Determinants of Interest Rates

MATH 372 *Financial Mathematics I*

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Fall 2021

Objectives

In this lesson we will learn:

- the factors which determine the level of interest rates charged in the real world, and
- the influence of central banks on interest rates.

Background

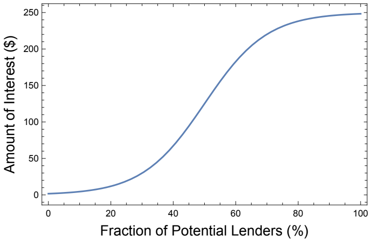
Consider a group of people each with $1000. Each individual can either spend the money now or save it for the future.

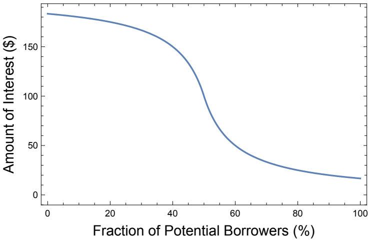
- From the perspective of a **lender**, interest is compensation received for delaying consumption.
- From the perspective of a **borrower**, interest is the cost incurred to make a purchase when the funds for that purchase are not available.

Incentives Affecting Borrowing/Lending

- Generally the number of individuals willing to lend is an increasing function of the interest rate.
- Generally the number of individuals willing to borrow is a decreasing function of the interest rate.

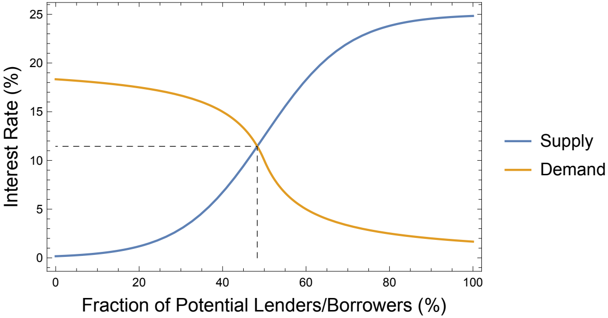
Suppose lenders have $1000 to loan and borrowers wish to borrow $1000.





Supply and Demand

Let the **price** of money be the interest rate and the **quantity** of money be the volume of loans made.



The **equilibrium point** is point at which supply and demand for loans is in balance. The interest rate associated with the equilibrium point is called the **equilibrium price for money**.

Quotation Bases for Interest Rates

- Interest rates have been expressed as **annual effective rates** (or effective rates for other time periods), **nominal rates**, **rates of discount**, and **continuously compounded rates** (**force of interest**).
- As long as two interest rates are expressed using the same basis, the two rates can be compared.
- In this section we will express interest rates as a continuously compounded rate in order to simplify certain calculations.

Averaging Interest Rates

Let δ_k be the force of interest for year k with $k = 1, 2, \ldots, n$ and let i_k be the corresponding annual effective rate.

Recall: $i_k = e^{\delta_k} - 1$.

The n-year accumulation factor is

$$a(n) = e^{\delta_1 + \delta_2 + \cdots + \delta_n} = (1 + i_1)(1 + i_2) \cdots (1 + i_n).$$

The average rate of return over n years is

$$\begin{aligned} \bar{\delta} &= \frac{\delta_1 + \delta_2 + \cdots + \delta_n}{n} \\ \bar{i} &= [(1 + i_1)(1 + i_2) \cdots (1 + i_n)]^{1/n} - 1. \end{aligned}$$

Components of Interest Rates

Several factors influence the supply and demand for loans (and hence the equilibrium point where supply and demand are in balance) including:

- term of a loan,
- probability a loan will not be fully repaid,
- inflation.

A continuously compounded interest rate can be considered the sum of it components: compensation for deferred consumption, compensation for the risk of default, and compensation for the loss of purchasing power.

Loan Term

In our discussion of the term structure of interest rates we saw that the yield curve is normally increasing with the term of the loan.

