Applications of Taylor Series

MATH 211, *Calculus II*

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Introduction (1 of 2)

The two important results we have learned regarding Taylor polynomials and Taylor series are:

Theorem (Taylor's Theorem)

Suppose that f *has* $n+1$ *derivatives on the interval* $(c-r, c+r)$ *for some* $r > 0$*. Then for* $x \in (c-r, c+r)$*,* $f(x) \approx P_n(x)$ *and the error in using* $P_n(x)$ *to approximate* $f(x)$ *is* $R_n(x)$ *and*

$$f(x) = P_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

for some z *between* x *and* c*.*

Introduction (2 of 2)

and:

Theorem

If $f(x)$ has derivatives of all orders in the interval $(c - r, c + r)$ for some $r > 0$ and if $\lim_{n \to \infty} R_n(x) = 0$ for all $x \in (c - r, c + r)$, then the Taylor series for f expanded about $x = c$ converges to $f(x)$, i.e.,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k.$$

Example

- Estimate $\ln(0.95)$ using a Taylor polynomial of degree 5.
- What is the maximum error in the approximation?
- What is the actual error?
- How many terms are needed to estimate $\ln(0.95)$ to an accuracy of 10^{-16}?

Solution (1 of 3)

Estimate $\ln(0.95)$ using a Taylor polynomial of degree 5.

If $f(x) = \ln(1-x)$ then $f(0.05) = \ln(0.95)$. Find the Taylor polynomial representation for $f(x)$.

k	$f^{(k)}(x)$	$f^{(k)}(0)$	$\frac{f^{(k)}(0)}{k!}$
0	$\ln(1-x)$	0	0
1	$-(1-x)^{-1}$	-1	-1
2	$-(1-x)^{-2}$	-1	$-1/2$
3	$-2(1-x)^{-3}$	-2	$-1/3$
4	$-6(1-x)^{-4}$	-6	$-1/4$
5	$-24(1-x)^{-5}$	-24	$-1/5$

$$\begin{aligned}
P_5(x) &= 0 + (-1)x + \left(-\frac{1}{2}\right)x^2 + \left(-\frac{1}{3}\right)x^3 + \left(-\frac{1}{4}\right)x^4 + \left(-\frac{1}{5}\right)x^5 \\
&= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 \\
P_5(0.05) &\approx -0.0512933
\end{aligned}$$

Solution (2 of 3)

What is the maximum error in the approximation?

Using the Taylor remainder we have

$$|R_5(0.05)| = \left| \frac{f^{(6)}(z)}{6!}(0.05)^6 \right| = \left| \frac{-1}{6(1-z)^6}(0.05)^6 \right|$$

for some $0 \le z \le 0.05$.

$$|R_5(0.05)| \le \frac{1}{6(1-0.05)^6}(0.05)^6 \approx 3.54 \times 10^{-9}$$

What is the actual error?

$$|\ln 0.95 - P_5(0.05)| = 2.27 \times 10^{-9}$$

Solution (3 of 3)

How many terms are needed to estimate ln(0.95) to an accuracy of 10^{-16}?

n	$\lvert R_n(0.05)\rvert \leq \dfrac{1}{(n+1)(1-0.05)^{n+1}}(0.05)^{n+1}$
6	1.59818×10^{-10}
7	7.36006×10^{-12}
8	3.44330×10^{-13}
9	1.63104×10^{-14}
10	7.80401×10^{-16}
11	3.76509×10^{-17}
12	1.82919×10^{-18}

If $n \geq 11$ then $\lvert P_n(0.05) - \ln(0.95)\rvert \leq 10^{-16}$.

