

Sequences of Real Numbers

MATH 211, *Calculus II*

J. Robert Buchanan

Department of Mathematics

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Introduction

We have used sequences before in the context of

- ▶ finding limits of a function using numerical evidence,

x	$\frac{\sin x}{x}$
0.1	0.998334
0.01	0.999983
0.001	1.000000
0.0001	1.000000

- ▶ approximating the solution to an equation using Newton's Method

n	x_n
0	0.500000
1	0.755222
2	0.739142
3	0.739085

Formal Definition

Definition

A **sequence** is a function whose domain is the set of integers starting at some i_0 (usually $i_0 = 0$ or $i_0 = 1$)

Notation:

▶ $a_1, a_2, \dots, a_n, \dots$

▶ $\{a_j\}_{j=1}^{\infty}$

Remark: the expression a_j (or $a(i)$) is referred to as the **general j^{th} term of the sequence.**

Examples

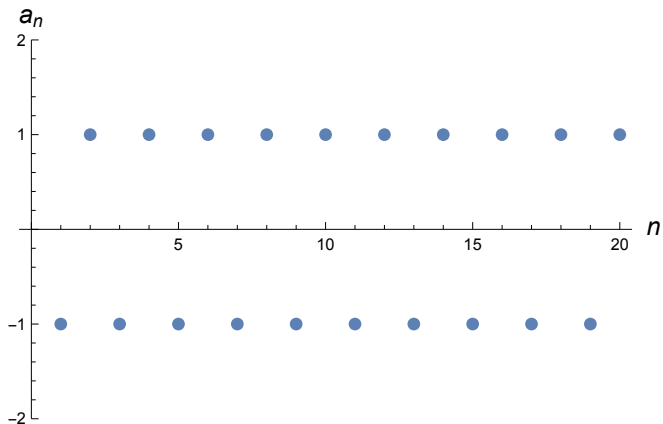
Write out the first four terms (starting with $i = 1$) of each of these sequences:

1. $a_i = (-1)^i$ $a_1 = -1, a_2 = 1, a_3 = -1, a_4 = 1, \dots$

2. $\left\{ \frac{i}{2i-1} \right\}_{i=1}^{\infty}$ $\left\{ 1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \dots \right\}$

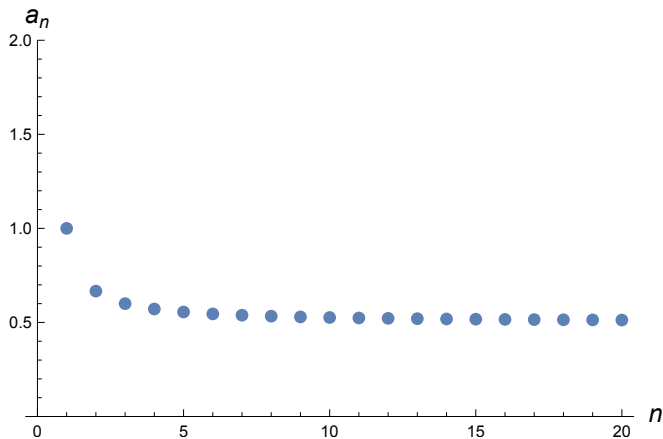
Graph of a Sequence (1 of 2)

$$a_i = (-1)^i$$



Graph of a Sequence (2 of 2)

$$\left\{ \frac{i}{2i-1} \right\}_{i=1}^{\infty}$$



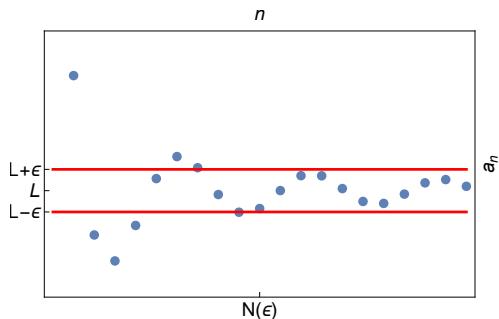
Convergence and Divergence

Definition

The sequence $\{a_n\}_{n=n_0}^{\infty}$ **converges** to L if given $\epsilon > 0$, there is an integer N for which

$$|a_n - L| < \epsilon, \quad \text{for every } n > N.$$

If there is no such number L , then we say the sequence **diverges**.



Example

Determine the limit of the sequence $\left\{ \frac{1}{2^n} \right\}_{n=1}^{\infty}$ and verify the limit using the definition of convergence.

Solution

We claim that $\lim_{n \rightarrow \infty} \frac{1}{2^n} = L = 0$. Let $\epsilon > 0$ then

$$\left| \frac{1}{2^n} - L \right| = \left| \frac{1}{2^n} - 0 \right| = \frac{1}{2^n} < \epsilon$$

$$2^n > \frac{1}{\epsilon}$$

$$n \ln 2 > \ln \frac{1}{\epsilon}$$

$$n > \frac{\ln \frac{1}{\epsilon}}{\ln 2}.$$

Examples

Determine the limit of the sequence $\left\{ \frac{1}{n} \cos n \right\}_{n=1}^{\infty}$ and verify the limit using the definition of convergence.