There are several explanations for this behavior:

- market segmentation theory,
- liquidity preference theory,
- expectations theory, and
- preferred habitat theory.

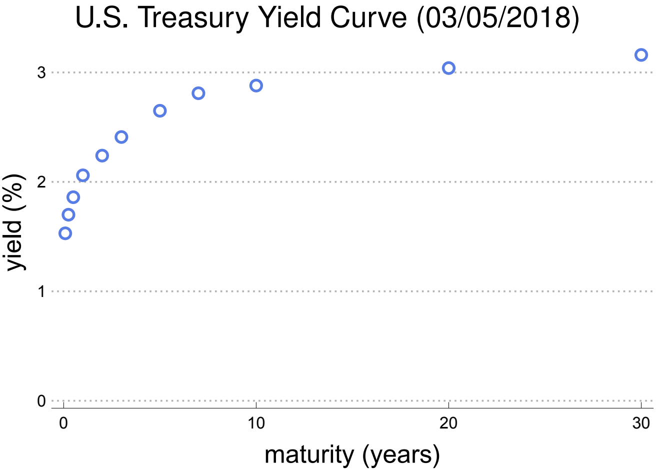
Theories (1 of 2)

- **Market segmentation theory** is based on the idea that the lenders who make short-term loans are a separate group from the lenders who make long-term loans. A similar statement holds for borrowers.
- **Liquidity preference theory** (or **opportunity cost theory**) implies that lenders have a natural preference for shorter-term loans since longer-term loans tie up money for a longer time and may prevent lenders from taking advantage of other investment opportunities which occur during the term of the loan.

Theories (2 of 2)

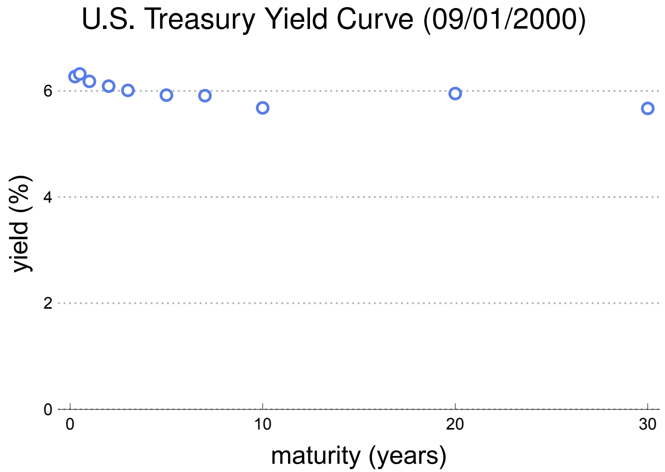
- **Expectations theory** argues that rates charged for longer-term loans provides information about the expected interest rates for future short-term loans (recall our calculation of forward rates).
- **Preferred habitat theory** implies that borrowers and lenders have preferred terms for which they would like to borrow or lend but can be persuaded to borrow or lend for a different term if the interest rate is sufficiently attractive.

Illustration (Normal Yield Curve)



Generally the longer the term of the loan, the higher the interest rate.

Inverted Yield Curve

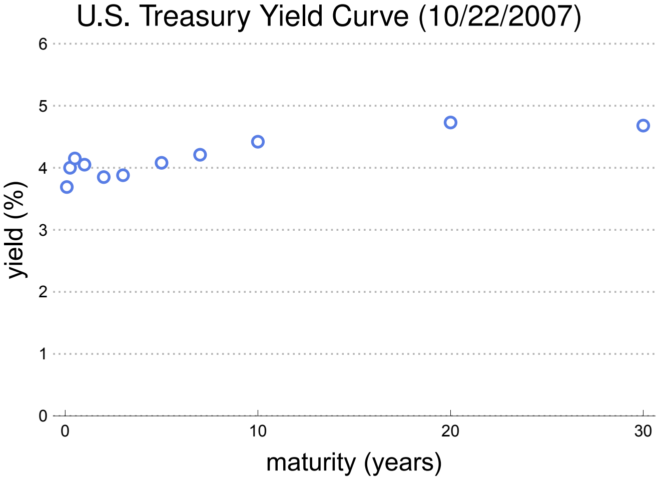


Comments

Causes of an inverted yield curve:

- shortage of lenders willing to make short-term loans,
- increase in demand for short-term loans,
- decrease in demand for longer-term loans (borrowers are pessimistic about the future and unwilling to take on debt).

