

Valuation of Annuities (Part II)

MATH 372 Financial Mathematics I

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Differing Interest and Payment Periods

If the payment period of an annuity is different from the compounding period of the interest, find the equivalent compound interest rate for the payment period of the annuity.

Example

On the last day of every March, June, September, and December, Walt makes a deposit of \$1,750 into a savings account. The first deposit was on March 31, 2004 and the final one will be on December 31, 2018. Find the balance in Walt's account on January 1, 2019 when

1. interest is quoted at 10% compounded monthly,
2. interest is 9% effective annually with compound interest paid for amounts on deposit for the fraction of a year.

Solution

1. The equivalent quarterly compounded interest rate is j where

$$(1 + j)^4 = \left(1 + \frac{0.10}{12}\right)^{12}$$
$$j = 0.0252.$$

Deposits are made for 15 years (60 quarters). The accumulated value is

$$1750s_{\overline{60}|0.0252} = 239,770.73.$$

2. The equivalent quarterly compounded interest rate is j where

$$(1 + j)^4 = 1 + 0.09$$
$$j = 0.0218$$

The accumulated value is

$$1750s_{\overline{60}|0.0218} = 212,338.41.$$

*m*th-ly Payable Annuities

If the quoted interest rate is an effective annual rate and annuity payments are made more frequently than once per year, this situation is referred to as an ***m*th-ly payable annuity**. The actuarial notation for the accumulated value is

$$K s_{\overline{n}|i}^{(m)} = \frac{K}{m} s_{\overline{m n}|i} = K s_{\overline{n}|i} \cdot \frac{i}{j^{(m)}}.$$

where

- ▶ i is the annual effective interest rate,
- ▶ K/m is the payment made at the end of every $1/m$ -year period

The present value one payment period before the first payment is

$$K a_{\overline{n}|i}^{(m)} = \frac{K}{m} a_{\overline{m n}|i} = K a_{\overline{n}|i} \cdot \frac{i}{j^{(m)}}.$$

Example

Joan wishes to accumulate \$1500 by means of quarterly deposits earning interest at the nominal rate $i^{(4)} = 0.08$. The regular deposits will be \$75 each. Find the number of regular deposits required and the additional fractional deposit if

1. the additional fractional deposit is made at the time of the last regular deposit,
2. the additional fractional deposit is made three months after the last regular deposit.

Solution

Let the number of deposits be n .

$$1500 = 75 s_{\overline{n}|0.02}$$

$$n = 16.9913$$

Sixteen regular deposits will be made.

1. At the time of the 16th deposit,

$$75 s_{\overline{16}|0.02} = 1397.95.$$

The additional fractional deposit made at the time of the last regular deposit will be

$$1500 - 1397.95 = 102.05.$$

2. Three months after the last regular deposit,

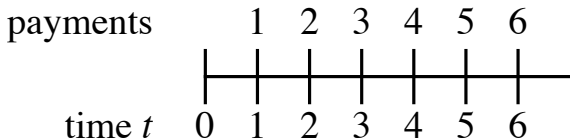
$$75 \ddot{s}_{\overline{16}|0.02} = (1 + 0.02) \left(75 s_{\overline{16}|0.02} \right) = 1425.91.$$

The additional fractional deposit made three months after the last regular deposit will be

$$1500 - 1425.91 = 74.09.$$

Payments Forming an Arithmetic Progression

- ▶ Consider an annuity whose payments are \$1, \$2, \$3, and so on.
- ▶ Suppose the payments are equally spaced and the interest rate per period is i .



Unit Increasing Annuity

The present value of the payments one period before the first payment is

$$\begin{aligned}X &= v + 2v^2 + 3v^3 + \dots + nv^n \\(1+i)X &= 1 + 2v^1 + 3v^2 + \dots + nv^{n-1} \\(1+i)X - X &= 1 + v + v^2 + \dots + v^{n-1} - nv^n \\X &= \frac{1 + v + v^2 + \dots + v^{n-1} - nv^n}{i} \\(Ia)_{\overline{n}|i} &= \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i}\end{aligned}$$

Related Formulas for Present Value

Present value of increasing annuity-due:

$$(\ddot{l}\ddot{a})_{\overline{n}|i} = (1 + i)(la)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{d}$$

Present value of continuously payable increasing annuity:

$$(\bar{l}\bar{a})_{\overline{n}|i} = \frac{i}{\delta}(la)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{\delta}$$

Related Formulas for Perpetuities

Present value of increasing perpetuity:

$$(Ia)_{\infty|i} = \lim_{n \rightarrow \infty} \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i} = \frac{\frac{1}{d} - 0}{i} = \frac{1}{id}$$

