Techniques of Trigonometric Integration

MATH 211, *Calculus II*

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Outline

Today's discussion will focus on evaluating integrals of the forms:

- $\int \sin^m x \cos^n x \, dx$
- $\int \tan^m x \sec^n x \, dx$
- $\int \cot^m x \csc^n x \, dx$

where $m = 0, 1, \ldots$ and $n = 0, 1, \ldots$.

Powers of Sine and Cosine (m odd)

$$\int \sin^m x \cos^n x \, dx$$

If m is odd, then $m - 1$ is even, *i.e.* $m - 1 = 2k$ and

$$\begin{aligned}
\int \sin^m x \cos^n x \, dx &= \int \sin^{m-1} x \cos^n x \sin x \, dx \\
&= \int \left(\sin^2 x\right)^k \cos^n x \sin x \, dx \\
&= \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x \, dx \\
&= -\int \left(1 - u^2\right)^k u^n \, du
\end{aligned}$$

where $u = \cos x$ and $du = -\sin x \, dx$.

Powers of Sine and Cosine (n odd)

$$\int \sin^m x \cos^n x \, dx$$

If n is odd, then $n-1$ is even, *i.e.* $n-1=2j$ and

$$\begin{aligned}
\int \sin^m x \cos^n x \, dx &= \int \sin^m x \cos^{n-1} x \cos x \, dx \\
&= \int \sin^m x \left(\cos^2 x\right)^j \cos x \, dx \\
&= \int \sin^m x \left(1-\sin^2 x\right)^j \cos x \, dx \\
&= \int u^m \left(1-u^2\right)^j \, du
\end{aligned}$$

where $u = \sin x$ and $du = \cos x \, dx$.

Examples

Evaluate the following indefinite integrals.

- $\int \sin^3 x \, dx$
- $\int \sin^2 x \cos^3 x \, dx$

$\int \sin^3 x \, dx$

$$\begin{aligned}\int \sin^3 x \, dx &= \int \sin^2 x \sin x \, dx \\ &= \int (1 - \cos^2 x) \sin x \, dx\end{aligned}$$

Let $u = \cos x$ and $du = -\sin x \, dx$ then

$$\begin{aligned}\int \sin^3 x \, dx &= -\int (1 - u^2) \, du = \int (u^2 - 1) \, du \\ &= \frac{1}{3} u^3 - u + C \\ &= \frac{1}{3} \cos^3 x - \cos x + C.\end{aligned}$$

Even Powers of Sine and Cosine

$$\int \sin^m x \cos^n x \, dx$$

If both *m* and *n* are even then we make use of a **half-angle identity**.

$$\begin{aligned} \sin^2 x &= \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x &= \frac{1}{2}(1 + \cos 2x) \end{aligned}$$

Examples

Evaluate the following indefinite integrals.

- $\int \sin^2 x \, dx$
- $\int \cos^4 x \, dx$
- $\int \sin^2 x \cos^2 x \, dx$

$\int \sin^2 x \, dx$

$$\begin{aligned}
\int \sin^2 x \, dx &= \frac{1}{2} \int (1 - \cos(2x)) \, dx \\
&= \frac{1}{2} \int 1 \, dx - \frac{1}{2} \int \cos(2x) \, dx \\
&= \frac{1}{2} x - \frac{1}{4} \sin(2x) + C
\end{aligned}$$

Products of Sines and Cosines

Occasionally we encounter integrals of the form:

$$\int \cos(ax)\cos(bx)\,dx$$
$$\int \cos(ax)\sin(bx)\,dx$$
$$\int \sin(ax)\sin(bx)\,dx$$

where a and b are constants. In particular these types of integrals are frequently find in applications such as Fourier series and partial differential equations.