Solution

We claim that $\lim_{n \rightarrow \infty} \frac{1}{n} \cos n = L = 0$. Let $\epsilon > 0$ then

$$\left| \frac{1}{n} \cos n - L \right| = \left| \frac{1}{n} \cos n - 0 \right| = \frac{1}{n} |\cos n| \leq \frac{1}{n} < \epsilon$$
$$n > \frac{1}{\epsilon}.$$

Sequence of Ratios

Theorem

Suppose $a_n = r^n$ where r is a constant.

1. If $|r| < 1$ then $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} r^n = 0$.
2. If $|r| > 1$ then $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} |r^n| = \infty$.

Fibonacci Sequence

Definition

The **Fibonacci Sequence** $\{F_n\}_{n=1}^{\infty}$ is defined as

$$F_1 = 1$$

$$F_2 = 1$$

$$F_n = F_{n-2} + F_{n-1} \quad \text{for } n \geq 3.$$

Question: what is the limit of the Fibonacci sequence?

Question: what is the limit of $a_n = F_{n+1}/F_n$?

Solution

Answer: $\lim_{n \rightarrow \infty} F_n = \infty$.

Answer: let $a_n = F_{n+1}/F_n$ and suppose $\lim_{n \rightarrow \infty} a_n = L$.

$$a_n = \frac{F_{n+1}}{F_n} = \frac{F_{n-1} + F_n}{F_n} = \frac{1}{F_n/F_{n-1}} + 1 = \frac{1}{a_{n-1}} + 1$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{1}{a_{n-1}} + 1 \right)$$

$$L = \frac{1}{L} + 1$$

$$L^2 - L - 1 = 0$$

$$L = \frac{1 + \sqrt{5}}{2}$$

Properties

Theorem

Suppose that $\{a_n\}_{n=n_0}^{\infty}$ and $\{b_n\}_{n=n_0}^{\infty}$ both converge, then

1. $\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n,$
2. $\lim_{n \rightarrow \infty} (a_n - b_n) = \lim_{n \rightarrow \infty} a_n - \lim_{n \rightarrow \infty} b_n,$
3. $\lim_{n \rightarrow \infty} (a_n b_n) = \left(\lim_{n \rightarrow \infty} a_n \right) \left(\lim_{n \rightarrow \infty} b_n \right),$
4. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n},$ provided $\{b_n\}_{n=n_0}^{\infty} \neq 0.$

Examples

Find the limits of the following sequences, if they exist

$$1. a_n = \frac{2n + 1}{n}$$

$$2. b_n = \frac{5n^3 - 1}{2n^3 + 1}$$

$$3. c_n = \frac{n^2 + 1}{n^3 + 1}$$

$$4. d_n = (-1)^n \frac{2n^2 + n + 3}{3n^2 + 1}$$

$$a_n = \frac{2n + 1}{n}$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{2n + 1}{n} &= \lim_{n \rightarrow \infty} \frac{(2n + 1)^{\frac{1}{n}}}{(n)^{\frac{1}{n}}} \\ &= \lim_{n \rightarrow \infty} \frac{2 + \frac{1}{n}}{1} \\ &= \lim_{n \rightarrow \infty} \left(2 + \frac{1}{n} \right) \\ &= 2 + 0 = 2\end{aligned}$$

$$b_n = \frac{5n^3 - 1}{2n^3 + 1}$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{5n^3 - 1}{2n^3 + 1} &= \lim_{n \rightarrow \infty} \frac{(5n^3 - 1) \frac{1}{n^3}}{(2n^3 + 1) \frac{1}{n^3}} \\ &= \lim_{n \rightarrow \infty} \frac{5 - \frac{1}{n^3}}{2 + \frac{1}{n^3}} \\ &= \frac{5 - 0}{2 + 0} = \frac{5}{2}\end{aligned}$$

$$C_n = \frac{n^2 + 1}{n^3 + 1}$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{n^2 + 1}{n^3 + 1} &= \lim_{n \rightarrow \infty} \frac{(n^2 + 1) \frac{1}{n^3}}{(n^3 + 1) \frac{1}{n^3}} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n} + \frac{1}{n^3}}{1 + \frac{1}{n^3}} \\ &= \frac{0 + 0}{1 + 0} = 0\end{aligned}$$

$$d_n = (-1)^n \frac{2n^2 + n + 3}{3n^2 + 1}$$

Note that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2n^2 + n + 3}{3n^2 + 1} &= \lim_{n \rightarrow \infty} \frac{(2n^2 + n + 3) \frac{1}{n^2}}{(3n^2 + 1) \frac{1}{n^2}} \\ &= \lim_{n \rightarrow \infty} \frac{2 + \frac{1}{n} + \frac{3}{n^2}}{3 + \frac{1}{n^2}} \\ &= \frac{2 + 0 + 0}{3 + 0} = \frac{2}{3} \end{aligned}$$

however,

$$(-1)^n \frac{2n^2 + n + 3}{3n^2 + 1} \rightarrow \frac{-2}{3} \text{ for large odd numbers } n, \text{ and}$$

$$(-1)^n \frac{2n^2 + n + 3}{3n^2 + 1} \rightarrow \frac{2}{3} \text{ for large even numbers } n$$

so $\lim_{n \rightarrow \infty} d_n$ does not exist.

