Bond Valuation

MATH 372 *Financial Mathematics I*

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Background

Corporations have two main ways of raising a large sum of money:

equity: selling stock, which usually gives the stockholders some voice in how the corporation is run,

debt: borrow the money, which requires periodic interest payments and the repayment of principal.

If corporations choose to borrow, they can borrow from banks or they can issue bonds.

Corporate Indebtedness

Definition

A **bond** is an interest-bearing certificate of public (used by the government) or private (used by corporations) indebtedness. The interest payments are called **coupons.** The risk of loss of repayment of the principal or the loss of an interest payment is referred to as **credit risk.**

Bond Terminology

face amount: the principal amount borrowed (denoted F),

coupon rate: rate at which interest is paid on the bond (denoted r),

maturity date: the time interval during which coupon payments will be made (denoted T), the number of coupons is n,

redemption amount: the amount to be paid in addition to any coupon on the maturity date (denoted C),

yield rate: the interest rate used to determine the price of the bond (denoted j).

Bond Redemption Value

- When $C = F$ the bond is **redeemed at par.**
- When $C > F$ the bond is **redeemed at a premium.**
- When $C < F$ the bond is **redeemed at a discount.**

Example

XYZ Corporation offers 10-year bonds for sale on a day when investors are demanding a yield rate of 10.4% convertible semiannually. Investors are willing to buy bonds at a price which will yield 10.4% convertible semiannually. Over the 10 years each bond will make 20 semiannual coupon payments of \$50 and a final payment of \$1000 (plus the last coupon payment). What is the price of the bond?

Solution

$$P = 50 a_{\overline{20}|\frac{0.104}{2}} + v_{0.104/2}^{20} 1000 = \$975.4929$$

Use the TI-BA II Plus calculator with $N = 20$, $I/Y = 10.4/2$, $PMT = 50$, *and* $FV = 1000$, *then CPT PV.*

Example

If investors in the previous example were demanding a yield rate of 9.9% convertible semiannually, what is the price of the bond?

General Behavior

The *price* of a bond moves in the opposite direction of the **interest rate**.

- A higher bond price means a lower bond yield.
- A lower bond price means a higher bond yield.
- The bond's coupon rate is *not* the interest rate the investor earns.

Example: US Treasury Bond

Issue Date:	January 1, 2019
Maturity Date:	January 1, 2021
Coupon Rate:	4.875%
Yield Rate:	4.930%
Face Amount:	\$100
Redemption Amount:	\$100

Coupons are paid semiannually, so the price of the bond is

$$P = \frac{4.875}{2} a_{\overline{4}|0.04930/2} + 100 v^4_{0.04930/2} = 99.8965.$$

Historical Note: US government bond prices are rounded to the nearest $1/32. The price calculated would be published as

$$99.29 \equiv 99 + \frac{29}{32} = 99.9063.$$

Price of Bond on Coupon Date

The price of a bond is the present value of the coupons and the redemption amount.

$$P = (F\,r)a_{\overline{n}|j} + Cv_j^n$$
$$P = (F\,r)a_{\overline{n}|j} + Fv_j^n \text{ (if } F = C\text{)}$$

Example

A bond will pay a 10% coupon convertible semiannually for 10 years and then return the face value of \$1000. If investors wish to price the bond to yield 10.2% convertible semiannually, what is the price of the bond?

Solution

Since $n = 20$, $r = 0.10/2$, $j = 0.102/2$, and $F = C = 1000$, then

$$\begin{aligned} P &= \left(1000 \cdot \frac{0.10}{2}\right) a_{\overline{20}|0.051} + 1000\, v_{0.051}^{20} \\ &= 50\, a_{\overline{20}|0.051} + 1000\, v_{0.051}^{20} \\ &= \$987.64. \end{aligned}$$

The \$1000 bond is sold for \$987.64 thus it is said to have sold at a **discount** of $1000 - 987.64 = \$12.36$.

