Techniques of Trigonometric Substitution

MATH 211, *Calculus II*

J. Robert Buchanan

Department of Mathematics

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Overview

Remark: We often see expressions of the forms

$$\sqrt{a^2 - x^2}, \quad \sqrt{x^2 - a^2}, \quad \text{or} \quad \sqrt{a^2 + x^2}$$

inside integrands.

In each case there is a trigonometric function which we can substitute for x which may enable us to evaluate the integral.

Case: $\sqrt{a^2 - x^2}$

Suppose the integrand contains an expression of the form $\sqrt{a^2 - x^2}$ with $a > 0$, then we can substitute $x = a \sin\theta$.

$$\begin{aligned}
\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a\sin\theta)^2} \\
&= \sqrt{a^2 - a^2\sin^2\theta} \\
&= \sqrt{a^2(1 - \sin^2\theta)} \\
&= \sqrt{a^2\cos^2\theta} \\
&= a\cos\theta
\end{aligned}$$

if $-\pi/2 \leq \theta \leq \pi/2$. (Why?)

Example

Evaluate the following indefinite integral.

$$\int \frac{\sqrt{4 - x^2}}{x^2}\, dx$$

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx$$

Make the substitution

$$\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta. \end{aligned}$$

$$\begin{aligned} \int \frac{\sqrt{4-x^2}}{x^2}\,dx &= \int \frac{\sqrt{4-(2\sin\theta)^2}}{(2\sin\theta)^2}(2\cos\theta)\,d\theta \\ &= \int \frac{\sqrt{4-4\sin^2\theta}}{4\sin^2\theta}(2\cos\theta)\,d\theta \\ &= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \cot^2\theta\,d\theta \\ &= \int (\csc^2\theta - 1)\,d\theta \\ &= -\cot\theta - \theta + C = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\frac{x}{2} + C \end{aligned}$$

Case: $\sqrt{a^2 + x^2}$

Suppose the integrand contains an expression of the form $\sqrt{a^2 + x^2}$ with $a > 0$, then we can substitute $x = a \tan \theta$.

$$\begin{aligned} \sqrt{a^2 + x^2} &= \sqrt{a^2 + (a \tan \theta)^2} \\ &= \sqrt{a^2 + a^2 \tan^2 \theta} \\ &= \sqrt{a^2 (1 + \tan^2 \theta)} \\ &= \sqrt{a^2 \sec^2 \theta} \\ &= a \sec \theta \end{aligned}$$

if $-\pi/2 < \theta < \pi/2$. (Why?)

Example

Evaluate the following indefinite integral.

$$\int \frac{1}{\sqrt{9+x^2}}\, dx$$

$$\int \frac{1}{\sqrt{9+x^2}}\,dx$$

Make the substitution

$$\begin{aligned} x &= 3\tan\theta \\ dx &= 3\sec^2\theta\, d\theta. \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\sqrt{9+x^2}}\,dx &= \int \frac{1}{\sqrt{9+(3\tan\theta)^2}} 3\sec^2\theta\, d\theta \\ &= \int \frac{1}{\sqrt{9+9\tan^2\theta}} 3\sec^2\theta\, d\theta \\ &= \int \frac{1}{3\sec\theta} 3\sec^2\theta\, d\theta \\ &= \int \sec\theta\, d\theta \\ &= \ln|\sec\theta+\tan\theta| + C = \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C \end{aligned}$$

Case: $\sqrt{x^2 - a^2}$

Suppose the integrand contains an expression of the form $\sqrt{x^2 - a^2}$ with $a > 0$, then we can substitute $x = a \sec \theta$.

$$\begin{aligned} \sqrt{x^2 - a^2} &= \sqrt{(a \sec \theta)^2 - a^2} \\ &= \sqrt{a^2 \sec^2 \theta - a^2} \\ &= \sqrt{a^2 (\sec^2 \theta - 1)} \\ &= \sqrt{a^2 \tan^2 \theta} \\ &= a \tan \theta \end{aligned}$$

if $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$. (Why?)

Example

Evaluate the following indefinite integral.

$$\int \frac{\sqrt{x^2 - 4}}{x} \, dx$$

$$\int \frac{\sqrt{x^2-4}}{x}\,dx$$

Make the substitution

$$\begin{aligned} x &= 2\sec\theta \\ dx &= 2\sec\theta\tan\theta\,d\theta. \end{aligned}$$

$$\begin{aligned} \int \frac{\sqrt{x^2-4}}{x}\,dx &= \int \frac{\sqrt{(2\sec\theta)^2-4}}{2\sec\theta}(2\sec\theta\tan\theta)\,d\theta \\ &= \int \sqrt{4\sec^2\theta-4}(\tan\theta)\,d\theta \\ &= 2\int \tan^2\theta\,d\theta \\ &= 2\int (\sec^2\theta-1)\,d\theta \\ &= 2\tan\theta-2\theta+C = \sqrt{x^2-4}-2\cos^{-1}\frac{2}{x}+C \end{aligned}$$

Summary of Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$, $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Examples

Evaluate the following indefinite integrals.

- $\displaystyle\int \frac{1}{x\sqrt{x^2+16}}\,dx$
- $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$
- $\displaystyle\int \frac{1}{(x^2-9)^{3/2}}\,dx$

Putting Them All Together

Evaluate the following indefinite integrals.

- $$\int \frac{1}{x\sqrt{x^2+16}}\,dx = -\frac{1}{4}\ln\left|\frac{\sqrt{x^2+16}}{x}+\frac{4}{x}\right|+C$$
- $$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \frac{9}{2}\sin^{-1}\frac{x}{3}-\frac{x\sqrt{9-x^2}}{2}+C$$
- $$\int \frac{1}{(x^2-9)^{3/2}}\,dx = -\frac{x}{9\sqrt{x^2-9}}+C$$

Homework

- Read Section 3.3.
- Exercises: 135, 139, 143, 147, ..., 163/handout