Evaluating Limits

Use a Taylor series to help evaluate the limit

$$\lim_{x \to 0} \frac{\tan^{-1} x - x}{x^3}.$$

Solution

We have showed that $\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$.

$$
\begin{aligned}
\frac{\tan^{-1} x - x}{x^3} &= \frac{\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} - x}{x^3} \\
&= \frac{\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots\right) - x}{x^3} \\
&= \frac{-\frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots}{x^3} \\
&= -\frac{1}{3} + \frac{1}{5}x^2 - \cdots \\
\lim_{x \to 0} \frac{\tan^{-1} x - x}{x^3} &= \lim_{x \to 0} \left(-\frac{1}{3} + \frac{1}{5}x^2 - \cdots\right) = -\frac{1}{3}
\end{aligned}
$$

Evaluating Limits

Use a Taylor series to help evaluate the limit

$$\lim_{x \to 0} \frac{\cos(x^2) - 1}{\ln(1 + x) - x}.$$

Solution (1 of 2)

Using the common Taylor series:

$$\begin{aligned}
\cos(x^2) - 1 &= -1 + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}(x^2)^{2k} \\
&= -\frac{1}{2!}x^4 + \frac{1}{4!}x^8 - \frac{1}{6!}x^{12} + \cdots \\
\ln(1+x) - x &= -x + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}x^k \\
&= -\frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots
\end{aligned}$$

Solution (2 of 2)

$$
\begin{aligned}
\lim_{x\to 0} \frac{\cos(x^2)-1}{\ln(1+x)-x} &= \lim_{x\to 0} \frac{-\frac{1}{2!}x^4+\frac{1}{4!}x^8-\frac{1}{6!}x^{12}+\cdots}{-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots} \\
&= \lim_{x\to 0} \frac{-\frac{1}{2!}x^2+\frac{1}{4!}x^6-\frac{1}{6!}x^{10}+\cdots}{-\frac{1}{2}+\frac{1}{3}x-\frac{1}{4}x^2+\cdots} \\
&= \frac{0}{-1/2} = 0
\end{aligned}
$$

Estimating Definite Integrals (1 of 3)

Use a Taylor polynomial of degree 8 to estimate

$$\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \cos(x^2)\, dx.$$

- What is the maximum theoretical error in the estimation?
- What is the actual error assuming that the exact value is $\sqrt{2\pi}C(\sqrt{2}) \approx 1.325734627$?

Estimating Definite Integrals (2 of 3)

Recall that $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$.

$$\begin{aligned}
\cos(x^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k} = 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8 - \frac{1}{720}x^{12} + \cdots \\
P_8(x) &= 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8 \\
\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \cos(x^2)\,dx &\approx \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \left(1 - \frac{1}{2}x^4 + \frac{1}{24}x^8\right)\,dx \\
&= \left[x - \frac{1}{10}x^5 + \frac{1}{216}x^9\right]_{x=-\sqrt{\pi}}^{x=\sqrt{\pi}} \\
&= 2\sqrt{\pi} - \frac{1}{5}\pi^{5/2} + \frac{1}{108}\pi^{9/2} \approx 1.64486
\end{aligned}$$

Estimating Definite Integrals (3 of 3)

Since the series for the definite integral is an alterating series, the Alternating Series Estimation Theorem states

$$\begin{aligned} |R_8| &\leq \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{1}{720} x^{12}\, dx \\ &= \frac{1}{4680} \pi^{13/2} \approx 0.364106 \end{aligned}$$

Actual error:

$$\left| \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \left[\cos(x^2) - P_8(x) \right]\, dx \right| \approx |1.32573 - 1.64486| \approx 0.31913$$

Estimating Functions

- Use a Taylor series to estimate the function:
  $f(x) = \int_0^x t\, e^{-t^3}\, dt$.
- Find an error formula for the estimation of this function.
- Estimate the error in calculating $f(0.4)$.

Solution (1 of 3)

$$\begin{aligned}
t\,e^{-t^3} &= t\sum_{k=0}^{\infty}\frac{(-t^3)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k t^{3k+1}}{k!} \\
f(x) &= \int_0^x \sum_{k=0}^{\infty}\frac{(-1)^k t^{3k+1}}{k!}\,dt = \sum_{k=0}^{\infty}\int_0^x \frac{(-1)^k t^{3k+1}}{k!}\,dt \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k x^{3k+2}}{(3k+2)k!} \\
&= \frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{16} - \frac{x^{11}}{66} + \cdots
\end{aligned}$$

Solution (2 of 3)

