Taylor and Maclaurin Series

MATH 211, *Calculus II*

J. Robert Buchanan

Department of Mathematics

Fall 2021

Background

- We have seen that some power series converge.
  - When they do, we can think of them as functions, say $f(x)$.
  - The derivatives and antiderivatives of $f(x)$ are power series too.
- Suppose we start with a function, $f(x)$. Is there a power series representation for $f(x)$?
- Which functions have power series representations?
- How do we find the power series representation?

Differentiating a Power Series

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c) + \cdots \\
f(c) &= a_0 \\
f'(x) &= \sum_{k=1}^{\infty} k\, a_k(x-c)^{k-1} = a_1 + 2a_2(x-c) + \cdots \\
f'(c) &= a_1 \\
f''(x) &= \sum_{k=2}^{\infty} k(k-1)a_k(x-c)^{k-2} = 2a_2 + 6a_3(x-c) + \cdots \\
f''(c) &= 2a_2 \\
&\vdots
\end{aligned}$$

Observation: in general $f^{(k)}(c) = k!\, a_k \iff a_k = \dfrac{f^{(k)}(c)}{k!}$.

Summary

If $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges with radius of convergence $r > 0$ to a function $f(x)$, then

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Big Question

Question: suppose $f(x)$ is infinitely differentiable, *i.e.*, $f^{(k)}(x)$ exists for all $k = 1, 2, \ldots$, then

- does $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ converge?
- is its radius of convergence positive?
- does it converge to $f(x)$?

If the answers are all yes, we say $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ is the **Taylor series** expansion of $f(x)$ about $x = c$.

Examples (1 of 3)

Verify that $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is the Taylor series expansion about $c = 0$ of e^x.

- Let $f(x) = e^x$, then $f^{(k)}(x) = e^x$ for all $k = 0, 1, \ldots$.
- If $c = 0$ then $f^{(k)}(c) = e^c = e^0 = 1$ for all $k = 0, 1, \ldots$.
- Consequently $\frac{f^{(k)}(c)}{k!} = \frac{1}{k!}$ for all $k = 0, 1, \ldots$.
- According to the Ratio Test, the power series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges absolutely for $-\infty < x < \infty$, therefore

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ for all } x.$$

Examples (2 of 3)

Find Taylor series expansions for the following functions.

1. $e^{3x} \quad = \sum_{k=0}^{\infty} \frac{(3x)^k}{k!} = \sum_{k=0}^{\infty} \frac{3^k x^k}{k!}$
2. $e^{-x} \quad = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$
3. $2xe^{-3x} \quad = 2x \sum_{k=0}^{\infty} \frac{(-3x)^k}{k!} = \sum_{k=0}^{\infty} \frac{2(-3)^k x^{k+1}}{k!}$

Examples (3 of 3)

Find an infinite series of positive terms whose sum is e.

Since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for $-\infty < x < \infty$ then

$$e^1 = \sum_{k=0}^{\infty} \frac{1^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Terminology

Definition

A Taylor series for which $c = 0$, *i.e.*, $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ is called a **Maclaurin series.**

Definition

The nth partial sum of a Taylor series is a polynomial of degree n

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \\ &= f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \end{aligned}$$

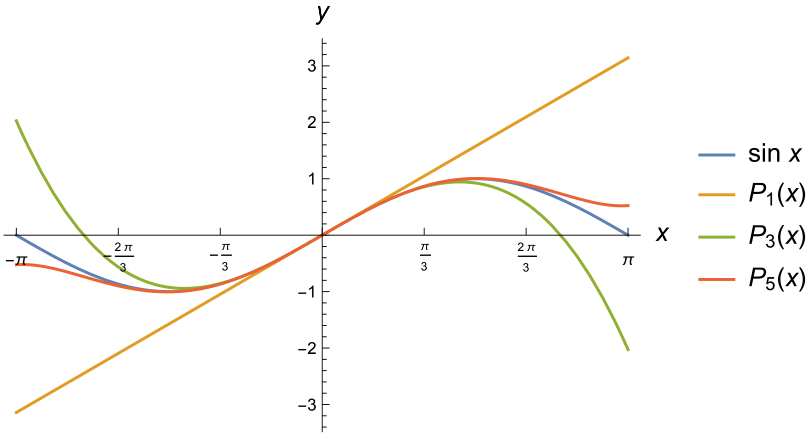
called the **Taylor polynomial of degree** n for f expanded about $x = c$.

