Volume: Method of Cylindrical Shells

MATH 211, *Calculus II*

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Fall 2021

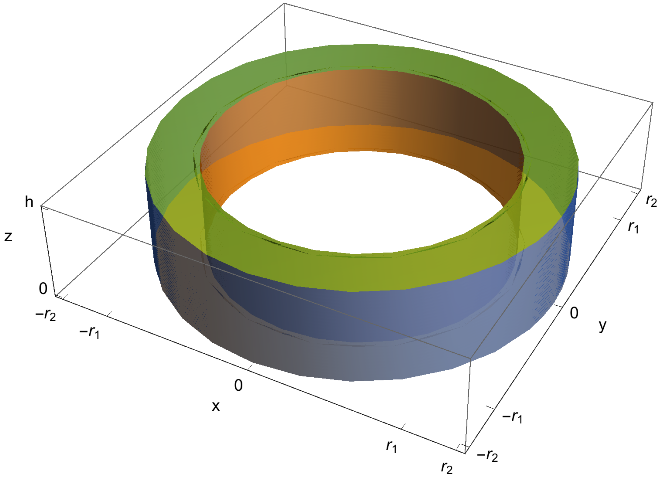
Background

Yesterday we used **disks** and **washers** to find the volume of a solid of revolution.

For some types of solid regions decomposing the volume into **cylindrical shells** may be more convenient.

Cylindrical Shells

Consider a cylindrical shell of **height** h, **inner radius** r_1, and **outer radius** r_2.



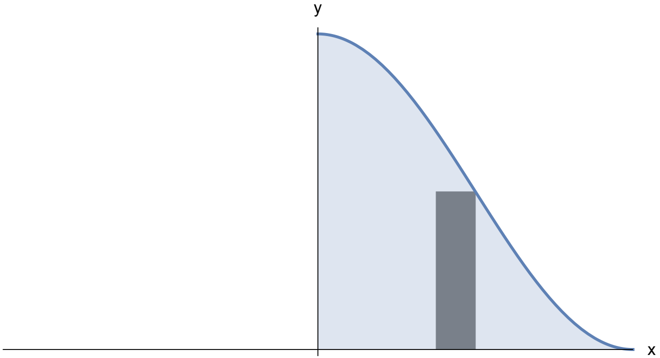
Volume of a Cylindrical Shell

$$\begin{aligned}\Delta V &= \pi r_2^2 h - \pi r_1^2 h = \pi h (r_2^2 - r_1^2) \\ &= \pi h (r_2 + r_1)(r_2 - r_1) = \pi h (r_1 + r_2) \Delta r \\ &= 2\pi h \frac{r_1 + r_2}{2} \Delta r = 2\pi r \, h \, \Delta r\end{aligned}$$

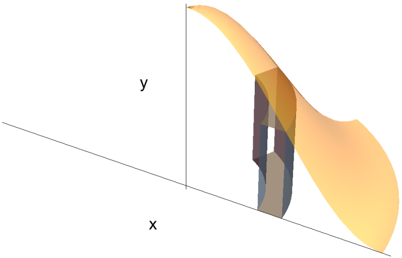
where $\Delta r = r_2 - r_1$ is the **thickness** of the shell and $r = \dfrac{r_1 + r_2}{2}$ is the **average radius** of the shell.

Solid of Revolution

Imagine that the shaded region and the rectangle pictured below are rotated completely around the y-axis.



Solid (Different Perspective)



Riemann Sum Approach

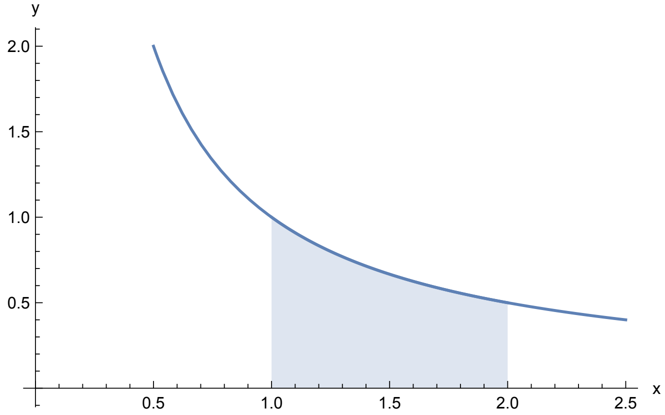
Suppose f is continuous on $[a, b]$ and $f(x) \geq 0$ for $a \leq x \leq b$ and the region bounded below the graph of $y = f(x)$, above the x-axis and between $x = a$ and $x = b$ is revolved around the y-axis.