Present value of increasing perpetuity-due:

$$(I\ddot{a})_{\infty|i} = \lim_{n \rightarrow \infty} \frac{\ddot{a}_{\overline{n}|i} - nv^n}{d} = \frac{\frac{1}{d} - 0}{d} = \frac{1}{d^2}$$

Present value of continuously payable increasing annuity:

$$(I\bar{a})_{\infty|i} = \lim_{n \rightarrow \infty} \frac{\ddot{a}_{\overline{n}|i} - nv^n}{\delta} = \frac{\frac{1}{d} - 0}{\delta} = \frac{1}{\delta d}$$

Related Formulas for Future Value

Accumulated value:

$$(Is)_{\overline{n}|i} = (1+i)^n (Ia)_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{i}$$

Accumulated value of increasing annuity-due:

$$(I\ddot{s})_{\overline{n}|i} = (1+i)^n (I\ddot{a})_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{d}$$

Accumulated value of continuously payable increasing annuity:

$$(I\bar{s})_{\overline{n}|i} = (1+i)^n (I\bar{a})_{\overline{n}|i} = \frac{\ddot{s}_{\overline{n}|i} - n}{\delta} = \frac{i}{\delta} (Is)_{\overline{n}|i}$$

Example

- ▶ Find $(Ia)_{\overline{10}|0.05}$
- ▶ Find $(I\ddot{a})_{\overline{10}|0.05}$
- ▶ Find $(I\bar{a})_{\overline{10}|0.05}$
- ▶ Find $(Is)_{\overline{10}|0.05}$
- ▶ Find $(I\ddot{s})_{\overline{10}|0.05}$
- ▶ Find $(I\bar{s})_{\overline{10}|0.05}$

Solution

$$(Ia)_{\overline{10}|0.05} = \frac{\ddot{a}_{\overline{10}|0.05} - 10v^{10}}{i} = \frac{8.1078 - 10(1 + 0.05)^{-10}}{0.05}$$

$$= 39.3738$$

$$(I\ddot{a})_{\overline{10}|0.05} = (1 + 0.05)(Ia)_{\overline{10}|0.05} = 41.3425$$

$$(I\bar{a})_{\overline{10}|0.05} = \frac{0.05}{\ln(1 + 0.05)}(Ia)_{\overline{10}|0.05} = 40.3501$$

$$(Is)_{\overline{10}|0.05} = (1 + 0.05)^{10}(Ia)_{\overline{10}|0.05} = 64.1357$$

$$(I\ddot{s})_{\overline{10}|0.05} = (1 + 0.05)^{10}(I\ddot{a})_{\overline{10}|0.05} = 67.3426$$

$$(I\bar{s})_{\overline{10}|0.05} = (1 + 0.05)^{10}(I\bar{a})_{\overline{10}|0.05} = 65.7261$$

Example

Mary sets up a retirement fund by making annual payments at the end of each year for 30 years. The first payment is \$1,000 and each subsequent payment is \$100 larger than the previous one. The fund earns 5.8% annually. How much is accumulated in the fund at the end of the 30 years?

Example

Don invests payments of \$2,500 per year at the beginning of each year for 10 years. The payments earn 7% interest, and the interest received on the payments is reinvested at 4%. How much will Don have at the end of 10 years?

Solution

- ▶ At the end of 10 years Don will have his total of \$25,000 in deposits plus interest and interest on interest.
- ▶ At the beginning of year k the account total deposit is $2500k$ and thus the interest earned during the k th year is

$$(2500)(0.07)k = 175k,$$

for $k = 1, 2, \dots, 10$. This amount is invested at 4%.

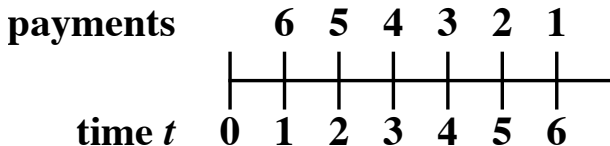
- ▶ Don's total accumulated value is

$$\begin{aligned} X &= 25,000 + 175(Is)_{\overline{10}|0.04} \\ &= 25,000 + 175 \frac{\ddot{S}_{\overline{10}|0.04} - 10}{0.04} \\ &= 25,000 + 175(62.1588) = \$35,877.79. \end{aligned}$$

Decreasing Annuities with Terms in Arithmetic Progression

Definition

The n -year **unit decreasing annuity-immediate** has n payments of $n, n - 1, \dots, 2, 1$ payable at the end of each year (at times $1, 2, \dots, n$).