Bow-Shaped Yield Curve



Comments

Bow-shaped yield curves occur when the interest rates for medium-term loans are higher (lower) than the interest rates for short-term and longer-term loans.

Bow-shaped yield curves do not usually last for long.

Default

- Whenever money is lent there is a chance that the loan will not be repaid.
- Lenders want to be compensated for the **risk of default.**

δ: continuously compounded interest rate in the absence of default risk.

r: continuously compounded interest rate actually charged by a lender.

s: compensation to the lender for the risk of default.

$$r = \delta + s$$

Example

A lender is considering lending \$5,000 for four years. To compensate for the loss of liquidity the lender requires a continuously compounded yield of 0.05 per year.

1. If the loan is to be repaid in a lump sum in four years, what is the repayment amount?
2. If there is a 3% chance the loan will not be repaid (not even partially), what is the yield rate on the loan?
3. What is the minimum continuously compounded interest rate the lender should charge to achieve an expected return of 0.05?

Solution

1. Repayment amount:

$$P = 5000e^{(0.05)(4)} = \$6,107.0138.$$

2. Yield rate with 3% chance of default:

$$\begin{aligned} \text{Expected repayment} &= (0.03)(0) + (0.97)(6107.0138) \\ 5923.8034 &= 5000e^{\delta(4)} \\ \delta &= 0.0424. \end{aligned}$$

3. Minimum yield rate to be repaid $6,107.0138:

$$\begin{aligned} (0.03)(0) + (0.97)P &= 6107.0138 \\ P &= 6295.8905 \\ 6295.8905 &= 5000e^{(\delta+s)(4)} \\ \delta + s &= 0.0576. \end{aligned}$$

Example

A lender requires a continuously compounded yield of 6% per year for a 20-year loan. There is a 5% chance the loan will not be repaid (not even partially). What is the minimum continuously compounded interest rate the lender should charge to achieve an expected return of 6%?

Partial Repayment and Collateral

If a borrower defaults on a loan by not repaying the full amount owed, they may still make a partial payment or forfeit some collateral (foreclosing on a house or repossessing a car).

Example

A lender is considering lending \$5,000 for four years. There is a 3% chance of default, but in the event of default the lender will be able to recover 75% of the amount owed. If the lender expects continuously compounded yield of 5%, what is the minimum interest rate that should be charged?

Solution

Expected value of amount repaid:

$$\begin{aligned} 0.97x + 0.03(0.75x) &= 5000e^{(0.05)(4)} \\ x &= \$6,153.1625. \end{aligned}$$

Minimum yield to be repaid $\$6,153.1625$:

$$\begin{aligned} 5000e^{(\delta+s)(4)} &= 6153.1625 \\ \delta + s &= 0.0519. \end{aligned}$$

Example

A lender is considering lending $4,000 for five years. There is a 7% chance of default, but in the event of default the lender will be able to recover 50% of the amount owed. If the lender expects continuously compounded yield of 6%, what is the minimum interest rate that should be charged?

Inflation

Definition

Inflation refers to the increase in the prices of goods and services over time (other terms include **monetary inflation** and **price inflation**).