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+2}}{(3k+2)k!} \\
P_n(x) &= \sum_{k=0}^{n} \frac{(-1)^k x^{3k+2}}{(3k+2)k!} \\
P_3(x) &= \frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{16} - \frac{x^{11}}{66}
\end{aligned}
$$

Since the Taylor series for $f(x)$ is an alternating series then the Alternating Series Estimation Theorem states

$$|R_3(x)| \leq \frac{x^{3(4)+2}}{(3(4)+2)4!} = \frac{x^{14}}{336}.$$

Solution (3 of 3)

Estimation of $f(0.4)$ and $R_3(0.4)$:

$$
\begin{aligned}
f(0.4) &\approx P_3(0.4) \\
&= \frac{(0.4)^2}{2} - \frac{(0.4)^5}{5} + \frac{(0.4)^8}{16} - \frac{(0.4)^{11}}{66} \approx 0.0779923 \\
|R_3(0.4)| &\leq \frac{(0.4)^{14}}{336} \\
&\approx 7.98915 \times 10^{-9}
\end{aligned}
$$

Binomial Series

If n is any real number than for $|x| < 1$,

$$\begin{aligned} (1+x)^n &= \sum_{k=0}^{\infty} \binom{n}{k} x^k \\ &= 1 + n\,x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \end{aligned}$$

Application to Physics

In Einstein's theory of special relativity the mass of an object moving with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the object at rest and c is the speed of light. The kinetic energy of an object is the difference between its total energy and its energy at rest,

$$K = m\,c^2 - m_0 c^2.$$

- Show that when v is small compared to c then K given above is approximately $\frac{1}{2} m_0 v^2$ (the expression for kinetic energy given by classical Newtonian physics).
- Estimate the difference in the classical and relativistic expressions for K when $|v| \leq 100$ m/s.

Solution (1 of 3)

$$K = m\,c^2 - m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2 = m_0 c^2 \left[\left(1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right]$$

Use the Maclaurin series for $(1 + x)^{-1/2}$:

$$\begin{aligned}
(1+x)^{-1/2} &= 1 - \frac{1}{2}x + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}x^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}x^3 + \cdots \\
&= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots
\end{aligned}$$

Solution (2 of 3)

$$\begin{aligned}
K &= m_0 c^2 \left[\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1 \right] \\
&= m_0 c^2 \left[\left(1 - \frac{1}{2}\left(\frac{-v^2}{c^2}\right) + \frac{3}{8}\left(\frac{-v^2}{c^2}\right)^2 - \frac{5}{16}\left(\frac{-v^2}{c^2}\right)^3 + \cdots \right) - 1 \right] \\
&= m_0 c^2 \left[\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots \right) - 1 \right] \\
&= m_0 c^2 \left[\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots \right] \\
&= \frac{1}{2} m_0 v^2 \left[1 + \frac{3}{4}\frac{v^2}{c^2} + \frac{5}{8}\frac{v^4}{c^4} + \cdots \right] \\
&\approx \frac{1}{2} m_0 v^2
\end{aligned}$$

Solution (3 of 3)

Let $f(x) = m_0 c^2((1+x)^{-1/2} - 1)$, then

$$\begin{aligned} f''(x) &= \frac{3}{4} m_0 c^2 (1+x)^{-5/2} \\ f''\left(\frac{-v^2}{c^2}\right) &= \frac{3}{4} m_0 c^2 (1 - v^2/c^2)^{-5/2} \\ \left| f''\left(\frac{-v^2}{c^2}\right) \right| &\leq \frac{3 m_0 c^2}{4(1 - 100^2/c^2)^{5/2}} \end{aligned}$$

when $|v| \leq 100$ m/s.

Taylor's Inequality states the error in the approximation is $R_1(x) = \frac{f''(z)}{2!} x^2$. Thus

$$|R_1(-v^2/c^2)| \leq (4.17 \times 10^{-10}) m_0$$

when $|v| \leq 100$ m/s.

Homework

- Read Section 6.4
- Exercises: 175, 179, 195, 199, 213, 217, 227/handout