Example

A bond will pay a 10% coupon convertible semiannually for 10 years and then return the face value of $1000. If investors wish to price the bond to yield 9.7% convertible semiannually, what is the price of the bond?

Example

A bond with face value \$1000 will pay a 10% coupon convertible semiannually for 10 years and then return the redemption value of \$1100.

1. If investors wish to price the bond to yield 10.2% convertible semiannually, what is the price of the bond?
2. If investors wish to price the bond to yield 9.8% convertible semiannually, what is the price of the bond?

Example

A $1000 face value bond with a term of 10 years and a coupon of 10% convertible semiannually is priced at $990. What is the yield convertible semiannually?

Solution

Since $n = 20$, $r = 0.10/2$, $P = 990$, and $F = C = 1000$, then

$$990 = \left(1000 \cdot \frac{0.10}{2}\right) a_{\overline{20}|j/2} + 1000\, v_{j/2}^{20} = 50\, a_{\overline{20}|j/2} + 1000\, v_{j/2}^{20},$$

and $j/2 = 0.050808$ which implies $j^{(2)} = 10.1616\%$.

Bond Purchase Value

- If $P > F$, the bond is said to be bought **at a premium**,
- If $P = F$, the bond is said to be bought **at par**,
- If $P < F$, the bond is said to be bought **at a discount.**

Example

Suppose a company issues a 10-year bond that produces a required yield of 9.8%. The face value of the bond is \$1,000 and produces semiannual coupons at a coupon rate of 10%. What is the redemption value of the bond if the price of the bond is \$1,000?

Premium-Discount Formula

Remark: If the redemption value of a bond is not specifically mentioned, assume the bond will be redeemed at par.

If a bond is redeemed at par,

$$\begin{aligned} P &= (F\,r)a_{\overline{n}|j} + Fv_j^n \\ &= (F\,r)a_{\overline{n}|j} + F - F(1 - v_j^n) \\ &= (F\,r)a_{\overline{n}|j} + F - F\,j\left(\frac{1 - v_j^n}{j}\right) \\ &= (F\,r)a_{\overline{n}|j} + F - (F\,j)a_{\overline{n}|j} \\ &= F + F(r - j)a_{\overline{n}|j}. \end{aligned}$$

The price is the face value F plus the premium or discount $F(r - j)a_{\overline{n}|j}$.

Makeham's Formula

Recall that

$$a_{\overline{n}|j} = \frac{1 - v_j^n}{j} \iff v_j^n = 1 - j\, a_{\overline{n}|j},$$

thus

$$\begin{aligned}
P &= F(r-j)\,a_{\overline{n}|j} + F \text{ (premium-discount formula)} \\
&= F(r-j)\left(\frac{1-v_j^n}{j}\right) + F \\
&= F(-j)\left(\frac{1-v_j^n}{j}\right) + F + r\,F\left(\frac{1-v_j^n}{j}\right) \\
&= -F(1-v_j^n) + F + \frac{rF}{j}(1-v_j^n) \\
&= F\,v_j^n + \frac{r}{j}(F - Fv_j^n) \\
P &= K + \frac{r}{j}(F-K)
\end{aligned}$$

where $K = F\,v_j^n$.

Example

Suppose a $1000 bond is redeemable at par in 5 years with a nominal coupon rate of 12% payable semiannually. The bond is purchased to yield 10.5% convertible semiannually. What is the price of the bond?

Bond Amortization

Since a bond is in effect a loan, it makes sense to think of the make of coupons as payment of interest and the redemption amount as the repayment of principal. Thus we can construct an amortization table for a bond.

BV_t: **book value** of bond, the present value immediately after the *t*th coupon is paid of the remaining coupons and redemption amount.

I_t: interest paid in *t*th coupon.

PR_t: principal repaid in the *t*th coupon.

