Physical Applications

MATH 211, *Calculus II*

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Mass and Weight

mass: quantity of matter (units: kg or g (metric) or slugs (English))

gravity: gravitational acceleration (notation, g)

- Metric units $g = 9.8$ m/s^2 or $g = 980$ cm/s^2.
- English units, $g = 32$ ft/s^2.

weight: a force related to mass through **Newton's Second Law**

$$\begin{aligned} \text{weight} &= \text{(mass)(gravitational acceleration)} \\ \text{1 pound} &= \text{(1 slug)(32 ft/s}^2) \\ \text{1 Newton} &= \text{(1 kg)(9.8 m/s}^2) \end{aligned}$$

Work

Work is what is accomplished by moving a **force** through some **distance.**

If the force F is constant and is moved in a straight line a distance d, the work is $W = F\,d$.

Units of work:

force	distance	work
pound (lb)	foot (ft)	foot-pound (ft-lb)
	inch (in)	inch-pound (in-lb)
Newton (N)	meter (m)	Newton-meter (N-m)

Note: 1 Newton-meter $=$ 1 Joule $=$ 1 J.

Variable Forces

Suppose the force f depends on position x, *i.e.*, the force $f(x)$ is moved in a straight line from $x = a$ to $x = b$.

Let $n \in \mathbb{N}$ and define $\Delta x = (b - a)/n$ and $x_i = a + i\Delta x$ for $i = 1, 2, \ldots, n$.

The work done moving the force from x_{k-1} to x_k is approximately $\Delta W_k = f(x_k)\Delta x$.

A Riemann sum for the work is then

$$\begin{aligned} W &\approx \sum_{k=1}^{n} f(x_k)\Delta x \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \\ W &= \int_a^b f(x)\, dx. \end{aligned}$$

Hooke's Law Spring

Hooke's Law: the force required to stretch or compress a spring beyond its natural length is $f(x) = k\,x$ where k is called the **spring constant**.

Example

A force of 10 lb is required to stretch a spring from its natural length of 7 in to 7.5 in. Find the work done in stretching the spring. How much work is done stretching the spring from 7.5 in to 9 in?

Solution

- Since $10 = k(7.5 - 7)$ then the spring constant $k = 20$.
- The work done to stretch the spring the initial $1/2$ inch is

$$W = \int_0^{1/2} 20x\,dx = \left[10x^2\right]_{x=0}^{x=1/2} = \frac{5}{2} \text{ in-lb.}$$

- The work done to stretch the spring the next 1.5 inches is

$$W = \int_{1/2}^{2} 20x\,dx = \left[10x^2\right]_{x=1/2}^{x=2} = \frac{75}{2} \text{ in-lb.}$$

Work Done Against a Spring

Suppose that 2 J of work is done to stretch a spring from its natural length of 30 cm to a length of 42 cm.

1. How much work is done to stretch the spring from 35 cm to 45 cm?
2. How far beyond its natural length will a force of 35 N stretch the spring?

Solution (1 of 2)

Recall that $W = \int_{x_1}^{x_2} k\,x\,dx$, where k is the spring constant.

$$
\begin{aligned}
2 &= \int_0^{(42-30)/100} k\,x\,dx = \left[\frac{k}{2}x^2\right]_{x=0}^{x=3/25} = \frac{9k}{1250} \\
k &= \frac{2500}{9} \text{ N/m}
\end{aligned}
$$

1. How much work is done to stretch the spring from 35 cm to 45 cm?

$$
\begin{aligned}
W &= \int_{(35-30)/100}^{(45-30)/100} \frac{2500}{9} x\,dx = \left[\frac{1250}{9}x^2\right]_{x=1/20}^{x=3/20} \\
&= \frac{1250}{9}\left(\frac{9}{400} - \frac{1}{400}\right) = \frac{25}{9} \text{ J}
\end{aligned}
$$

Solution (2 of 2)

2. How far beyond its natural length will a force of 35 N stretch the spring?

$$\begin{aligned} F &= k\,x \\ 35 &= \frac{2500}{9} x \\ x &= \frac{63}{500}\text{ m} = \frac{63}{5}\text{ cm} = 12.6\text{ cm} \end{aligned}$$

Winding a Cable

A 50-ft cable weighing a total of 25 lbs is attached to a 600 lb object. Find the work done in using the cable to lift the object 30 ft.