Product-to-Sum Formulas

Recall,

$$
\begin{aligned}
\sin(ax+bx) &= \sin(ax)\cos(bx)+\cos(ax)\sin(bx)\\
\sin(ax-bx) &= \sin(ax)\cos(bx)-\cos(ax)\sin(bx)\\
\cos(ax+bx) &= \cos(ax)\cos(bx)-\sin(ax)\sin(bx)\\
\cos(ax-bx) &= \cos(ax)\cos(bx)+\sin(ax)\sin(bx).
\end{aligned}
$$

Add the first two equations and divide the sum by 2:

$$\sin(ax)\cos(bx)=\frac{1}{2}\left[\sin(a+b)x+\sin(a-b)x\right].$$

Add the last two equations and divide the sum by 2:

$$\cos(ax)\cos(bx)=\frac{1}{2}\left[\cos(a+b)x+\cos(a-b)x\right].$$

Subtract the 3rd equation from the 4th equation and divide by 2:

$$\sin(ax)\sin(bx)=\frac{1}{2}\left[\cos(a-b)x-\cos(a+b)x\right].$$

Examples

Use the product-to-sum formulas:

$$\begin{aligned}
\sin(ax)\cos(bx) &= \frac{1}{2}\left[\sin(a+b)x + \sin(a-b)x\right] \\
\cos(ax)\cos(bx) &= \frac{1}{2}\left[\cos(a+b)x + \cos(a-b)x\right] \\
\sin(ax)\sin(bx) &= \frac{1}{2}\left[\cos(a-b)x - \cos(a+b)x\right]
\end{aligned}$$

to evaluate the following integrals.

- $\int \sin(3x)\cos(5x)\,dx$
- $\int \sin(2x)\sin(4x)\,dx$

Solution

$$\begin{aligned}
\int \sin(3x)\cos(5x)\,dx &= \frac{1}{2}\int [\sin(3+5)x + \sin(3-5)x]\ dx \\
&= \frac{1}{2}\int [\sin(8x) - \sin(2x)]\ dx \\
&= \frac{-1}{16}\cos(8x) + \frac{1}{4}\cos(2x) + C \\
\int \sin(2x)\sin(4x)\,dx &= \frac{1}{2}\int [\cos(2-4)x - \cos(2+4)x]\ dx \\
&= \frac{1}{2}\int [\cos(2x) - \cos(6x)]\ dx \\
&= \frac{1}{4}\sin(2x) - \frac{1}{12}\sin(6x) + C
\end{aligned}$$

Powers of Tangent and Secant

$$\int \tan^m x \sec^n x \, dx$$

If m is odd then $m-1$ is even, *i.e.*, $m-1=2k$.

$$\begin{aligned}
\int \tan^m x \sec^n x \, dx &= \int \tan^{m-1} x \sec^{n-1} x \sec x \tan x \, dx \\
&= \int \left(\tan^2 x\right)^k \sec^{n-1} x \sec x \tan x \, dx \\
&= \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x \sec x \tan x \, dx \\
&= \int (u^2-1)^k u^{n-1} \, du
\end{aligned}$$

where $u = \sec x$ and $du = \sec x \tan x \, dx$.

Powers of Tangent and Secant

$$\int \tan^m x \sec^n x \, dx$$

If n is even then $n-2$ is also even, *i.e.*, $n-2=2j$.

$$\begin{aligned}
\int \tan^m x \sec^n x \, dx &= \int \tan^m x \sec^{n-2} x \sec^2 x \, dx \\
&= \int \tan^m x \left(\sec^2 x\right)^j \sec^2 x \, dx \\
&= \int \tan^m x \left(1+\tan^2 x\right)^j \sec^2 x \, dx \\
&= \int u^m (1+u^2)^j \, du
\end{aligned}$$

where $u = \tan x$ and $du = \sec^2 x \, dx$.

Examples

Evaluate the following indefinite integrals.

- $\int \tan^3 x \sec^3 x \, dx$
- $\int \tan^2 x \sec^4 x \, dx$

$\int \tan^3 x \sec^3 x \, dx$

$$\begin{aligned}\int \tan^3 x \sec^3 x \, dx &= \int \tan^2 x \sec^2 x (\sec x \tan x) \, dx \\ &= \int (\sec^2 x - 1) \sec^2 x (\sec x \tan x) \, dx\end{aligned}$$

Let $u = \sec x$ and $du = \sec x \tan x \, dx$ then

$$\begin{aligned}\int \tan^3 x \sec^3 x \, dx &= \int (u^2 - 1) u^2 \, du = \int (u^4 - u^2) \, du \\ &= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C \\ &= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C.\end{aligned}$$

Challenging Case

$$\int \tan^m x \sec^n x \, dx$$

If m is even and n is odd, replace $\tan^2 x$ by $\sec^2 x - 1$.
Integration by parts may be necessary.