Example

Determine whether the following sequence converges or diverges.

$$a_n = \sqrt{n^2 + n} - n$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \sqrt{n^2 + n} - n &= \lim_{n \rightarrow \infty} \frac{(\sqrt{n^2 + n} - n)(\sqrt{n^2 + n} + n)}{\sqrt{n^2 + n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + n} + n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \\ &= \frac{1}{2}\end{aligned}$$

Relationship with Functions on $[1, \infty)$

Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with $a_n = f(n)$ where f is a function defined for every real number $x \geq 1$.

1. If $\lim_{x \rightarrow \infty} f(x) = L$, then $\lim_{n \rightarrow \infty} a_n = L$.
2. If $\lim_{x \rightarrow \infty} f(x) = \pm\infty$, then $\lim_{n \rightarrow \infty} a_n = \pm\infty$.

Remark: It is *not* true that if $\lim_{n \rightarrow \infty} a_n = L$ then $\lim_{x \rightarrow \infty} f(x) = L$.

Example

Consider $f(x) = \sin(\pi x)$.

Example

Determine whether the following sequence converges or diverges.

$$b_n = \frac{2n}{e^n}$$

Let $f(x) = \frac{2x}{e^x}$ for $x \geq 1$.

$$\begin{aligned}\lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{2x}{e^x} \text{ (indeterminate } \infty/\infty\text{)} \\ &= \lim_{x \rightarrow \infty} \frac{2}{e^x} \\ &= 0\end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} b_n = 0$.

Squeeze Theorem

Theorem

If $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ are sequences and if there exists an integer $n_1 \geq 1$ such that $a_n \leq b_n \leq c_n$ for all $n \geq n_1$, then if $\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} c_n$, then $\lim_{n \rightarrow \infty} b_n = L$.

Example

Determine if the following sequence converges.

$$b_n = \frac{\cos n\pi}{n^2}$$

Corollary

If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$.

Factorial

Definition

For any integer $n \geq 1$, the **factorial**, $n!$ is defined as the product of the first n positive integers,

$$n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n.$$

We define $0! = 1$.

Increasing and Decreasing Sequences

Definition

1. The sequence $\{a_n\}_{n=1}^{\infty}$ is **increasing** if

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots .$$

2. The sequence $\{a_n\}_{n=1}^{\infty}$ is **decreasing** if

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots .$$

If a sequence is either increasing or decreasing, it is called **monotonic**.

Examples

Determine whether the following sequences are increasing, decreasing, or neither.

1. $a_n = \frac{2n - 3}{3n + 4}$ (increasing)

2. $b_n = 3 + \frac{(-1)^n}{n}$ (neither)

3. $c_n = \frac{\sqrt{n}}{n + 2}$ (decreasing for $n \geq 2$)

Bounded Sequences

Definition

The sequence $\{a_n\}_{n=n_0}^{\infty}$ is **bounded** if there is a number $M > 0$ (called a **bound**) for which $|a_n| \leq M$, for all n .

Example

If the following sequences are bounded, find their bounds.

1. $a_n = \frac{2n-3}{3n+4}$ ($|a_n| \leq 2/3$)

2. $b_n = 3 + \frac{(-1)^n}{n}$ ($|b_n| \leq 7/2$)

3. $c_n = \frac{\sqrt{n}}{n+2}$ ($|c_n| \leq \sqrt{2}/4$)

Theorem

Every bounded, monotonic sequence converges.

Example (1 of 2)

Determine if the following sequence converges.

$$\left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}$$

1. Show that the sequence is decreasing.
2. Show that the sequence is bounded.
3. Apply the previous theorem.

Solution

Decreasing:

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n+1)n!}{(n+1)(n+1)^n} = \frac{n!}{(n+1)^n} < \frac{n!}{n^n} = a_n$$

Bounded:

$$\frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots(3)(2)(1)}{nnn\cdots nnn} = \frac{(n-1)(n-2)\cdots(3)(2)(1)}{nn\cdots nnn} < 1$$

Every bounded, monotonic sequence converges.

Example (2 of 2)

Investigate the convergence of the sequence defined by the recurrence relation,

$$a_1 = 2, \quad \text{and} \quad a_n = \frac{1}{2}(a_{n-1} + 6) \quad \text{for } n = 2, 3, 4, \dots$$

Solution

The first several terms of the sequence are $a_1 = 2$, $a_2 = 4$, $a_3 = 5$, $a_4 = 11/2$, \dots

- ▶ The sequence is bounded since if $a_{n-1} \leq 6$ then $a_n \leq 6$.
- ▶ The sequence is increasing since if $a_{n-1} < 6$ then $a_n \geq a_{n-1}$.
- ▶ The sequence must therefore converge to a limit L .

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{2}(a_{n-1} + 6)$$

$$L = \frac{1}{2}(L + 6)$$

$$L = 6$$

Homework

- ▶ Read Section 5.1
- ▶ Exercises: 1, 5, 9, \dots , 49, 53/handout