Valuation of Unit Decreasing Annuities

$$\begin{aligned}(Da)_{\overline{n}|i} + (Ia)_{\overline{n}|i} &= (n+1)a_{\overline{n}|i} \\(Da)_{\overline{n}|i} &= (n+1)a_{\overline{n}|i} - (Ia)_{\overline{n}|i} \\&= (n+1)\frac{1-v^n}{i} - \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i} \\&= \frac{(n+1) - (n+1)v^n - \ddot{a}_{\overline{n}|i} + nv^n}{i} \\&= \frac{n - (-1 + v^n + \ddot{a}_{\overline{n}|i})}{i} \\(Da)_{\overline{n}|i} &= \frac{n - a_{\overline{n}|i}}{i}\end{aligned}$$

Unit Decreasing Annuities

The present value of an annuity that pays $n, n - 1, \dots, 2, 1$ with payment starting one period from now is

$$(Da)_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{i}.$$

If the payments start now the present value is

$$(D\ddot{a})_{\overline{n}|i} = (1 + i)(Da)_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{d}.$$

If the decreasing annuity is continuously payable then the present value is

$$(D\bar{a})_{\overline{n}|i} = \frac{i}{\delta}(Da)_{\overline{n}|i} = \frac{n - a_{\overline{n}|i}}{\delta}.$$

Accumulated Value of Unit Decreasing Annuities

The accumulated value at the time of the final payment is

$$(Ds)_{\overline{n}|i} = (1+i)^n(Da)_{\overline{n}|i} = \frac{n(1+i)^n - s_{\overline{n}|i}}{i}.$$

If the payments start now the future value is

$$(D\ddot{s})_{\overline{n}|i} = (1+i)^n(D\ddot{a})_{\overline{n}|i} = \frac{n(1+i)^n - s_{\overline{n}|i}}{d}.$$

If the decreasing annuity is continuously payable then the future value is

$$(D\bar{s})_{\overline{n}|i} = \frac{i}{\delta}(D\bar{a})_{\overline{n}|i} = \frac{n(1+i)^n - s_{\overline{n}|i}}{\delta}.$$

Example

- ▶ Find $(Da)_{\overline{10}|0.05}$
- ▶ Find $(D\ddot{a})_{\overline{10}|0.05}$
- ▶ Find $(D\bar{a})_{\overline{10}|0.05}$
- ▶ Find $(Ds)_{\overline{10}|0.05}$
- ▶ Find $(D\ddot{s})_{\overline{10}|0.05}$
- ▶ Find $(D\bar{s})_{\overline{10}|0.05}$

Solution

$$(Da)_{\overline{10}|0.05} = \frac{10 - a_{\overline{10}|0.05}}{i} = \frac{10 - 7.7217}{0.05}$$

$$= 45.5653$$

$$(D\ddot{a})_{\overline{10}|0.05} = (1 + 0.05)(Da)_{\overline{10}|0.05} = 47.8436$$

$$(D\bar{a})_{\overline{10}|0.05} = \frac{0.05}{\ln(1 + 0.05)}(Da)_{\overline{10}|0.05} = 46.6952$$

$$(Ds)_{\overline{10}|0.05} = (1 + 0.05)^{10}(Da)_{\overline{10}|0.05} = 74.2211$$

$$(D\ddot{s})_{\overline{10}|0.05} = (1 + 0.05)^{10}(D\ddot{a})_{\overline{10}|0.05} = 77.9321$$

$$(D\bar{s})_{\overline{10}|0.05} = (1 + 0.05)^{10}(D\bar{a})_{\overline{10}|0.05} = 76.0616$$

Example

Hank purchases an increasing perpetuity-due with annual payments of 10 and increasing by 10 each year until the payment amount reaches 300. The payments stay at 300 after that. The effective annual interest rate is 8%. Determine the present value of the annuity.

Solution

$$\begin{aligned}PV &= 10 + 20v + 30v^2 + \dots + 290v^{28} + 300(v^{29} + v^{30} + \dots) \\&= (300 - 290) + (300 - 280)v + (300 - 270)v^2 + \dots + (300 - 10)v^{28} \\&\quad + 300(v^{29} + v^{30} + \dots) \\&= 300(1 + v + v^2 + \dots) - 10(29 + 28v + 27v^2 + \dots + 1v^{28}) \\&= 300\ddot{a}_{\infty|0.08} - 10(D\ddot{a})_{\overline{29}|0.08} \\&= \frac{300}{d} - 10\left(\frac{29 - a_{\overline{29}|0.08}}{d}\right) = 4050 - 2408.62 = \$1,641.38\end{aligned}$$