Example

Find the Taylor polynomials centered at $c = 0$ of degree 1, 3, and 5 for $f(x) = \sin x$.

$$
\begin{aligned}
P_1(x) &= x \\
P_3(x) &= x - \frac{x}{3!} \\
P_5(x) &= x - \frac{x}{3!} + \frac{x^5}{5!}
\end{aligned}
$$

Illustration



Taylor's Theorem

Theorem (Taylor's Theorem)

Suppose that f has $n+1$ derivatives on the interval $(c-r, c+r)$ for some $r>0$. Then for $x \in (c-r, c+r)$, $f(x) \approx P_n(x)$ and the error in using $P_n(x)$ to approximate $f(x)$ is

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

for some z between x and c.

Remarks:

- $R_n(x)$ is called the n**th Taylor remainder**.
- $f(x) = P_n(x) + R_n(x)$.

Proof (1 of 4)

Define $g(t) = f(x) - R_n(x)\dfrac{(x-t)^{n+1}}{(x-c)^{n+1}} - \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$.

Verify that $g(x) = 0$ and that $g(c) = 0$.

$$
\begin{aligned}
g(x) &= f(x) - R_n(x)\frac{(x-x)^{n+1}}{(x-c)^{n+1}} - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(x-x)^k \\
&= f(x) - R_n(x)\frac{(0)^{n+1}}{(x-c)^{n+1}} - \frac{f^{(0)}(x)}{0!} - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}(0)^k \\
&= f(x) - f(x) = 0 \\
g(c) &= f(x) - R_n(x)\frac{(x-c)^{n+1}}{(x-c)^{n+1}} - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \\
&= f(x) - R_n(x) - P_n(x) = 0
\end{aligned}
$$

Proof (2 of 4)

Since $g(x) = 0 = g(c)$, $g(t)$ is continuous on the closed interval from x to c, and $g(t)$ is differentiable on the open interval from x to c, then by Rolle's Theorem there is a number z between x and c for which $g'(z) = 0$.

Differentiate $g(t)$ with respect to t and set the derivative equal to zero.

Proof (3 of 4)

$$
\begin{aligned}
g(t) &= f(x) - R_n(x)\frac{(x-t)^{n+1}}{(x-c)^{n+1}} - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k \\
g'(t) &= -R_n(x)\frac{(-1)(n+1)(x-t)^n}{(x-c)^{n+1}} \\
&\quad - \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \sum_{k=1}^{n}(-1)k\,\frac{f^{(k)}(t)}{k!}(x-t)^{k-1} \\
&= (n+1)R_n(x)\frac{(x-t)^n}{(x-c)^{n+1}} \\
&\quad - \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} \\
&= (n+1)R_n(x)\frac{(x-t)^n}{(x-c)^{n+1}} \\
&\quad - \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!}(x-t)^k
\end{aligned}
$$

Proof (4 of 4)

$$\begin{aligned} g'(t) &= (n+1)R_n(x)\frac{(x-t)^n}{(x-c)^{n+1}} \\ &\quad -\sum_{k=0}^{n}\frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=0}^{n-1}\frac{f^{(k+1)}(t)}{k!}(x-t)^k \\ g'(t) &= (n+1)R_n(x)\frac{(x-t)^n}{(x-c)^{n+1}} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n \end{aligned}$$

For some z between x and c, $g'(z) = 0$, so

$$\begin{aligned} (n+1)R_n(x)\frac{(x-z)^n}{(x-c)^{n+1}} - \frac{f^{(n+1)}(z)}{n!}(x-z)^n &= 0 \\ (n+1)R_n(x)\frac{1}{(x-c)^{n+1}} &= \frac{f^{(n+1)}(z)}{n!} \\ R_n(x) &= \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}. \end{aligned}$$

Big Answer

Using Taylor's Theorem we can now answer the big questions posed at the beginning of this discussion.

