Cashflow Duration and Immunization (Part I)

MATH 372 *Financial Mathematics I*

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Fall 2021

Objectives

In this lesson we will

- explore the relationship between the market value of a fixed series of payments and the yield rate or term structure of interest rates used to value the payments,
- define and calculate the **duration** of a bond or general sequence of payments,
- determine a method for matching assets and liabilities (**immunization**).

Most of the formulas developed in this lesson will make liberal use of Taylor's Theorem.

Basic Asset-Liability Management

- Insurance companies collect premiums and invest them.
- The **assets** of the insurance company are premiums and interest earned.
- The future claims in insurance policies are the **liabilities** if the insurance company.
- The insurance company must manage its assets in such a way as to ensure that liabilities can be paid.

Simple Example

A company pay liabilities of $1,000 in one year and $3,000 in two years. The company may invest in the following two zero-coupon bonds.

Maturity (years)	Annual Effective Yield	Par Value
1	5%	1,000
2	6%	1,000

What is the cost of matching the liability cash flows?

Solution

The company can purchase one 1-year bond and three 2-year bonds. The total cost will be

$$\frac{1000}{1+0.05} + \frac{3(1000)}{(1+0.06)^2} = \$3,633.3702.$$

Remark: the company can cover its liabilities exactly with the bonds. This process is called **cash flow matching** or **dedication.**

Matching with Coupon Bonds

Matching is more complicated when the bonds pay coupons.

Example

Joanne has liabilities that require payments of $2,000 six months from now and $4,000 one year from now. There are two available investments:

- **Bond A:** a 6-month bond with a face amount of 1,000, a 4% nominal annual coupon rate payable semiannually, and a 5% nominal annual yield rate convertible semiannually.
- **Bond B:** a 1-year bond with a face amount of 1,000, a 6% nominal annual coupon rate payable semiannually, and a 8% nominal annual yield rate convertible semiannually.

Find the amount of each bond to purchase and the total cost needed to match the cash flows exactly. Assume that Joanne can purchase fractions of a bond.

Solution (1 of 2)

- Start with the payment of \$4,000 to be made one year from now. This cash flow must be matched by the 1-year bond, since the 6-month bond will have already matured.
- One of the 1-year bonds will pay \$1,030 at maturity (redemption amount plus coupon), thus Joanne must purchase

$$\frac{4000}{1030} = 3.8835 \text{ one-year bonds.}$$

- At 6-months each purchased 1-year bond will make a coupon payment of \$30 for a total of $(3.8835)(30) = \$116.5049$, thus

$$2000 - 116.5049 = \$1,883.4951$$

  must still be matched.
- One of the 6-month bonds will pay \$1,020 at maturity, thus Joanne should purchase

$$\frac{1883.4951}{1020} = 1.8466 \text{ six-month bonds.}$$

Solution (2 of 2)

Joanne's cost to match the cash flows is

$$\begin{aligned} P &= 1.8466\left(\frac{1020}{1+0.025}\right)+3.8835\left(\frac{30}{1+0.04}+\frac{1030}{(1+0.04)^2}\right) \\ &= (1.8466)(995.1220)+(3.8835)(981.139) \\ &= \$5647.85 \end{aligned}$$

Asset-Liability Matching

- In business a company may make commitments involving future income and outgo of funds.
- At time $t = 0$ the company may project a net outgoing payment L_t at time $t > 0$ called a **liability due.**
- At time $t = 0$ the company may expect revenue or investment income A_t maturing at time $t > 0$ called **asset income.**
- If it can be arranged that $L_t = A_t$ for all t, then projected liabilities and asset income are **exactly matched**.

Example

A company is terminating three employees of ages 53, 58, and 60. The company will provide a severance package that pays $20,000 per year to each employee until age 65 plus $100,000 at age 65. Find the cost of the severance package if the company plans to exactly match the liabilities with the purchase of bonds with redemption amounts of $100,000 and 20% annual coupons. The yield rate for a 12-year coupon bond is 10%, for a 7-year coupon bond is 11%, and for a 5-year coupon bond is 12%.

Solution

Cost of the bonds:

$$
\begin{aligned}
P_{12} &= 100,000\left(0.20\,a_{\overline{12}|0.10} + v_{0.10}^{12}\right) = \$168,136.92 \\
P_{7} &= 100,000\left(0.20\,a_{\overline{7}|0.11} + v_{0.11}^{7}\right) = \$142,409.77 \\
P_{5} &= 100,000\left(0.20\,a_{\overline{5}|0.12} + v_{0.12}^{5}\right) = \$128,838.21 \\
\text{Total} &= \$439,384.90
\end{aligned}
$$

Interest Rate Risk

Interest rate risk occurs because the market value of a company's investments

decreases when interest rate rise, and

increases when interest rates fall.

An investment manager uses the concept of **duration** to measure the company's interest rate risk.

There are two related types of duration:

- Macaulay duration D_{mac},
- modified duration D_{mod}.

Macaulay Duration

Definition

The **Macaulay duration** denoted D_{mac} of an investment is the weighted average time at which the investment's payments will occur. The weights are the present values of the payments.