Unified Formula

Suppose the first payment of an annuity-immediate is P and subsequent payments change by Q per period (Q can be positive or negative). If the annuity has n payments, then the sequence of payments is

$$P, P + Q, P + 2Q, P + 3Q, \dots, P + (n - 1)Q.$$

The present value of the annuity is,

$$\begin{aligned} PV &= Pa_{\overline{n}|} + vQ(la)_{\overline{n-1}|} \\ &= Pa_{\overline{n}|} + vQ \frac{\ddot{a}_{\overline{n-1}|} - (n-1)v^{n-1}}{i} \\ &= Pa_{\overline{n}|} + Q \frac{a_{\overline{n-1}|} - (n-1)v^n}{i} \\ &= Pa_{\overline{n}|} + Q \frac{(a_{\overline{n-1}|} + v^n) - nv^n}{i} \\ PV &= Pa_{\overline{n}|} + Q \frac{a_{\overline{n}|} - nv^n}{i}. \end{aligned}$$

Related Formulas

The first payment of an annuity-immediate is P and subsequent payments change by Q per period. If the annuity has n payments, then

$$PV = Pa_{\overline{n}|i} + Q \frac{a_{\overline{n}|i} - nv^n}{i}$$

$$FV = Ps_{\overline{n}|i} + Q \frac{s_{\overline{n}|i} - n}{i}$$

If the number of payments is infinite, then

$$PV = \frac{P}{i} + \frac{Q}{i^2}$$

Related Formulas for Annuity-Due

The first payment of an annuity-due is P and subsequent payments change by Q per period. If the annuity has n payments, then

$$PV = P\ddot{a}_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{d}$$

$$FV = P\ddot{s}_{\overline{n}|} + Q\frac{s_{\overline{n}|} - n}{d}.$$

If the number of payments is infinite, then

$$PV = \frac{P}{d} + \frac{Q}{id}.$$

Example

A 15-year annuity-immediate has a first-year payment of \$750. The subsequent payments increase by \$150 per year. Find the present value of this annuity based on an annual effective rate of 6%.

Geometric Progression of Payments

Some annuities have payments which are adjusted periodically (perhaps to account for inflation). If the annual inflation rate is r the payments would increase annually by a factor $1 + r$.

Example

Roger purchases a 30-year annuity with annual payments beginning one year from now. The annuity is valued at an effective annual rate of 10%. Roger anticipates an effective annual inflation rate of 3% over the 30 years of the annuity and thus would like each payment after the first to be 3% larger than the previous one. The first payment is \$50,000. What is the present value of the annuity?

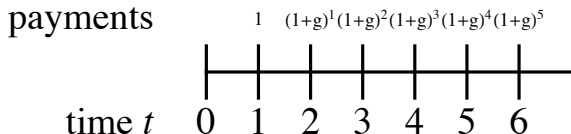
Solution

The present value of all the payments is

$$\begin{aligned}PV &= 50,000v_{0.10} + 50,000(1.03)v_{0.10}^2 + \cdots + 50,000(1.03)^{29}v_{0.10}^{30} \\&= \frac{50,000}{1.10} \left[1 + \frac{1.03}{1.10} + \cdots + \left(\frac{1.03}{1.10} \right)^{29} \right] \\&= \frac{50,000}{1.10} \frac{1 - \left(\frac{1.03}{1.10} \right)^{30}}{1 - \frac{1.03}{1.10}} \\&= (45,454.55) \frac{1 - (0.9364)^{30}}{1 - 0.9364} \\PV &= \$614,926.56.\end{aligned}$$

Annuities with Terms in Geometric Progression

A **geometric annuity** with growth rate g is an annuity which pays $\{1, 1 + g, (1 + g)^2, (1 + g)^3, \dots\}$.



$$\begin{aligned} PV &= \frac{1}{1+i} + \frac{(1+g)}{(1+i)^2} + \dots + \frac{(1+g)^{n-1}}{(1+i)^n} \\ &= \frac{1}{1+i} \left[1 + \left(\frac{1+g}{1+i} \right) + \dots + \left(\frac{1+g}{1+i} \right)^{n-1} \right] \\ a_{\overline{n}|i}^g &= \frac{1 - \left(\frac{1+g}{1+i} \right)^n}{i-g} \text{ if } i \neq g. \end{aligned}$$