Amortization Formulas

$$
\begin{aligned}
\mathrm{BV}_0 &= (F\,r)\,a_{\overline{n}|j} + F v_j^n \\
\mathrm{BV}_t &= \mathrm{BV}_{t-1}(1+j) - F\,r \quad (\text{for } t = 1, \ldots, n-1) \\
\mathrm{I}_t &= \mathrm{BV}_{t-1} j \\
\mathrm{PR}_t &= F\,r - I_t
\end{aligned}
$$

t	**Book Value**	**Payment**	**Interest Due**	**Principal Repaid**
0	BV_0	—	—	—
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	BV_t	$F\,r$	I_t	PR_t
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	0	$F\,r + F$	I_n	$F + \mathrm{PR}_n$

Example

An 8% coupon bond with face amount \$1000 matures 4 years after issue. Construct the amortization schedule for the bond over its term for a nominal yield rate of 10%.

Solution

t	**Book Value**	**Payment**	**Interest Due**	**Principal Repaid**
0	935.37	—	—	—
1	942.14	40	46.77	-6.77
2	949.24	40	47.11	-7.11
3	956.71	40	47.46	-7.46
4	964.54	40	47.84	-7.84
5	972.77	40	48.23	-8.23
6	981.41	40	48.64	-8.64
7	990.48	40	49.07	-9.07
8	0.00	1040	49.52	990.48

Amortization of Premium

If a bond is redeemable at face value the **amortization of premium in period** k is defined to be

$$F(r-j)v^{n-k+1}.$$

Example

A 10% coupon bond with face value $1000 matures 4 years after issue. The bond is priced to yield 8%.

- Find the premium paid for the bond.
- Construct a table showing the premium amortized in each coupon payment.

Solution

Using the premium-discount formula, the premium is

$$1000(0.05-0.04)a_{\overline{8}|0.04} = 67.33.$$

The premium amortized in the kth coupon will be

$$\frac{1000(0.05-0.04)}{(1+0.04)^{8-k+1}}.$$

Period k	Remaining Premium	Premium Amortized
0	67.33	
1	60.02	7.31
2	52.42	7.60
3	44.52	7.90
4	36.30	8.22
5	27.75	8.55
6	18.86	8.89
7	9.61	9.25
8	-0.01	9.62

Example: Amortization of Discount

A 10% coupon bond with face value \$1000 matures 4 years after issue. The bond is priced to yield 12%.

- Find the discount in the price of the bond.
- Construct a table showing the discount amortized in each coupon payment.

Solution

Using the premium-discount formula, the discount is

$$-1000(0.05-0.06)\,a_{\overline{8}|0.06} = 62.10.$$

The discount amortized in the kth coupon will be

$$\frac{-1000(0.05-0.06)}{(1+0.06)^{8-k+1}}.$$

Period k	Remaining Discount	Discount Amortized
0	62.10	
1	55.83	6.27
2	49.18	6.65
3	42.13	7.05
4	34.66	7.47
5	26.74	7.92
6	18.34	8.40
7	9.44	8.90
8	0.01	9.43

Callable Bond

Remark: sometimes a borrower would like to pay off a loan early to take advantage of new and lower interest rates. Some bonds are created with **call provisions** to allow the bonds to be paid off before the maturity date.

Definition

A **callable bond** is a bond for which there is a range of possible redemption dates, not known in advance to the bond purchaser. The redemption date is chosen by the bond issuer. The bond issuer will often pay a **call premium** to the bond holder if the bond is called early to compensate for the lost interest.

Pricing a Callable Bond

The price of a callable bond is based on the **earliest call date** or the **maturity date** whichever gives the lower price.

Redemption date to use for pricing a callable bond:

Type of Bond	**Find *n* using:**
Premium Bond ($r > j$)	Earliest possible redemption date
Discount Bond ($r < j$)	Latest possible redemption date

Example

A 10-year \$1000-face bond has an 8% coupon payable semiannually. The bond is callable in 7 years. An investor wishes to purchase the bond to yield 8.5% convertible semiannually. Find the price of the bond.

