Valuation of Annuities (Part III)

MATH 372 *Financial Mathematics I*

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Continuous Annuities with Varying Payments

- Suppose the instantaneous rate of payment per unit time is $h(t)$.
- In infinitesimal time dt the payment is $h(t)\,dt$.
- The present value over interval $[0, n]$ is

$$\mathrm{PV} = \int_0^n h(t) v^t \, dt.$$

- The accumulated value over interval $[0, n]$ is

$$\mathrm{FV} = \int_0^n h(t)(1+i)^{n-t} \, dt.$$

Varying Force of Interest

If the force of interest at time t is δ_t then

$$
\begin{aligned}
\text{PV} &= \int_0^n h(t) e^{-\int_0^t \delta_s \, ds} \, dt \\
\text{FV} &= \int_0^n h(t) e^{\int_t^n \delta_s \, ds} \, dt.
\end{aligned}
$$

Special Cases (Continuously Increasing)

If $h(t) = t$ then

$$\begin{aligned}
\text{PV} &= \int_0^n t\, v^t\, dt = \int_0^n t(1+i)^{-t}\, dt \\
&= \left[-\frac{t(1+i)^{-t}}{\ln(1+i)}\right]_{t=0}^{t=n} + \int_0^n \frac{(1+i)^{-t}}{\ln(1+i)}\, dt \\
(\bar{I}\bar{a})_{\overline{n}|i} &= \frac{1}{\delta}\left(\frac{1 - v^n - \delta n v^n}{\delta}\right) = \frac{\bar{a}_{\overline{n}|i} - n v^n}{\delta} \\
(\bar{I}\bar{s})_{\overline{n}|i} &= (1+i)^n (\bar{I}\bar{a})_{\overline{n}|i} = \frac{\bar{s}_{\overline{n}|i} - n}{\delta} \\
(\bar{I}\bar{a})_{\overline{\infty}|i} &= \frac{1}{\delta^2}
\end{aligned}$$

Special Cases (Continuously Decreasing)

Note that $(\bar{I}\bar{a})_{\overline{n}|i} + (\bar{D}\bar{a})_{\overline{n}|i} = n\,\bar{a}_{\overline{n}|i}$ and thus

$$\begin{aligned} (\bar{D}\bar{a})_{\overline{n}|i} &= n\frac{1-v^n}{\delta} - \frac{\bar{a}_{\overline{n}|i} - nv^n}{\delta} = \frac{n - \bar{a}_{\overline{n}|i}}{\delta} \\ (\bar{D}\bar{s})_{\overline{n}|i} &= (1+i)^n (\bar{D}\bar{a})_{\overline{n}|i} = \frac{n(1+i)^n - \bar{s}_{\overline{n}|i}}{\delta}. \end{aligned}$$

Example

Find the present value of a 15-year continuously increasing annuity. The annuity's payment rate begins at 0 and increases continuously at a rate of \$500 per year. The effective annual interest rate is 6%.

Solution

Let $n = 15$ and $i = 0.06$.

$$\begin{aligned} PV &= 500(\bar{I}\bar{a})_{\overline{15}|0.06} = 500\left(\frac{\bar{a}_{\overline{15}|0.06} - 15(1+0.06)^{-15}}{\ln(1+0.06)}\right) \\ &= 500\left(\frac{\frac{0.06}{\ln(1+0.06)}a_{\overline{15}|0.06} - 6.2590}{0.0583}\right) \\ PV &= \$32,108.13 \end{aligned}$$

Example

An annuity provides continuous payments at a rate of \$500 per year starting at $t = 0$ and its payment rate increases continuously at a rate of \$50 per year for 20 years. Calculate the accumulated (future) value of this annuity at the end of 20 years assuming the annual effective rate of interest is 8%.

Continuously Payable Geometric Annuities

A continuously payable geometric annuity can have a continuously changing rate of payment.

- Let $\bar{g}$ be the continuously compounded rate of change in the payment rate.
- The payment rate at time t will be $e^{\bar{g}t}$.

$$
\begin{aligned}
\bar{a}^{\bar{g}}_{\overline{n}|} &= \int_0^n e^{\bar{g}t} e^{-\delta t}\,dt = \int_0^n e^{-(\delta-\bar{g})t}\,dt = \bar{a}_{\overline{n}|(\delta-\bar{g})} = \frac{1-e^{-n(\delta-\bar{g})}}{\delta-\bar{g}} \\
\bar{s}^{\bar{g}}_{\overline{n}|} &= e^{n\delta}\,\bar{a}^{\bar{g}}_{\overline{n}|} = \frac{e^{n\delta}-e^{n\bar{g}}}{\delta-\bar{g}} \\
\bar{a}^{\bar{g}}_{\overline{\infty}|} &= \frac{1}{\delta-\bar{g}} \text{ (if } \delta > \bar{g}\text{)}
\end{aligned}
$$

Continuously Varying Payments and Force of Interest

For an annuity with a continuously varying rate of payment $\rho(t)$ and a continuously varying force of interest $\delta(t)$, the present value of the annuity can be expressed as the integral:

$$\text{PV} = \int_0^n \rho(t) e^{-\int_0^t \delta(s)\, ds}\, dt$$

Example

A 10-year annuity makes continuous payments at a rate of $5000 + 100t$ per year. Calculate the present value of this annuity at a continuously varying force of interest $\delta(t) = \frac{1}{10+t}$.