Geometric Annuities

$$a_{\overline{n}|i}^g = \frac{1 - \left(\frac{1+g}{1+i}\right)^n}{i - g} \text{ if } i \neq g$$

$$s_{\overline{n}|i}^g = (1+i)^n a_{\overline{n}|i}^g = \frac{(1+i)^n - (1+g)^n}{i - g}$$

$$\ddot{a}_{\overline{n}|i}^g = (1+i) a_{\overline{n}|i}^g = \frac{1 - \left(\frac{1+g}{1+i}\right)^n}{d - g v}$$

$$\ddot{s}_{\overline{n}|i}^g = (1+i)^{n+1} a_{\overline{n}|i}^g = \frac{(1+i)^{n+1} - (1+g)^{n+1}}{d - g v}$$

$$a_{\infty|i}^g = \frac{1}{i - g} \text{ if } i > g$$

$$\ddot{a}_{\infty|i}^g = \frac{1+i}{i - g}$$

Further Formulas ($i = g$)

Recall that

$$PV = \frac{1}{1+i} \left[1 + \left(\frac{1+g}{1+i} \right) + \cdots + \left(\frac{1+g}{1+i} \right)^{n-1} \right].$$

If $i = g$ then,

$$PV = \frac{1}{1+i} \left[1 + \left(\frac{1+i}{1+i} \right) + \cdots + \left(\frac{1+i}{1+i} \right)^{n-1} \right]$$

$$a_{\overline{n}|i}^g = \frac{n}{1+i}$$

$$\ddot{a}_{\overline{n}|i}^g = n.$$

Example

Find the present value of the sequence of payments $\{1, 1.07, (1.07)^2, \dots, (1.07)^{11}\}$ if $i = 6\%$ and the payments are made at the end of each year.

Example

Find the present value of the perpetuity paying $\{1.08, (1.08)^2, (1.08)^3, \dots\}$ if $i = 10\%$ and the payments are made at the beginning of each year.

Example

You want to save \$1,000,000 for retirement. You plan to make annual deposits at the end of each year into an account earning an effective annual rate of 6.5%. You will increase the amount of the deposit by 3% each year. If you retire in 40 years, what should be the amount of the first deposit?

Example

Maggie's child was born January 1, 2003. Maggie receives monthly child support payments on the last day of every month beginning January 31, 2003. The payments are increased by 11% each calendar year to meet cost-of-living increases. Monthly payments are constant during each calendar year at \$650 in 2003. Immediately upon receiving a payment, Maggie deposits it in an account earning $i^{(12)} = 0.04$ with interest credited on the last day of the month. Find the accumulated value of the payments on the child's 18th birthday.

Solution (1 of 2)

- ▶ Since payments are deposited monthly and interest is compounded monthly but payment adjustments occur yearly, first consider the accumulated value of the first year of payments:

$$650s_{\overline{12}|} = \$7,944.60.$$

- ▶ This the accumulated value of payments made during the child's j th year will be $7,944.60(1.11)^{j-1}$.
- ▶ The annual effective rate of interest is $(1 + 0.04/12)^{12} - 1 = 0.0407$.

Solution (2 of 2)

On the child's 18th birthday the accumulated value of all payments will be

$$\begin{aligned} \text{FV} &= 7,944.60 s_{\overline{18}|0.0407}^{0.11} \\ &= 7,944.60 \left(\frac{(1 + 0.0407)^{18} - (1 + 0.11)^{18}}{0.0407 - 0.11} \right) \\ &= 7,944.60(64.8524) \\ &= \$515,226.53. \end{aligned}$$

Equation of Value and Loan Payments

Important Principle: the set of payments made by a lender and the set of payments made by a borrower must have the same value at every possible valuation date.

The valuation date can be any convenient date.

Solution

Let the valuation date be the beginning of the 4th year, then

$$P\ddot{a}_{\overline{10}|0.06} = 15,000\ddot{s}_{\overline{3}|0.06}$$

$$7.8017P = 50,619.2400$$

$$P = \$6,488.24.$$

Annuities with Varying Discrete Payments

Suppose we will receive payments of $\{P_1, P_2, \dots, P_n\}$ at the ends of the next n years. If the effective annual interest rate is constantly i , then we can use the **Cash Flow Worksheet** of the TI BA II Plus calculator to calculate the present value.

The following linked video will explain the steps involved.

<https://www.youtube.com/16KkkzxSy2I>

Example

Suppose $i = 6.5\%$ and at $t = 0$ we make an investment of \$250 and then receive payments at the end of the next four years in the amounts of $\{100, 50, 75, 125\}$. What is the present value of the cashflows?

Example

Suppose $i = 7.5\%$ and an annuity-immediate has a first payment of \$200 and the payments increase by \$100 each year until they reach \$600. There are 20 further payments of \$600. What is the present value of the annuity?

Homework

- ▶ Read Chapter 2
- ▶ Exercises: on handout