Let $\Delta x = (b - a)/n$ for some $n \in \mathbb{N}$ and $x_i = a + \left(i - \frac{1}{2}\right) \Delta x$ for $i = 1, 2, \ldots, n$, then

$$
\begin{aligned}
V &\approx \sum_{i=1}^{n} 2\pi x_i f(x_i) \Delta x \\
V &= \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i f(x_i) \Delta x \\
V &= \int_a^b 2\pi \underbrace{x}_{\text{(radius)}} \underbrace{f(x)}_{\text{(height)}} \underbrace{dx}_{\text{(thickness)}}
\end{aligned}
$$

Example

Find the volume of the solid of revolution generated when the region bounded between $y = 1/x$, the x-axis, and the lines $x = 1$ and $x = 2$ is revolved around the y-axis.

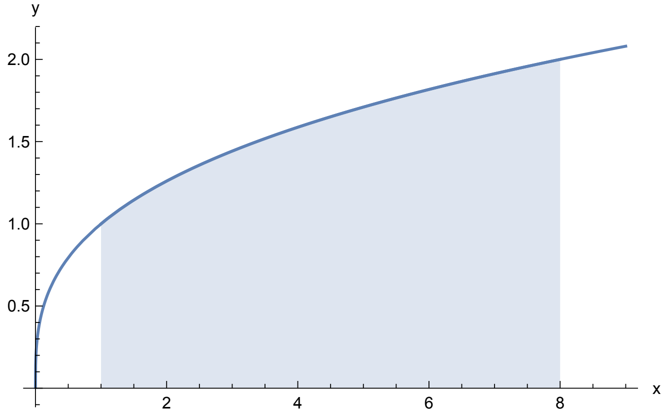


Solution

$$
\begin{aligned}
V &= 2\pi \int_1^2 x \left(\frac{1}{x}\right) dx \\
&= 2\pi \int_1^2 1 \, dx \\
&= \left[2\pi x\right]_{x=1}^{x=2} \\
&= 2\pi(2-1) \\
V &= 2\pi
\end{aligned}
$$

Example

Find the volume of the solid of revolution generated when the region bounded between $y = \sqrt[3]{x}$, the x-axis, and the lines $x = 1$ and $x = 8$ is revolved around the y-axis.

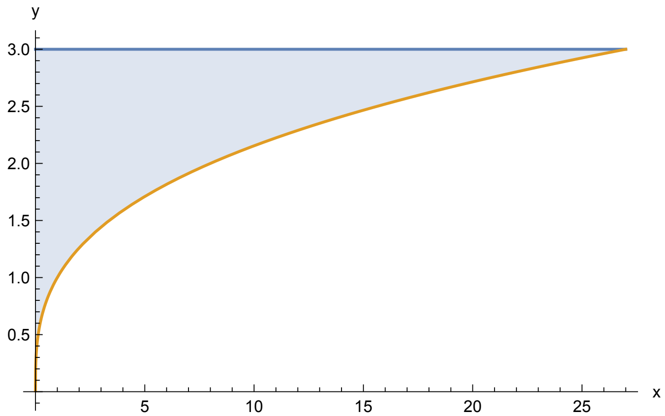


Solution

$$
\begin{aligned}
V &= 2\pi \int_1^8 x\left(\sqrt[3]{x}\right)\, dx \\
&= 2\pi \int_1^8 x^{4/3}\, dx \\
&= \left[2\pi\left(\frac{3}{7}\right)x^{7/3}\right]_{x=1}^{x=8} \\
&= \frac{6}{7}\pi\left(8^{7/3} - 1^{7/3}\right) \\
&= \frac{6}{7}\pi(128 - 1) \\
V &= \frac{762}{7}\pi
\end{aligned}
$$

Revolving Around the x-axis

Find the volume of the solid of revolution generated when the region bounded between the graph of $y^3 = x$, the y-axis, and the line $y = 3$ is revolved around the x-axis.

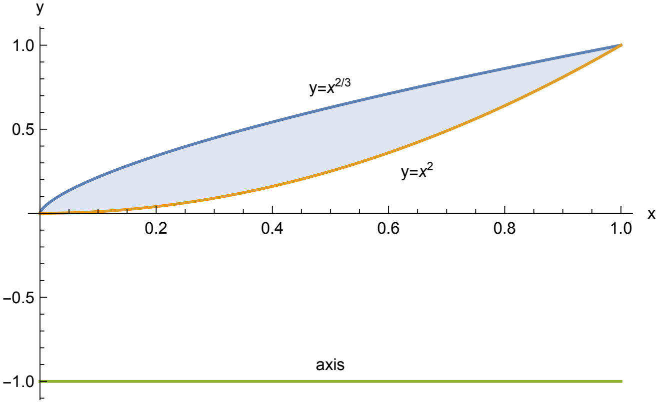


Solution

$$\begin{aligned}
V &= 2\pi \int_0^3 y\left(y^3\right)\, dy \\
&= 2\pi \int_0^3 y^4\, dy \\
&= \left[2\pi\left(\frac{1}{5}\right) y^5\right]_{y=0}^{y=3} \\
&= \frac{2}{5}\pi\left(3^5 - 0^5\right) \\
&= \frac{2}{5}\pi(243) \\
V &= \frac{486}{5}\pi
\end{aligned}$$

Revolving Around Other Lines

Find the volume of the solid of revolution generated when the region bounded between the graphs of $y = x^{2/3}$ and $y = x^2$ is revolved around the line $y = -1$.