The US Bureau of Labor Statistics measures the inflation rate by calculating the

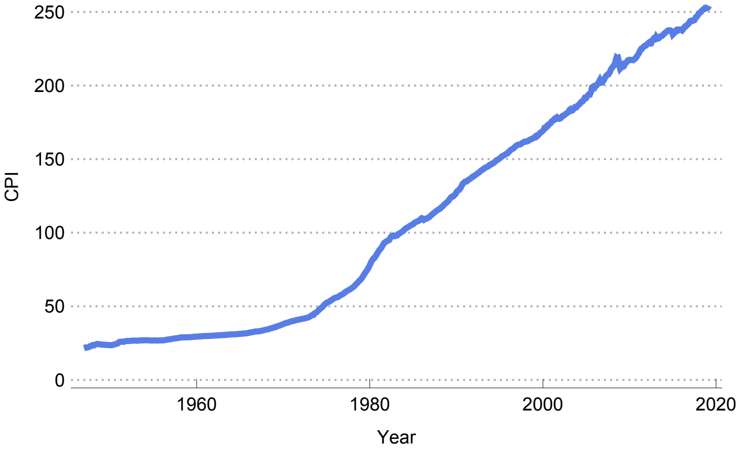
- **Consumer Price Index** (CPI), and
- **Producer Price Index** (PPI).

Consumer Price Index

The CPI measures the change over time in the prices paid for a standard collection of consumer goods and services including,

- food,
- clothing,
- medical services, and
- transportation.

Historical CPI (1947–2019)



Producer Price Index

The PPI measures the change over time in the prices of goods and services provided by US producers.

- The PPI is made of approximately 10,000 indices which measure the prices of goods and services in a variety of industries.
- Alphabetically these industries range from “abrasive product manufacturing”, “accessories and other apparel manufacturing”, *etc*.

Effect of Inflation on Borrowing and Lending

- Inflation means lenders will be repaid with dollars worth less than at the start of the loan.
- Lenders will want a higher interest rate than in the absence of inflation, to compensate for the lower value of future dollars.
- Borrowers will repay loans with inflated dollars and therefore will be willing to borrow at higher interest rates.
- Interest rates are set at the beginning of a loan and inflation can only be measured in retrospect.
- Lenders must set interest rates based on their expectation of the inflation rate.

Compensating for Inflation

Suppose the continuously compounded annual inflation rate is i, then the continuous compounded interest rate that incorporates deferred consumption, risk of default, and inflation can be expressed as

$$r^* = \delta + s + i.$$

Example

A lender is considering a 4-year loan of $3,000 and requires a continuously compounded yield of 5% if there were no inflation. Assuming there is no chance of default and that the continuously compounded rate of inflation is 3%, find the rate of interest the lender should charge.

Solution

Without inflation the lender would be repaid

$$3000e^{(0.05)(4)} = \$3,664.2083.$$

To have the same purchasing power when the inflation rate is 3%, the lender would have to be repaid

$$\left(3000e^{(0.05)(4)}\right) e^{(0.03)(4)} = 3000e^{(0.08)(4)} = \$4,131.3833.$$

Thus $r^* = 0.08$.

Example

An investor plans to purchase a 15-year zero-coupon bond with a face amount of $10,000. The continuously compounded rate of inflation is expected to be 2%. Assuming there is no risk of default, what continuously compounded rate must the investor earn in order to realize 6% continuously compounded growth in purchasing power?

Uncertainty About the Inflation Rate

Definition

An **inflation-protected loan** is a loan structured so that payments reflect the actual inflation during the term of the loan.

Example

A lender is considering a 4-year loan of $3,000 and specifies a continuously compounded interest rate of 5%. The repayment amount will be adjusted by a factor equal to the value of particular price index on the repayment date, divided by the value of that index on the date of the loan.

Repayment of Loan

- Suppose the index has a value of 2.7183 at the inception of the loan
- After 4 years the index has a value of 3.3201.
- The repayment amount for the loan would be

$$3000e^{(0.05)(4)}\frac{3.3201}{2.7183} = \$4,475.4214.$$

- Thus the nominal interest rate charged on the loan is

$$\frac{1}{4}\ln\frac{4475.4214}{3000} = 0.1000,$$

though the real interest rate of 5%.