Solution

- The cable weighs $1/2$ lb/ft.
- When the object has been lifted x feet, the total weight of the object and the remaining length of cable is $f(x) = 600 + 25 - x/2 = 625 - x/2$.
- If this weight is lifted a distance Δx, the increment of work done to lift this weight is $\Delta W = (625 - x/2)\Delta x$.
- The work done is

$$W = \int_0^{30} \left(625 - \frac{x}{2}\right) dx = \left[625x - \frac{x^2}{4}\right]_{x=0}^{x=30} = 18525 \text{ ft-lb.}$$

Stopped Elevator

The elevator in Wickersham Hall has stopped between the first and second floors. The cable attached to the elevator car weighs 6 pounds/foot and is 40 feet long. The elevator car by itself weighs 1000 pounds and there are two students weighing 100 pounds each inside. To rescue the students, the elevator car must be manually lifted 5 feet by winding the cable onto a pulley at the top of the building. How much work must be done?

Solution

The force which must be moved can be expressed as

$$F = 1000 + 2(100) + 6(40 - x) = 1440 - 6x$$

when x feet of the cable have been wound onto the pulley.

If this force is moved a distance dx, the differential work done is

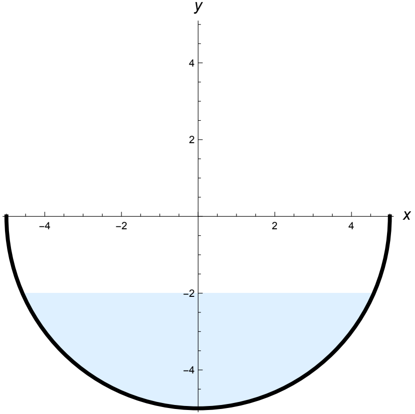
$$F\,dx = (1440 - 6x)\,dx.$$

If the elevator must be lifted 5 feet, then

$$W = \int_0^5 (1440 - 6x)\,dx = \Big[1440x - 3x^2\Big]_{x=0}^{x=5} = 7125 \text{ ft-lb.}$$

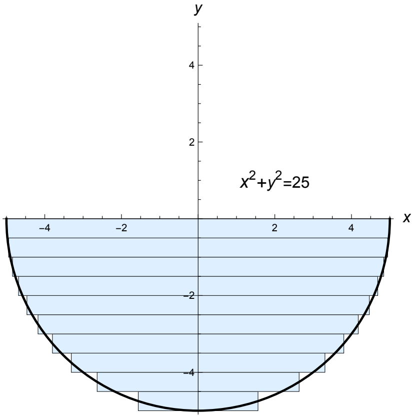
Pumping Out a Tank

A hemispherical tank of radius 5 m is full of water with a density of 10^3 kg/m^3. Find the work done in pumping the water out of the tank.



Solution (1 of 3)

Imagine a thin layer of water parallel to the surface of the tank at a depth of y.



The radius of "slice" of the water is $r = \sqrt{25 - y^2}$.

Solution (2 of 3)

- The volume of the thin layer of water can be expressed as

$$\pi(25 - y^2)\,dy.$$

- The weight of the thin layer of water is therefore

$$1000(9.8)\pi(25 - y^2)\,dy = 9800\pi(25 - y^2)\,dy.$$

- The distance this thin layer must be lifted to remove it from the tank is $0 - y = -y$.