Solution

$$
\begin{aligned}
PV &= \int_0^{10} (5000 + 100t) e^{-\int_0^t \frac{1}{10+s}\, ds}\, dt \\
&= \int_0^{10} (5000 + 100t) e^{-\ln(10+t) + \ln 10}\, dt \\
&= \int_0^{10} (5000 + 100t) \frac{10}{10+t}\, dt \\
&= \int_0^{10} \frac{50000 + 1000t}{10+t}\, dt \\
&= 1000 \int_0^{10} \left(1 + \frac{40}{10+t}\right) dt \\
&= 1000 \left[t + 40 \ln(10+t)\right]_{t=0}^{t=10} \\
PV &= \$37,725.89
\end{aligned}
$$

Reinvestment Situations

- Suppose an investment provides periodic interest payments.
- The investor reinvests the money provided by the interest payments in another investment which may pay a different interest rate than the first investment.

Example

Suppose Alice lends Bob $\$5,000$ for six years. Bob agrees to pay Alice 5% interest at the end of each of the six years. At the end of the six years Bob repays the full amount of the amount borrowed. Alice reinvests the interest payments at 4% interest.

1. What is the future value of Alice's investment?
2. What overall rate of interest will Alice have earned?

Solution

1. At the end of 6 years Alice has $\$5,000$ and the future value of 6 interest payments of $(5000)(0.05) = 250$ reinvested at 4%.

   $$FV = 5000 + 250 s_{\overline{6}|0.04} = 5000 + 1658.2439 = \$6,658.2439$$

2. Alice's original $\$5,000$ has grown to $\$6,658.2439$ in 6 years. The overall rate of interest is

   $$\left(\frac{6,658.2439}{5000}\right)^{1/6} - 1 = 0.0489.$$

Example

Suppose you can deposit \$500 into Bank A at the beginning of each year for the next 10 years and earn an annual effective interest rate of 6%. The interest paid by Bank A is invested in Bank B at an effective annual interest rate of 5%. How much will you have in the two accounts at the end of 10 years?

Inflation

- **Inflation** changes the purchasing power of funds.
- Inflation can have an effect on interest rates charged or offered by lenders.
- We will introduce the notions of “real” and “nominal” rates of return.

Motivating Example

- Suppose next year after passing SOA Exam FM you want to celebrate with a bottle of champagne which currently costs \$200.
- You put aside \$200 dollars today at an annual effective rate of interest of 5%. In one year you will have \$210.
- Suppose next year inflation has increased the price of the champagne to \$207.50. The real increase in the purchasing power is

$$\frac{210}{207.50} - 1 = 0.0120 \implies 1.2048\%.$$

Notation

i: annual effective interest rate

r: inflation rate

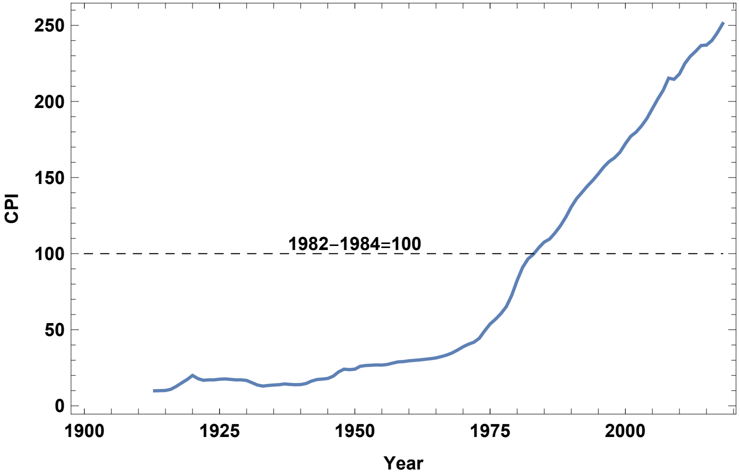
j: real rate of interest (or real rate of return)

$$1 + i = (1 + j)(1 + r) \iff 1 + j = \frac{1 + i}{1 + r}$$

Note: some people approximate $j = i - r$, but we should use the more accurate formula,

$$j = \frac{1 + i}{1 + r} - 1 = \frac{i - r}{1 + r}.$$

Historical Consumer Price Index



Dividend Discount Model for Valuing a Stock

One method of find the value of a stock is assign its value to be the present value of all future dividends that will be paid.

Assumptions:

- The next dividend payable one year from now is K.
- The annual compound growth rate for the dividend is r.
- The interest rate is i.

$$\text{PV} = K\left[\frac{1}{1+i} + \frac{1+r}{(1+i)^2} + \cdots\right] = \frac{K}{1+i}\,\frac{1}{1-\frac{1+r}{1+i}} = \frac{K}{i-r}$$

Example

Common stock X pays a dividend of \$30 at the end of the first year, with each subsequent annual dividend being 1% greater than the preceding one. Mike purchases the stock at a price to earn an expected effective annual yield of 5%. Immediately after receiving the 8th dividend, Mike sells the stock for P. His effective annual yield over the 8-year period was 4%. Calculate P.

Solution

- The price Mike paid for the stock was

$$\frac{K}{i-r} = \frac{30}{0.05-0.01} = 750.$$

- For the investment of \$750 Mike received 8 dividend payments and the sale price P in 8 years.
- Equating the present values at 4% rate of return yields

$$\begin{aligned}
750 &= 30\left[\frac{1}{1.04} + \frac{1.01}{(1.04)^2} + \cdots + \frac{(1.01)^7}{(1.04)^8}\right] + \frac{P}{(1.04)^8} \\
&= \frac{30}{1.04}\,\frac{1-\left(\frac{1.01}{1.04}\right)^8}{1-\frac{1.01}{1.04}} + (0.7307)P \\
&= 208.77 + (0.7307)P \\
P &= \$740.70.
\end{aligned}$$

Homework

- Read Chapter 2
- Exercises: on handout