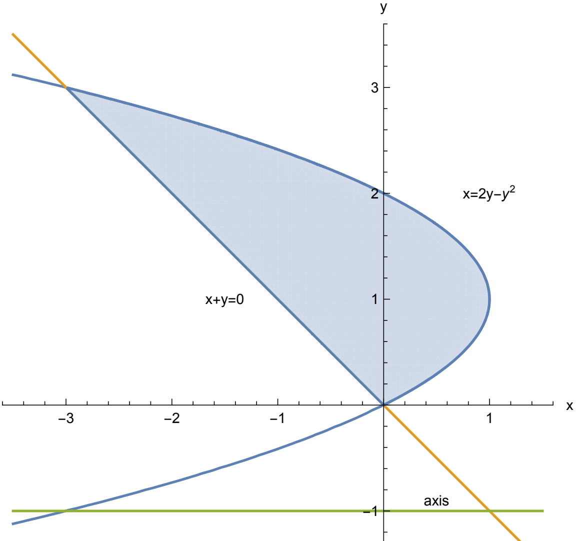


Solution

$$
\begin{aligned}
V &= 2\pi \int_0^1 (y+1)(y^{1/2} - y^{3/2})\, dy \\
&= 2\pi \int_0^1 (y^{3/2} - y^{5/2} + y^{1/2} - y^{3/2})\, dy \\
&= 2\pi \int_0^1 (y^{1/2} - y^{5/2})\, dy \\
&= \left[2\pi \left(\frac{2}{3} y^{3/2} - \frac{2}{7} y^{7/2} \right) \right]_{y=0}^{y=1} \\
&= 2\pi \left(\frac{2}{3} - \frac{2}{7} \right) \\
&= 2\pi \left(\frac{14-6}{21} \right) \\
V &= \frac{16\pi}{21}
\end{aligned}
$$

Example

Find the volume of the solid of revolution generated when the region bounded between the graphs of $x = 2y - y^2$ and $x + y = 0$ is revolved around the line $y = -1$.



Solution

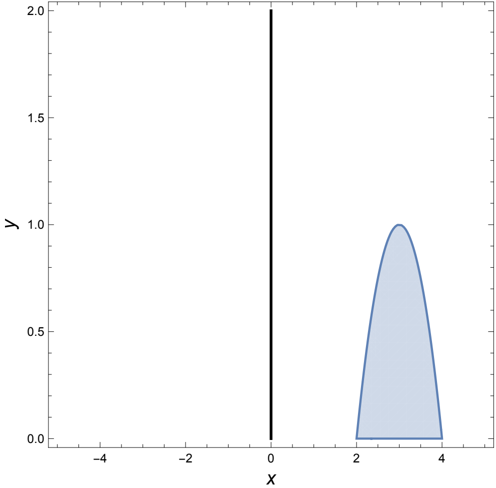
$$
\begin{aligned}
V &= 2\pi \int_0^3 (y+1)(2y - y^2 - [-y])\, dy \\
&= 2\pi \int_0^3 (y+1)(3y - y^2)\, dy \\
&= 2\pi \int_0^3 (2y^2 - y^3 + 3y)\, dy \\
&= \left[2\pi \left(\frac{2}{3}y^3 - \frac{1}{4}y^4 + \frac{3}{2}y^2 \right) \right]_{y=0}^{y=3} \\
&= 2\pi \left(18 - \frac{81}{4} + \frac{27}{2} \right) \\
&= 2\pi \left(\frac{72 - 81 + 54}{4} \right) \\
&= \frac{45\pi}{2}
\end{aligned}
$$