Theorem

If $f(x)$ has derivatives of all orders in the interval $(c - r, c + r)$ for some $r > 0$ and if $\lim_{n \to \infty} R_n(x) = 0$ for all $x \in (c - r, c + r)$, then the Taylor series for f expanded about $x = c$ converges to $f(x)$, i.e.,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k.$$

Taylor's Inequality

Theorem (Taylor's Inequality)

If $\left|f^{(n+1)}(x)\right| \leq M$ *for* $|x - c| \leq r$*, then the remainder* $R_n(x)$ *of the Taylor series satisfies the inequality*

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - c|^{n+1} \text{ for } |x - c| \leq r.$$

Useful Result

The following theorem will frequently be of use when trying to prove that $\lim_{n\to\infty} R_n(x) = 0$ for a potential Taylor series.

Theorem

If x is a real number then $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$.

Verifying $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

- We have seen that if $f(x) = e^x$ then $a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!}$ and the power series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges (by the Ratio Test) for all x.
- An important detail is to show that it converges to e^x.
- Let $R_n = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} = \frac{e^z}{(n+1)!} x^{n+1}$ where z is between 0 and x.
- If $|x| \leq d$ then

$$
\begin{aligned}
|R_n(x)| &= \left| \frac{e^z}{(n+1)!} x^{n+1} \right| \leq \frac{e^d}{(n+1)!} |d|^{n+1} \\
\lim_{n\to\infty} |R_n(x)| &\leq e^d \lim_{n\to\infty} \frac{|d|^{n+1}}{(n+1)!} = 0.
\end{aligned}
$$

Example

Find the Maclaurin series expansion for $\cos x$.

Solution (1 of 2)

Let $f(x) = \cos x$, then

$$
\begin{aligned}
f(0) &= \cos 0 = 1 \\
f'(0) &= -\sin 0 = 0 \\
f''(0) &= -\cos 0 = -1 \\
f'''(0) &= \sin 0 = 0 \\
f^{(4)}(0) &= \cos 0 = 1 \\
f^{(5)}(0) &= -\sin 0 = 0 \\
&\vdots \\
f^{(n)}(0) &= \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n/2 \text{ is even,} \\ -1 & \text{if } n/2 \text{ is odd.} \end{cases}
\end{aligned}
$$

Solution (2 of 2)

$$
\begin{aligned}
\cos x &= P_{2n}(x) + R_{2n}(x) \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} + \frac{\frac{d^{2n+1}}{dx^{2n+1}} [\cos x]_{x=z}}{(2n+1)!} x^{2n+1} \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} + \frac{\pm \sin z}{(2n+1)!} x^{2n+1}
\end{aligned}
$$

Note that

$$
\begin{aligned}
|R_{2n}(x)| &= \left| \frac{\pm \sin z}{(2n+1)!} x^{2n+1} \right| \leq \frac{|x^{2n+1}|}{(2n+1)!} \\
\lim_{n\to\infty} |R_{2n}(x)| &\leq \lim_{n\to\infty} \frac{|x^{2n+1}|}{(2n+1)!} = 0
\end{aligned}
$$

Thus $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$.

Error Estimates (1 of 4)

1. Find the Taylor series expansion for $\ln x$ about $x = 1$.
2. Estimate the error in using $P_4(x)$ to approximate $\ln 1.2$.
3. Compare this with the actual error.

Error Estimates (2 of 4)

Recall that $\frac{d}{dx}\left[\ln x\right] = \frac{1}{x}$.

