is no conclusion.*

Examples

Use the Ratio Test to determine the convergence or divergence of the following series.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{k^{20}}{2^k}$
2. $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{k!}$
3. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$
4. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

$$\sum_{k=1}^{\infty} \frac{k^{20}}{2^k}$$

$$\begin{aligned} \lim_{k\to\infty} \left| \frac{\frac{(k+1)^{20}}{2^{k+1}}}{\frac{k^{20}}{2^k}} \right| &= \lim_{k\to\infty} \frac{2^k}{2^{k+1}} \left(\frac{k+1}{k} \right)^{20} \\ &= \lim_{k\to\infty} \frac{1}{2} \left(1 + \frac{1}{k} \right)^{20} \\ &= \frac{1}{2} < 1 \end{aligned}$$

$\sum_{k=1}^{\infty} \frac{k^{20}}{2^k}$ **converges absolutely.**

$$\sum_{k=1}^{\infty} \frac{k^k}{k!}$$

$$\begin{aligned}
\lim_{k\to\infty} \left| \frac{\frac{(k+1)^{k+1}}{(k+1)!}}{\frac{k^k}{k!}} \right| &= \lim_{k\to\infty} \frac{k!}{(k+1)!} \frac{(k+1)^{k+1}}{k^k} \\
&= \lim_{k\to\infty} \frac{1}{k+1} \frac{(k+1)^{k+1}}{k^k} \\
&= \lim_{k\to\infty} \frac{(k+1)^k}{k^k} \\
&= \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e > 1
\end{aligned}$$

$\sum_{k=1}^{\infty} \frac{k^k}{k!}$ **diverges.**

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

$$\lim_{k\to\infty} \left| \frac{\frac{1}{k+1}}{\frac{1}{k}} \right| = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

- The Ratio Test reaches **no conclusion.**
- The series is recognized as the harmonic series and therefore **diverges.**

$\sum_{k=1}^{\infty} \frac{1}{k^2}$

$$\lim_{k\to\infty} \left| \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} \right| = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} = 1$$

- The Ratio Test reaches **no conclusion.**
- The series is recognized as a convergent *p*-series and therefore **converges absolutely**.

Root Test

Theorem (Root Test)

Given $\sum_{k=1}^{\infty} a_k$*, suppose that*

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = L.$$

Then

1. *if* $L < 1$*, the series converges absolutely,*
2. *if* $L > 1$*, the series diverges,*
3. *if* $L = 1$*, there is no conclusion.*

Examples

Use the Root Test to determine the convergence or divergence of the following series.

1. $\displaystyle\sum_{k=1}^{\infty} \frac{(\ln k)^{k/2}}{k^k}$
2. $\displaystyle\sum_{k=2}^{\infty} \frac{2^{k+1}}{(\ln k)^k}$
3. $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{2^k}$

$$\sum_{k=1}^{\infty} \frac{(\ln k)^{k/2}}{k^k}$$

$$\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{\left|\frac{(\ln k)^{k/2}}{k^k}\right|} &= \lim_{k\to\infty} \sqrt[k]{\frac{(\ln k)^{k/2}}{k^k}} \\
&= \lim_{k\to\infty} \frac{(\ln k)^{1/2}}{k} \\
&= 0 < 1
\end{aligned}$$

$\sum_{k=1}^{\infty} \frac{(\ln k)^{k/2}}{k^k}$ **converges absolutely.**

$$\sum_{k=2}^{\infty} \frac{2^{k+1}}{(\ln k)^k}$$

$$\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{\left|\frac{2^{k+1}}{(\ln k)^k}\right|} &= \lim_{k\to\infty} \sqrt[k]{\frac{2^{k+1}}{(\ln k)^k}} \\
&= \lim_{k\to\infty} \frac{2^{1+1/k}}{\ln k} \\
&= 0 < 1
\end{aligned}$$