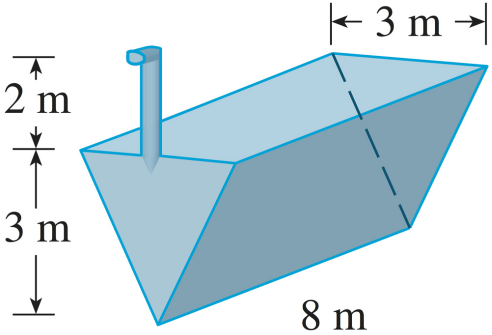
Solution (3 of 3)

The work done is

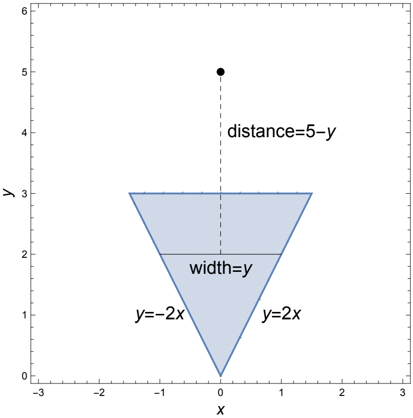
$$\begin{aligned}
W &= \int_{-5}^{0} 9800\pi(25 - y^2)(-y)\,dy \\
&= 9800\pi \int_{-5}^{0} (y^3 - 25y)\,dy \\
&= \left[9800\pi\left(\frac{1}{4}y^4 - \frac{25}{2}y^2\right)\right]_{y=-5}^{y=0} \\
&= -9800\pi\left(\frac{1}{4}(-5)^4 - \frac{25}{2}(-5)^2\right) \\
W &= 1531250\pi \text{ J.}
\end{aligned}$$

Emptying a Tank

The tank shown below is full of water. Find the work done to pump the water out of the spout.



Solution (1 of 2)



The "slice" of water at altitude y has a volume of

$$l\,w\,h = 8y\,dy.$$

Solution (2 of 2)

The work done to pump out the tank is

$$\begin{aligned} W &= \int_0^3 1000(9.8)8y(5-y)\,dy \\ &= 78400\int_0^3 (5y-y^2)\,dy \\ &= 78400\left[\frac{5y^2}{2}-\frac{y^3}{3}\right]_{y=0}^{y=3} \\ &= 78400\cdot\frac{27}{2} = 1,058,400 \text{ J.} \end{aligned}$$

Hydrostatic Force

Terminology:

pressure: force exerted per unit area (notation, p)

gravity: gravitational acceleration (notation, g)

- Metric units $g = 9.8$ m/s^2 or $g = 980$ cm/s^2.
- English units, $g = 32$ ft/s^2.

density: mass per unit volume (notation, ρ)

- for water $\rho = 1000$ kg/m^3 or $\rho = 1$ g/cm^3.
- English units, $\rho\, g = 62.4$ lb/ft^3.

depth: distance to the surface of a fluid (notation, h)

Pascal's Principle

Pascal's Principle: the pressure exerted at a depth h in a fluid is the same in every direction.

If the area of a plate is A then the force on the plate is $\rho\, g\, h\, A$, provided the plate is entirely at depth h.

Question: what if the plate is not oriented horizontally?

Riemann Sum Approach

- Suppose the plate is oriented vertically (parallel to the xy-plane) so that the width of the plate is a function of y, call it $w(y)$.
- Suppose the plate lies in the interval (on the y-axis) $[a, b]$.
- Let $n \in \mathbb{N}$ and $\Delta y = (b-a)/n$ and $y_i = a + i\Delta y$, then the force on the portion of the plate between y_{k-1} and y_k is