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \sum_{k=0}^{\infty}(1-x)^k = \sum_{k=0}^{\infty}(-1)^k(x-1)^k$$

if $0 < x < 2$. Therefore

$$\begin{aligned}
\ln x &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}(x-1)^{k+1} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(x-1)^k \\
P_4(x) &= \sum_{k=1}^{4}\frac{(-1)^{k-1}}{k}(x-1)^k \\
&= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \\
\ln 1.2 &\approx P_4(1.2) \approx 0.182267
\end{aligned}$$

Error Estimates (3 of 4)

Since $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$ is an alternating series then

$$\begin{aligned} |\ln 1.2 - P_4(1.2)| &\leq \frac{1}{5}(1.2-1)^5 \\ &\approx 0.000064. \end{aligned}$$

The actual error is

$$|\ln 1.2 - P_4(1.2)| \approx 0.0000548901.$$

Error Estimates (4 of 4)

Find the smallest value of n so that the maximum theoretical error in using $P_n(x)$ to approximate e^x on the interval $[-\ln 10, \ln 10]$ is less than 10^{-6}.

We can calculate $R_n(x) = \frac{e^z}{(n+1)!}x^{n+1}$.

$$
\begin{aligned}
|R_n(x)| &= \left|\frac{e^z}{(n+1)!}x^{n+1}\right| \\
&\leq \left|\frac{e^{\ln 10}}{(n+1)!}(\ln 10)^{n+1}\right| = \frac{10}{(n+1)!}(\ln 10)^{n+1} < 10^{-6}
\end{aligned}
$$

when $n \geq 15$ (found by trial and error).

Binomial Series

Definition

If n is any real number and $|x| < 1$, then

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k = 1 + n\,x + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

where the expressions

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

are called **binomial coefficients.**

Remark: we are accustomed to using this formula when n is a positive integer, but it can be applied to any real number n.

Example

Find the Maclaurin series for $f(x) = \frac{1}{\sqrt{1-x}}$ and its radius of convergence.

Solution (1 of 3)

We can write $f(x) = (1 + (-x))^{-1/2}$ and use the Binomial Series formula with $n = -1/2$.

$$
\begin{aligned}
\binom{-1/2}{0} &= 1 \\
\binom{-1/2}{1} &= \frac{-1/2}{1!} = \frac{-1}{2} \\
\binom{-1/2}{2} &= \frac{(-1/2)(-3/2)}{2!} = \frac{(1)(3)}{2^2 2!} \\
\binom{-1/2}{3} &= \frac{(-1/2)(-3/2)(-5/2)}{3!} = \frac{-(1)(3)(5)}{2^3 3!} \\
&\vdots \\
\binom{-1/2}{k} &= \frac{(-1)^k (1)(3)(5)\cdots(2k-1)}{2^k k!}
\end{aligned}
$$

Solution (2 of 3)

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \binom{-1/2}{k} (-x)^k = \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} x^k \\
&= 1 - \frac{-1}{2} x + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!} x^2 - \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!} x^3 + \cdots \\
&= 1 + \frac{1}{2} x + \frac{(1)(3)}{2^2 2!} x^2 + \frac{(1)(3)(5)}{2^3 3!} x^3 + \cdots
\end{aligned}
$$

Solution (3 of 3)

Now check the interval of convergence. Using the Ratio Test,

$$\lim_{k\to\infty}\left|\frac{\frac{(-1)^{k+1}(1)(3)(5)\cdots(2k-1)(2k+1)}{2^{k+1}(k+1)!x^{k+1}}}{\frac{(-1)^k(1)(3)(5)\cdots(2k-1)x^k}{2^k k!}}\right|$$

$$= \lim_{k\to\infty}\frac{(1)(3)(5)\cdots(2k-1)(2k+1)}{2^{k+1}(k+1)!}\,\frac{2^k k!}{(1)(3)(5)\cdots(2k-1)}|x|$$

$$= \lim_{k\to\infty}\frac{2k+1}{2(k+1)}|x| = |x| < 1.$$

Thus if $-1 < x < 1$ the binomial series converges absolutely.