Summary

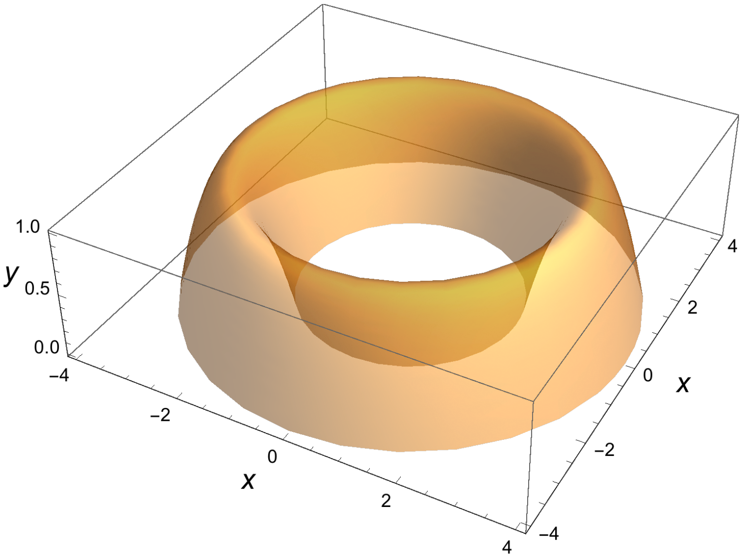
- Sketch the region to be revolved.
- Determine the variable of integration (x *vs.* y).
- Based on the variable of integration and the axis of revolution determine whether to use the method of disks or the method of shells.
- Label the inner and outer radii for washers or the radius and height for shells.
- Evaluate the appropriate integrals.

Example

The region bounded by $y = -x^2 + 6x - 8$ and $y = 0$ is rotated about the y-axis. Find the volume of the resulting solid by any method.



Solid of Revolution

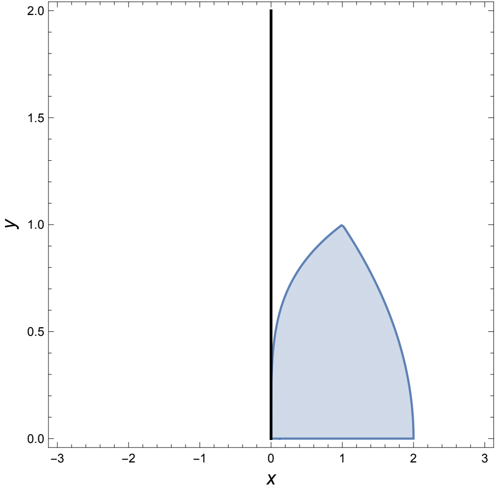


Volume: Method of Shells

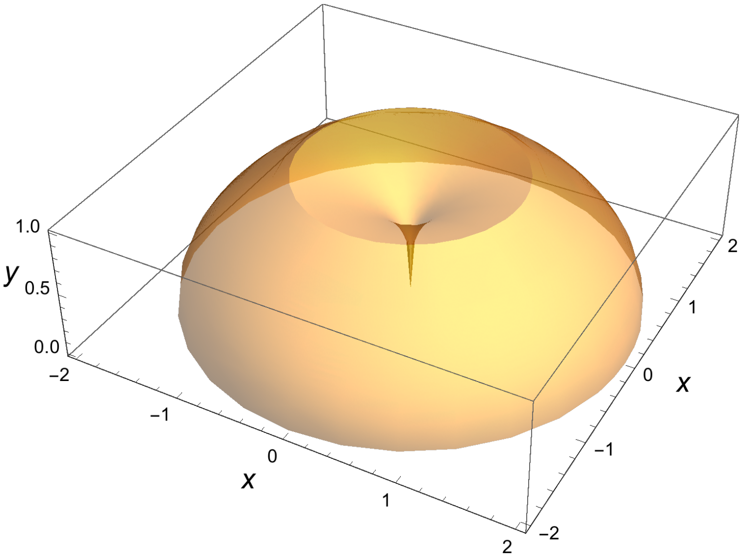
$$
\begin{aligned}
V &= 2\pi \int_2^4 x(-x^2 + 6x - 8)\,dx \\
&= 2\pi \int_2^4 (-x^3 + 6x^2 - 8x)\,dx \\
&= 2\pi \left[\frac{-x^4}{4} + 2x^3 - 4x^2 \right]_{x=2}^{x=4} \\
&= 2\pi \left(\frac{-(4^4)}{4} + 2(4)^3 - 4(4)^2 \right) - 2\pi \left(\frac{-(2^4)}{4} + 2(2)^3 - 4(2)^2 \right) \\
&= 8\pi
\end{aligned}
$$

Example

The region bounded by $x = 2 - y^2$ and $y = \sqrt[4]{x}$ is rotated about the line y-axis. Find the volume of the resulting solid by any method.



Solid of Revolution



Volume: Method of Disks

$$
\begin{aligned}
V &= \pi \int_0^1 (2 - y^2)^2 - (y^4)^2 \, dy \\
&= \pi \int_0^1 (4 - 4y^2 + y^4 - y^8) \, dy \\
&= \pi \left[4y - \frac{4y^3}{3} + \frac{y^5}{5} - \frac{y^9}{9} \right]_{y=0}^{y=1} \\
&= \pi \left(4 - \frac{4}{3} + \frac{1}{5} - \frac{1}{9} \right) \\
&= \frac{124\pi}{45}
\end{aligned}
$$

Homework

- Read Section 2.3
- Exercises: 121–149 odd/handout