Suppose an investment has n cash flows $CF_1, CF_2, \ldots, CF_n$ occurring at times $t_1, t_2, \ldots, t_n$.

The total present value of the cash flows is $P = \sum_{k=1}^{n} \frac{CF_k}{(1+i)^{t_k}}$.

Define the weight of the kth cash flow as

$$w_k = \frac{CF_k(1+i)^{-t_k}}{P} = \frac{CF_k(1+i)^{-t_k}}{\sum_{k=1}^{n} CF_k(1+i)^{-t_k}}.$$

The Macaulay duration is then

$$D_{mac} = \sum_{k=1}^{n} t_k w_k = \frac{\sum_{k=1}^{n} t_k CF_k(1+i)^{-t_k}}{\sum_{k=1}^{n} CF_k(1+i)^{-t_k}}.$$

Special Cases of Macaulay Duration

If the payments occur at $t = 1, 2, \ldots, n$, then

$$D_{mac} = \sum_{t=1}^{n} t\, w_t = \frac{\sum_{t=1}^{n} t\, CF_t (1+i)^{-t}}{\sum_{t=1}^{n} CF_t (1+i)^{-t}}.$$

Example

An investment pays \$2,500 in one year, \$3,000 in two years, and \$4,000 in three years. The investment was purchased to yield an annual effective rate of 10%. Find its Macaulay duration.

$$
\begin{aligned}
P &= \frac{2500}{1+0.10} + \frac{3000}{(1+0.10)^2} + \frac{4000}{(1+0.10)^3} = \$7,757.3253 \\
w_1 &= \frac{2500(1+0.10)^{-1}}{7757.3253} = 0.2930 \\
w_2 &= \frac{3000(1+0.10)^{-2}}{7757.3253} = 0.3196 \\
w_3 &= \frac{4000(1+0.10)^{-3}}{7757.3253} = 0.3874 \\
D_{mac} &= (1)(0.2930) + (2)(0.3196) + (3)(0.3874) = 2.0944
\end{aligned}
$$

Example

An investment pays \$1000 in one year, \$2000 in two years, and \$3000 in three years. The investment has been purchased to yield an annual effective rate of 9%. Find the investments Macaulay duration.

Modified Duration

Since the market value of a fixed series of payment is sensitive to changes in yield rate or term structure, one way to measure this sensitivity is to use the derivative of present value with respect to changes in yield or term structure.

Definition

The **modified duration** (or **volatility**) of an investment denoted D_{mod} is the negative of the derivative of the price of the investment with respect to the interest rate divided by the price of the investment.

$$D_{mod} = -\frac{dP/di}{P}$$

Remark: when the term "duration" is mentioned alone it typically means Macaulay duration.

Modified Duration of Zero Coupon Bond

The present value of an n-year zero coupon bond with redemption value 1 is

$$
\begin{aligned}
P &= (1+i)^{-n}. \\
\frac{dP}{di} &= -n(1+i)^{-n-1} \\
D_{mod} &= -\frac{dP/di}{P} = \frac{n(1+i)^{-n-1}}{(1+i)^{-n}} = n(1+i)^{-1} = n\,v
\end{aligned}
$$

Example

An investment pays \$2,500 in one year, \$3,000 in two years, and \$4,000 in three years. The investment was purchased to yield an annual effective rate of 10%. Find its modified duration.

$$
\begin{aligned}
P &= 2500(1+i)^{-1} + 3000(1+i)^{-2} + 4000(1+i)^{-3} \\
-\frac{dP}{di} &= 2500(1+i)^{-2} + (2)3000(1+i)^{-3} + (3)4000(1+i)^{-4} \\
D_{mod} &= \frac{2500(1.10)^{-2} + (2)3000(1.10)^{-3} + (3)4000(1.10)^{-4}}{2500(1.10)^{-1} + 3000(1.10)^{-2} + 4000(1.10)^{-3}} \\
&= \frac{14770.1660}{7757.3253} = 1.9040
\end{aligned}
$$

Special Cases of Modified Duration

If the cash flows are equally spaced in time then

$$\begin{aligned} P &= \sum_{t=1}^{n} \left(CF_t (1+i)^{-t} \right) \\ -\frac{dP}{di} &= \sum_{t=1}^{n} \left(t\, CF_t (1+i)^{-t-1} \right) \\ D_{mod} &= \frac{\sum_{t=1}^{n} \left(t\, CF_t (1+i)^{-(t+1)} \right)}{\sum_{t=1}^{n} \left(CF_t (1+i)^{-t} \right)}. \end{aligned}$$

Recall that the Macaulay duration for equally spaced cash flows was expressed as

$$D_{mac} = \frac{\sum_{t=1}^{n} \left(t\, CF_t (1+i)^{-t} \right)}{\sum_{t=1}^{n} \left(CF_t (1+i)^{-t} \right)} = (1+i) D_{mod}.$$

Macaulay *vs*. Modified Duration

The Macaulay duration denoted D_{mac} is

$$D_{mac} = (1+i)D_{mod} = \frac{-(1+i)}{P}\,\frac{dP}{di}.$$

Example

Show the Macaulay duration for an n-year zero coupon bond is n.