Solution

If the bond is priced using the earliest call date,

$$P = (1000 \cdot 0.04)a_{\overline{14}|\frac{0.085}{2}} + 1000v_{0.085/2}^{14} = \$974.02.$$

If the bond is priced using the maturity date,

$$P = (1000 \cdot 0.04)a_{\overline{20}|\frac{0.085}{2}} + 1000v_{0.085/2}^{20} = \$966.76.$$

The investor pays the lower price of \$966.76.

Example

A 12% coupon bond with semiannual coupons and a redemption amount of $1000 is issued with the condition that redemption can take place on any coupon date between 10 and 13 years from the issue date.

1. Find the price paid by an investor wishing a minimum yield of 10%.
2. Suppose the purchaser pays the maximum price for the range of redemption dates. Find the investor's yield rate if the bond issuer chooses a redemption date corresponding to the minimum price.
3. Suppose the purchaser pays the minimum price for the range of redemption dates. Find the investor's yield rate if the bond issuer chooses a redemption date corresponding to the maximum price.
4. Suppose the investor pays $1,130 for the bond. Find the minimum yield for the investor.

Solution (1 of 4)

- If the bond is called after the 20th coupon payment, the present value of the bond is

$$P_{20} = 60\, a_{\overline{20|}0.05} + 1000\, v_{0.05}^{20} = \$1,124.62.$$

- If the bond is called after the 26th coupon payment, the present value of the bond is

$$P_{26} = 60\, a_{\overline{26|}0.05} + 1000\, v_{0.05}^{26} = \$1,143.75.$$

- The investor should pay the lower price of $\$1,124.62$ for the bond.

Solution (2 of 4)

If the investor pays \$1,143.75 for the bond and the bond is called after the 20th coupon payment, the investor's yield is

$$
\begin{aligned}
1,143.75 &= 60 a_{\overline{20}|j/2} + 1000 v_{j/2}^{20} \\
j/2 &= 0.048602 \\
j &= 9.7203\%.
\end{aligned}
$$

Solution (3 of 4)

If the investor pays \$1,124.62 for the bond and the bond is called after the 26th coupon payment, the investor's yield is

$$\begin{aligned} 1,124.62 &= 60 a_{\overline{26}|j/2} + 1000 v_{j/2}^{26} \\ j/2 &= 0.051221 \\ j &= 10.2443\%. \end{aligned}$$

Solution (4 of 4)

- If the investor pays \$1,130 for the bond and the bond is called after the 20th coupon payment, the investor's yield is

$$\begin{aligned} 1,130 &= 60 a_{\overline{20}|j/2} + 1000 v_{j/2}^{20} \\ j/2 &= 0.049604 \\ j &= 9.9207\%. \end{aligned}$$

- If the investor pays \$1,130 for the bond and the bond is called after the 26th coupon payment, the investor's yield is

$$\begin{aligned} 1,130 &= 60 a_{\overline{26}|j/2} + 1000 v_{j/2}^{26} \\ j/2 &= 0.050875 \\ j &= 10.1750\%. \end{aligned}$$

The investor's minimum yield is 9.9207%.

Example

A ten-year \$1000-face bond pays semi-annual coupons at a 9% (annual) coupon rate. The bond is callable on or after its 3rd anniversary at a call price of \$1100. What is the maximum price an investor can pay for this bond and be assured of earning at least 8% convertible semi-annually?

Solution

Based on the earliest call date,

$$P = \left(1000 \cdot \frac{0.09}{2}\right) a_{\overline{6}|0.08/2} + 1100 v_{0.08/2}^{6} = \$1105.24.$$

Based on the maturity date,

$$P = \left(1000 \cdot \frac{0.09}{2}\right) a_{\overline{20}|0.08/2} + 1100 v_{0.08/2}^{20} = \$1113.59.$$

The maximum the investor should pay is \$1113.59.