Common Taylor Series

Taylor series	Interval of Convergence
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$	$(-\infty, \infty)$
$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$	$(-\infty, \infty)$
$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	$(-\infty, \infty)$
$\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k}$	$(0, 2]$
$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$	$(-1, 1)$
$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$	$(-1, 1)$

Further Examples

Find the Maclaurin series for the following functions.

- $h(x) = x^3 \sin x$
- $f(x) = (2 + x)^5$
- $g(x) = \ln \dfrac{1 + x}{1 - x}$

$h(x) = x^3 \sin x$

$$
\begin{aligned}
\sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \\
x^3 \sin x &= x^3 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+4}
\end{aligned}
$$

$f(x) = (2+x)^5$

k	$f^{(k)}(x)$	$f^{(k)}(c)$	$\frac{f^{(k)}(c)}{k!}$
0	$(2+x)^5$	32	32
1	$5(2+x)^4$	80	80
2	$20(2+x)^3$	160	80
3	$60(2+x)^2$	240	60
4	$120(2+x)$	240	10
5	120	120	1
6	0	0	0

Thus

$$f(x) = (2+x)^5 = 32 + 80x + 80x^2 + 60x^3 + 10x^4 + x^5$$

$g(x) = \ln \frac{1+x}{1-x}$

Note that

$$\begin{aligned} g(x) &= \ln(1+x) - \ln(1-x) \\ g'(x) &= \frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2} = 2\sum_{k=0}^{\infty} x^{2k} \\ g(x) &= \int_0^x \sum_{k=0}^{\infty} t^{2k}\, dt = \sum_{k=0}^{\infty} \int_0^x t^{2k}\, dt = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} \end{aligned}$$

if $|x| < 1$.

Limits and Maclaurin Series

Suppose $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Both Maclaurin series have a positive radius of convergence and $f(0) = 0 = g(0)$.

- What does $f(0) = 0 = g(0)$ imply about the coefficients of these two infinite series?
- Find $\lim_{x \to 0} \dfrac{f(x)}{g(x)}$.
- Find $\lim_{x \to 0} \dfrac{\cos(x^2) - 1}{\ln(1 + x) - x}$.

Function without Taylor Series Expansion

Observation: Some functions have derivatives of all orders, but their Taylor remainders do not limit on 0, and thus there is no convergent Taylor series for these functions.

Example

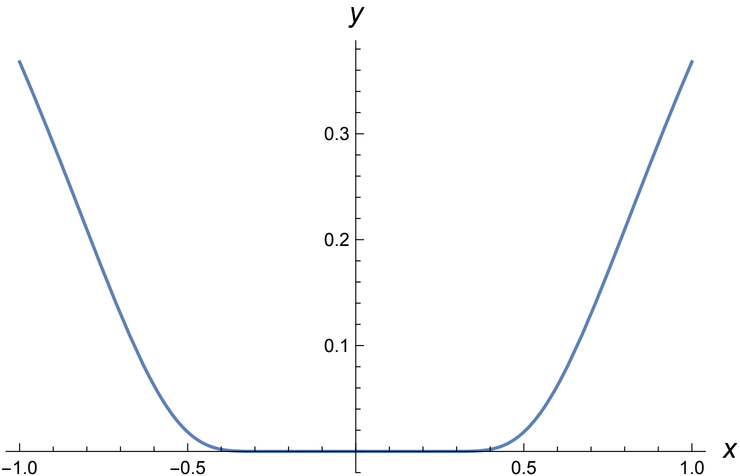
Consider $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

We can show that $f^{(k)}(0) = 0$ for all $k = 0, 1, 2, \ldots$.

However $f(x) \neq 0 = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$.

Illustration

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$



Homework

- Read Section 6.3
- Exercises: 117, 121, 125, . . . , 157/handout