Example

Evaluate the following indefinite integral.

$$\int \sec^3 x \, dx$$

$\int \sec^3 x \, dx$ (1 of 2)

$$\int \sec^3 x \, dx = \int \sec x \sec^2 x \, dx$$

Integrate by parts.

$$\begin{aligned} u &= \sec x & \qquad v &= \tan x \\ du &= \sec x \tan x \, dx & \qquad du &= \sec^2 x \, dx \end{aligned}$$

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \sec^3 x \tan x \, dx \\ &= \sec x \tan x - \int \sec^2 x \sec x \tan x \, dx \end{aligned}$$

Integrate by substitution with $u = \sec x$ and $du = \sec x \tan x \, dx$.

$\int \sec^3 x\,dx$ (2 of 2)

$$\begin{aligned}
\int \sec^3 x\,dx &= \sec x \tan x - \int \sec^2 x \sec x \tan x\,dx \\
&= \sec x \tan x - \int u^2\,du \\
&= \sec x \tan x - \frac{1}{3}u^3 + C \\
&= \sec x \tan x - \frac{1}{3}\sec^3 + C
\end{aligned}$$

Powers of Cotangent and Cosecant

$$\int \cot^m x \csc^n x \, dx$$

If m is odd then $m - 1$ is even, *i.e.*, $m - 1 = 2k$.

$$\begin{aligned}
\int \cot^m x \csc^n x \, dx &= \int \cot^{m-1} x \csc^{n-1} x \csc x \cot x \, dx \\
&= \int \left(\cot^2 x\right)^k \csc^{n-1} x \csc x \cot x \, dx \\
&= \int \left(\csc^2 x - 1\right)^k \csc^{n-1} x \csc x \cot x \, dx \\
&= -\int (u^2 - 1)^k u^{n-1} \, du
\end{aligned}$$

where $u = \csc x$ and $du = -\csc x \cot x \, dx$.

Powers of Cotangent and Cosecant

$$\int \cot^m x \csc^n x \, dx$$

If n is even then $n-2$ is also even, *i.e.*, $n-2=2j$.

$$\begin{aligned}
\int \cot^m x \csc^n x \, dx &= \int \cot^m x \csc^{n-2} x \csc^2 x \, dx \\
&= \int \cot^m x \left(\csc^2 x\right)^j \csc^2 x \, dx \\
&= \int \cot^m x \left(1+\cot^2 x\right)^j \csc^2 x \, dx \\
&= -\int u^m (1+u^2)^j \, du
\end{aligned}$$

where $u=\cot x$ and $du=-\csc^2 x \, dx$.

Examples

Evaluate the following indefinite integrals.

- $\int \cot^5 x \csc^3 x \, dx$
- $\int \cot^4 x \csc^4 x \, dx$

$\int \cot^5 x \csc^3 x \, dx$

$$\begin{aligned}\int \cot^5 x \csc^3 x \, dx &= \int \cot^4 x \csc^2 x \csc x \cot x \, dx \\ &= \int (\cot^2 x)^2 \csc^2 x \csc x \cot x \, dx \\ &= \int (\csc^2 x - 1)^2 \csc^2 x \csc x \cot x \, dx\end{aligned}$$

Integrate by substitution letting $u = \csc x$ and $-du = \csc x \cot x \, dx$.

$$\begin{aligned}\int \cot^5 x \csc^3 x \, dx &= -\int (u^2 - 1)^2 u^2 \, du \\ &= -\int (u^6 - 2u^4 + u^2) \, du \\ &= -\frac{1}{7}u^7 + \frac{2}{5}u^5 - \frac{1}{3}u^3 + C \\ &= -\frac{1}{7}\csc^7 x + \frac{2}{5}\csc^5 x - \frac{1}{3}\csc^3 x + C\end{aligned}$$

Challenging Case

$$\int \cot^m x \csc^n x \, dx$$

If m is even and n is odd, replace $\cot^2 x$ by $\csc^2 x - 1$.
Integration by parts may be necessary.

Example

Evaluate the following indefinite integral.

$$\int \cot^2 x \csc x \, dx$$

Homework

- Read Section 3.2.
- Exercises: 73, 77, 81, 85, . . . , 113, 119, 123/handout