Prices Between Coupon Dates

- If bonds are bought and sold only on the bonds' origination dates or coupon dates, the formula used earlier will price the bonds.
- If a bond is sold between coupon dates, the seller is entitled to compound interest for the fraction of the coupon period the bond was held.
- The selling price of the bond will be the accumulated value of the bond price on the last paid coupon date. This price includes accrued interest for a fraction of a coupon period and thus is called the **price-plus accrued** (or **purchase price** or **flat price**).
- Part of the flat price is the accrued interest in addition to principal value, the **true price** (or **market price**) removes the accrued interest.

Notation and Formulas

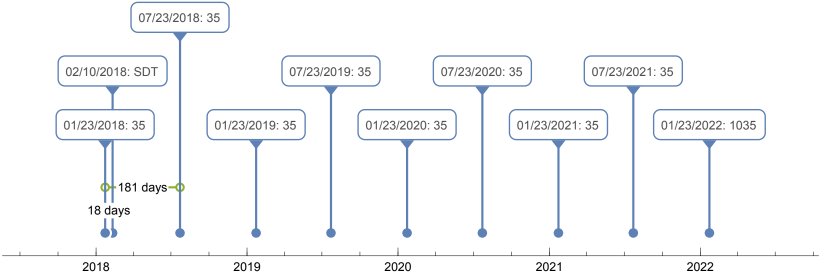
- t: fractional coupon period (number of days since last coupon divided by the number of days in a coupon period),
- P_0: price of bond immediately after last paid coupon,
- P_t: market price of bond t coupon periods after last coupon.

$$\begin{aligned}
\text{price-plus accrued} &= P_0(1+j)^t \text{ (compound interest)} \\
\text{accrued interest} &= t(F\,r) \text{ (simple interest)} \\
P_t &= \text{price-plus accrued} - \text{accrued interest} \\
P_t &= P_0(1+j)^t - t(F\,r)
\end{aligned}$$

Example

A bond with a par value of \$1000 has coupon payment dates of January 23 and July 23. The nominal coupon rate is 7% convertible semiannually. The bond matures on January 23, 2022. On January 23, 2018 a coupon payment of \$35 was made. The bond is sold on February 10, 2018 to yield 7.5% convertible semiannually. Find the market price and accrued interest of the bond on February 10, 2018.

Illustration



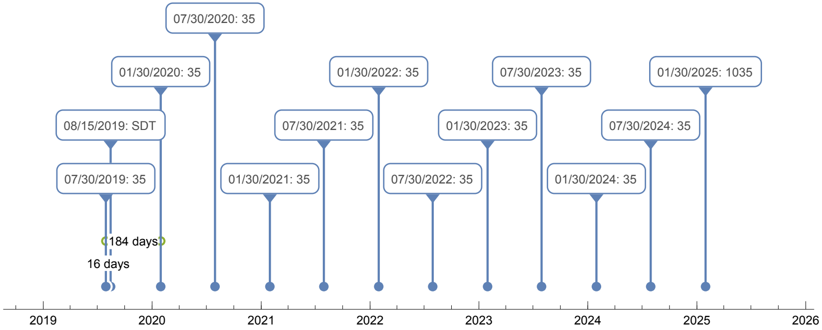
Solution

$$\begin{aligned}
t &= \frac{18}{181} \\
P_0 &= 35a_{\overline{8}|0.075/2} + 1000v^8_{0.075/2} = 982.9930 \\
\text{Flat price} &= 982.9930(1+0.075/2)^{18/181} = 986.5984 \\
\text{AI} &= \frac{18}{181}\cdot 35 = 3.4807 \\
P_t &= 982.9930(1+0.0375)^{18/181} - 3.4807 = 986.5984 - 3.
\end{aligned}$$

Example

A bond with a par value of $1000 has coupon dates of January 30 and July 30. The nominal coupon rate payable semiannually is 7%. The bond matures on January 30, 2025. The bond will be sold on August 15, 2019 to yield 10% convertible semiannually. Find the market price and the accrued interest of the bond.

Illustration



Using the DATE Worksheet

- 2ND 1 enters DATE worksheet
- DT1 *mm.ddyy* ENTER
- DT2 *mm.ddyy* ENTER
- DBD CPT

Note: $00 \leq yy \leq 49$ implies 2000–2049, while $50 \leq yy \leq 99$ implies 1950–1999.