Example

- Since an inflation-protected loan benefits the lender, this benefit must come at a cost.
- The cost of inflation protection is a reduction in the interest rate charged for the loan.
- Denote the continuously compounded reduction c.
- In the absence of default, the continuously compounded interest rate for an inflation-protected loan will be

$$r_1 = \delta - c.$$

- If i_a is the actual inflation rate during the term of the loan, then the actual rate earned on the loan is

$$r_1^{(a)} = \delta - c + i_a.$$

Non-Inflation-Protected Loans

- If a loan is not inflation-protected, the lender assumes the risk that inflation will diminish the value of the money repaid at the end of the loan term.
- The lender will require compensation for this risk in the form of a higher interest rate.
- The compensation can be partitioned as follows.

  i_e: compensation for expected inflation
  i_u: compensation for unexpected inflation

Rates of Non-Inflation-Protected Loans

- In the absence of default, the continuously compounded interest rate for a loan without inflation protection is

$$r_2 = \delta + i_e + i_u.$$

- The **"real" rate of interest** is the rate at which purchasing power of an investment grows.
- For an inflation-protected loan the real rate of interest is $r_1 = \delta - c$.
- For an inflation-protected loan the actual rate of interest is $r_1 = \delta - c + i_a$.
- The **"nominal" rate of interest** is the interest rate of a non-inflation-protected loan and is r_2.
- The difference between the nominal and real rates of interest is

$$r_2 - r_1 = i_e + i_u + c.$$

Comments

- The continuously compounded rate of compensation for unexpected inflation $i_u \geq 0$.
- In a time of decreasing prices $i_e < 0$. This situation is called **deflation** or **negative inflation**.
- If i_e is negative enough $r_2 < 0$. In these situations lenders will just store their money rather than lending it at a negative rate.
- The Bank of Japan and the European Central Bank pay a negative rate of interest in order to discourage member banks from storing their money and to encourage them to make loans and stimulate the economy.

Including Risk of Default and Inflation

Real-world loans have a risk of default as well as expected and unexpected inflation, thus

$$r^* = \delta + s + i_e + i_u.$$

- The **credit spread**, **spread for credit risk**, or simply the **spread** is the quantity s.
- The spread can be expressed as

  $$s = r^* - r_2,$$

  the difference between interest rate for a given loan minus the interest rate for the same loan assuming no risk of default.
- The credit spread for a loan will depend on the loan term. The curve created by plotting the credit spread versus the loan term is called the **spread curve**.

Summary

	interest rate with no default risk	**interest rate with default risk**
with no inflation	δ	$r = \delta + s$
with a known rate of inflation	$\delta + i$	$r^* = \delta + s + i$
with an unknown rate of inflation ("nominal" interest rate)	$r_2 = \delta + i_e + i_u$	$r^* = \delta + s + i_e + i_u$
inflation-protected loan: contractual rate ("real" interest rate)	$r_1 = \delta - c$	$\delta + s - c$
inflation-protected loan: actual rate paid ("nominal" interest rate)	$r_1^{(a)} = \delta - c + i_a$	$\delta + s - c + i_a$

Example

ABC Bank offers a six-year loan. It is repaid with a single payment of principal and interest at $t = 6$. ABC Bank wants to receive an annual rate of 7.50% compounded continuously to compensate for deferred consumption. One percent of the borrowers will default. When a borrower defaults, the bank will be able to recover 50% of the amount owed at time $t = 6$. Find s the credit spread calculated as annual rate compounded continuously so that ABC Bank can achieve its desired compensation rate.

Example

XYZ Bank offers a four-year loan. The annual interest rate that must be paid is 4.5% plus the rate of inflation compounded continuously. The PA state government borrows $100,000,000 from XYZ Bank. During the first year the actual rate of inflation 2.2% compounded continuously, while during the next three years the rates were 2.4%, 3.2%, and 2.8% respectively. Find the amount that PA will owe XYZ Bank at $t = 4$. Assume PA is a risk-free borrower.

Homework

- Read printed handout
- Exercises: distributed on handout