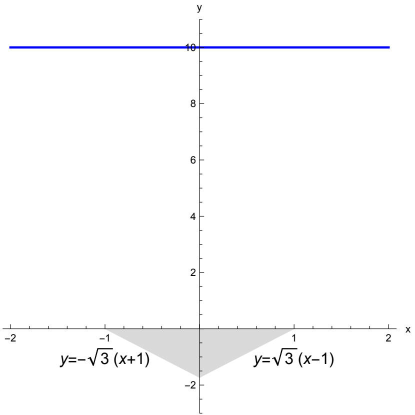
$$\Delta F_k \approx \rho\, g\, h(y_k) w(y_k) \Delta y.$$

- Total **hydrostatic force** is

$$\begin{aligned} F &\approx \sum_{k=1}^{n} \rho\, g\, h(y_k) w(y_k) \Delta y \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \rho\, g\, h(y_k) w(y_k) \Delta y \\ F &= \rho\, g \int_a^b h(y) w(y)\, dy \end{aligned}$$

Example

A dam has a submerged gate in the shape of an equilateral triangle, two feet on a side with the horizontal base nearest the surface of the water and ten feet below it. Find the force on the gate.



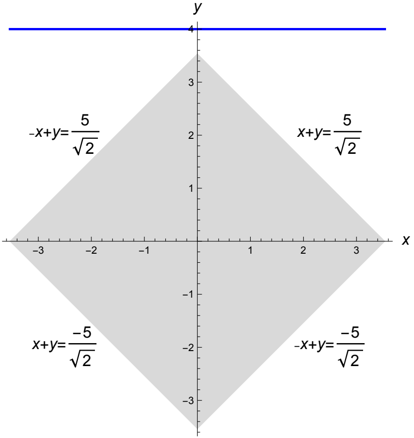
Solution

- The edges of the plate are described by the lines with equations $y = \sqrt{3}(x-1)$ and $y = -\sqrt{3}(x+1)$.
- The width of the plate is $w(y) = 2 + 2y/\sqrt{3}$
- The hydrostatic force is

$$
\begin{aligned}
F &= 62.4 \int_{-\sqrt{3}}^{0} (10-y)(2+2y/\sqrt{3})\,dy \\
&= 124.8 \int_{-\sqrt{3}}^{0} \left(10 + \frac{10}{\sqrt{3}}y - y - \frac{1}{\sqrt{3}}y^2\right) dy \\
&= 124.8 \left[10y + \frac{5}{\sqrt{3}}y^2 - \frac{1}{2}y^2 - \frac{1}{3\sqrt{3}}y^3\right]_{y=-\sqrt{3}}^{y=0} \\
&\approx 1143.2 \text{ lb.}
\end{aligned}
$$

Example

A square plate of sides 5 feet is submerged vertically in water such that one of the diagonals is parallel to the surface of the water. If the distance from the surface of the water to the center of the plate is 4 feet, find the force exerted by the water on one side of the plate.



Solution

- The edges of the plate are described by the lines with equations $y - x = 5/\sqrt{2}$ and $x + y = 5/\sqrt{2}$ (for $y \geq 0$) and by $x + y = -5/\sqrt{2}$ and $y - x = -5/\sqrt{2}$ (for $y < 0$).
- The width of the plate is

$$w(y) = \begin{cases} \frac{10}{\sqrt{2}} - 2y & \text{if } y \geq 0, \\ \frac{10}{\sqrt{2}} + 2y & \text{if } y < 0. \end{cases}$$

- The hydrostatic force is

$$\begin{aligned} F &= 62.4 \int_0^{5/\sqrt{2}} (4 - y) \left(\frac{10}{\sqrt{2}} - 2y \right) dy \\ &\quad + 62.4 \int_{-5/\sqrt{2}}^{0} (4 - y) \left(\frac{10}{\sqrt{2}} + 2y \right) dy \\ &= 62.4 \left(50 - \frac{125}{6\sqrt{2}} \right) + 62.4 \left(50 + \frac{125}{6\sqrt{2}} \right) = 6240 \text{ lb.} \end{aligned}$$

Homework

- Read Section 2.5
- Exercises: 219, 223, 227, $\ldots$, 251/handout