Using and BOND Worksheet

- 2ND 9 enters BOND worksheet
- SDT (settlement date) *mm.ddyy* ENTER
- CPN (total annual coupon) ENTER
- RDT (redemption date) *mm.ddyy* ENTER
- RV (redemption value) ENTER
- ACT (actual number of days *vs.* 360 days/year)
- 2/Y (2 coupons/year *vs.* 1 coupon/year)
- YLD (annual yield)
- PRI CPT (market price)
- AI (accrued interest)

Varying Redemption Amounts

A 10-year 6% coupon bond with a face amount of \$1000 is callable at the option of the issuer on a coupon date in the 5th through the 10th years. At the end of the 5th year the bond is callable at par, in the 6th through 8th years the bond is callable at \$1100, and in the 9th or 10th years it is callable at \$1200.

1. What price should a purchaser pay in order to ensure a minimum nominal yield of 8%?
2. What is the purchaser's minimum yield if the bond is purchased for \$1000?

Solution (1 of 2)

Calculate the price of the bond on the callable coupon dates.

t	n	C	P
5	10	1000	918.89
5.5	11	1100	977.35
6	12	1100	968.61
6.5	13	1100	960.20
7	14	1100	952.12
7.5	15	1100	944.34
8	16	1100	936.87
8.5	17	1200	981.02
9	18	1200	972.13
9.5	19	1200	963.59
10	20	1200	955.37

The bond should be purchased for 918.89.

Solution (2 of 2)

Calculate the yield of the bond on the callable coupon dates.

t	n	C	j
5	10	1000	6.000%
5.5	11	1100	7.5022%
6	12	1100	7.3558%
6.5	13	1100	7.2322%
7	14	1100	7.1265%
7.5	15	1100	7.0352%
8	16	1100	6.9554%
8.5	17	1200	7.7091%
9	18	1200	7.5894%
9.5	19	1200	7.4825%
10	20	1200	7.3866%

The purchaser's minimum yield is 6%.

Serial Bonds

Sometimes a bond's redemption amount is paid out over time instead of in one lump sum.

Assume:

- the bond has redemption amounts $F_1, F_2, \ldots, F_m$,
- redeemed after $n_1, n_2, \ldots, n_m$ coupon periods,
- with coupon rates $r_1, r_2, \ldots, r_m$, and
- the bond is purchased to yield $j_1, j_2, \ldots, j_m$ respectively on the m pieces.

$$
\begin{aligned}
P_t &= (F_t r_t) a_{\overline{n_t}|j_t} + F_t v_{j_t}^{n_t} \\
P &= \sum_{t=1}^{m} P_t
\end{aligned}
$$

Example

On June 30, 2015 a corporation issued a 12% serial bond with face amount \$5M. Redemption is scheduled to take place at \$500,000 every June 30 from 2025 to 2029 and \$2.5M on June 30, 2030. Find the price of the bond on the date of issue at a yield of 15%.

Solution

- Present value of redemption payments:

$$\begin{aligned} K &= 0.5M\left(v_{0.075}^{20} + v_{0.075}^{22} + v_{0.075}^{24} + v_{0.075}^{26} + v_{0.075}^{28}\right) \\ &\quad + 2.5Mv_{0.075}^{30} \\ &= 0.5Mv_{0.075}^{20}\left(\frac{1 - v_{0.075}^{10}}{1 - v_{0.075}^{2}}\right) + 2.5Mv_{0.075}^{30} \\ &= 0.7356M \end{aligned}$$

- Using Makeham's formula the price of the bond is

$$\begin{aligned} P &= K + \frac{r}{j}(F - K) \\ &= 0.7356M + \frac{0.06}{0.075}(5 - 0.7356)M \\ &= \$4.1471M. \end{aligned}$$

Homework

- Read Chapter 4.
- Exercises: distributed on handout