$\displaystyle\sum_{k=2}^{\infty} \frac{2^{k+1}}{(\ln k)^k}$ **converges absolutely.**

$$\sum_{k=1}^{\infty} \frac{k^2}{2^k}$$

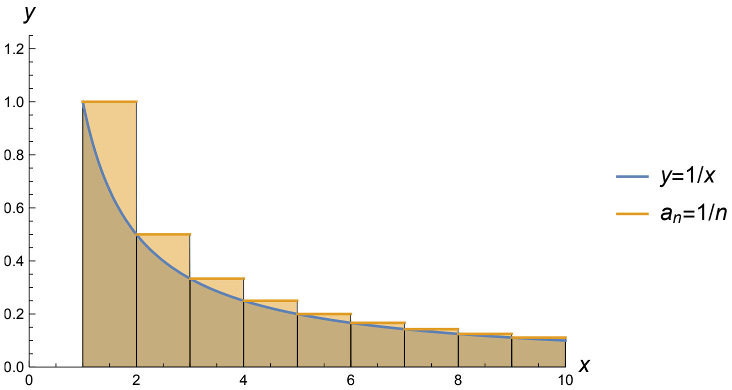
$$\begin{aligned}
\lim_{k\to\infty} \sqrt[k]{\left|\frac{k^2}{2^k}\right|} &= \lim_{k\to\infty} \sqrt[k]{\frac{k^2}{2^k}} \\
&= \lim_{k\to\infty} \frac{k^{2/k}}{2} \\
&= \frac{1}{2}
\end{aligned}$$

$\sum_{k=1}^{\infty} \frac{k^2}{2^k}$ **converges absolutely**,

Harmonic and Alternating Harmonic Series

- Consider the **harmonic series** $\sum_{k=1}^{\infty} \frac{1}{k}$ and the **alternating harmonic series** $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$.
- Define the *n*th partial sum of the harmonic series as $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and the n partial sum of the alternating harmonic series as $S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$.

Euler's Constant (1 of 2)



Define the sequence $\{t_n\}_{n=1}^{\infty}$ as

$$t_n = -\int_1^n \frac{1}{x}\,dx + \sum_{k=1}^{n} \frac{1}{k} = -\ln n + \sum_{k=1}^{n} \frac{1}{k} > 0.$$

Euler's Constant (2 of 2)

$\{t_n\}_{n=1}^{\infty}$ is a decreasing sequence since

$$\begin{aligned} t_n - t_{n+1} &= -\ln n + \sum_{k=1}^{n} \frac{1}{k} - \left(-\ln(n+1) + \sum_{k=1}^{n+1} \frac{1}{k} \right) \\ &= \ln(n+1) - \ln n - \frac{1}{n+1} = \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} > 0. \end{aligned}$$

Since $t_n > 0$ (bounded below) then $\lim_{n\to\infty} t_n = \gamma$ (called **Euler's constant**).

$$H_n - \ln n \to \gamma \text{ as } n \to \infty.$$

Alternating Harmonic Series Sum (1 of 2)

Claim:
$S_{2n} = H_{2n} - H_n$.

Proof.

$$
\begin{aligned}
& H_{2n} - H_n \\
&= H_{2n} - \frac{1}{2}H_n - \frac{1}{2}H_n \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) - \frac{1}{2}H_n \\
&= 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = S_{2n}
\end{aligned}
$$

□

Alternating Harmonic Series Sum (2 of 2)

$$\begin{aligned}\lim_{n\to\infty} S_{2n} &= \lim_{n\to\infty} (H_{2n} - H_n) \\ &= \gamma + \ln(2n) - (\gamma + \ln n) \\ &= \ln(2n) - \ln n = \ln \frac{2n}{n} = \ln 2\end{aligned}$$

Thus we have proved the Alternating Harmonic Series converges to $\ln 2$.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln 2$$

Rearrangements of Series

- Rearranging the order of terms in a **finite sum** leaves the sum unchanged.
- Rearranging the order of terms in an **absolutely convergent infinite sum** leaves the sum unchanged.
- Any **conditionally convergent infinite sum** can be rearranged to give a different sum.

Example: Alternating Harmonic Series

$$\begin{aligned}
\ln 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots \\
\frac{1}{2}\ln 2 &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots \\
&= 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \frac{1}{10} + 0 - \frac{1}{12} \\
&\quad + 0 + \frac{1}{14} + 0 - \frac{1}{16} + \cdots
\end{aligned}$$

Add the first and third equations.

$$\frac{3}{2}\ln 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$$

Homework

- Read Section 5.6
- Exercises: 317, 321, 325, . . . , 353/handout