Solution

$$\begin{aligned} D_{mod} &= n\,v \\ D_{mac} &= n\,v\,(1+i) = n \end{aligned}$$

Bonds and Duration

A bond's duration (both D_{mac} and D_{mod}) is affected by the bond's term, coupon rate, change in yield, and passage of time.

term: a bond with a longer term has a longer duration.

coupon rate: a bond with a higher coupon rate will have a shorter duration.

yield: a bond with a higher yield will have a shorter duration.

time: a bond's duration decreases with the passage of time.

Formulas for Duration

- n-payment level annuity

$$\begin{aligned} D_{mac} &= -(1+i)\frac{\frac{d}{di}\left[\sum_{k=1}^{n}(1+i)^{-k}\right]}{a_{\overline{n}|i}} \\ &= -(1+i)\frac{\sum_{k=1}^{n} -k(1+i)^{-k-1}}{a_{\overline{n}|i}} \\ &= \frac{\sum_{k=1}^{n} k\, v^k}{a_{\overline{n}|i}} \\ &= \frac{(Ia)_{\overline{n}|i}}{a_{\overline{n}|i}} \end{aligned}$$

Example

An annuity pays \$5,000 at the end of each of the next 10 years. Find the Macaulay duration at rate $i = 0.07$.

Solution

$$D_{mac} = \frac{(Ia)_{\overline{10}|0.07}}{a_{\overline{10}|0.07}} = \frac{34.7391}{7.0236} = 4.9461$$

Duration of a Coupon Bond (1 of 2)

Suppose a coupon bond has face amount F, redemption amount C, coupon rate r, n annual coupons until maturity, and is valued at yield rate j per year.

$$P = (F\,r)\,a_{\overline{n}|j} + C v_j^n = (F\,r)\sum_{t=1}^{n}(1+j)^{-t} + C(1+j)^{-n}$$

$$D_{mod} = -\frac{1}{P}\frac{dP}{dj} = \frac{(F\,r)\sum_{t=1}^{n} t(1+j)^{-t-1} + n\,C(1+j)^{-n-1}}{(F\,r)\sum_{t=1}^{n}(1+j)^{-t} + C(1+j)^{-n}}$$

$$D_{mac} = \frac{(F\,r)\sum_{t=1}^{n} t(1+j)^{-t} + n\,C(1+j)^{-n}}{(F\,r)\sum_{t=1}^{n}(1+j)^{-t} + C(1+j)^{-n}}$$

$$= \frac{F\,r\,(Ia)_{\overline{n}|j} + n\,C\,v_j^n}{F\,r\,a_{\overline{n}|j} + C\,v_j^n}$$

Duration of a Coupon Bond (2 of 2)

Suppose a coupon bond has face amount F, redemption amount F, coupon rate r, n annual coupons until maturity, and is valued at yield rate j per year.

$$D_{mac} = \frac{F\, r(Ia)_{\overline{n}|j} + n\, F\, v_j^n}{F\, r\, a_{\overline{n}|j} + F\, v_j^n} = \frac{r(Ia)_{\overline{n}|j} + n\, v_j^n}{r\, a_{\overline{n}|j} + v_j^n}$$

If the bond is priced at par so that $F = C$ and $r = j$ then

$$D_{mac} = \frac{j(Ia)_{\overline{n}|j} + n\, v_j^n}{j\, a_{\overline{n}|j} + v_j^n} = \frac{j\, \frac{\ddot{a}_{\overline{n}|j} - n\, v_j^n}{j} + n\, v_j^n}{j\, \frac{1 - v_j^n}{j} + v_j^n} = \ddot{a}_{\overline{n}|j}.$$

Example

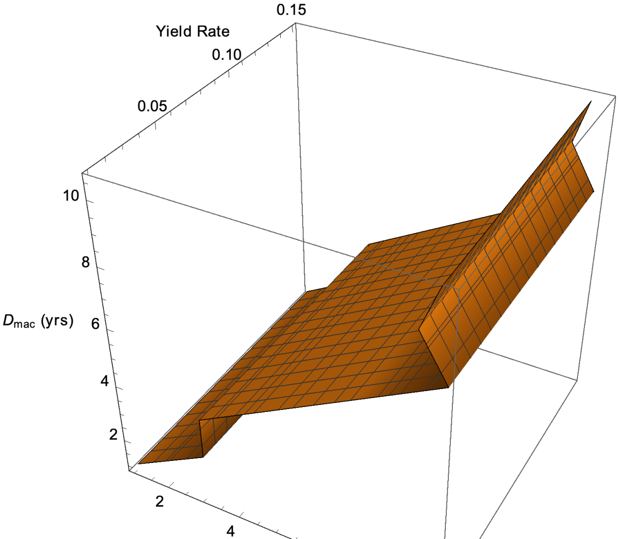
Find the Macaulay durations of coupon bonds with maturities of 3, 5, and 10 years with annual coupons of 10% for yields of 5%, 10%, and 15%.

$$D_{mac} = \frac{r\,(Ia)_{\overline{n}|j} + n\,v_j^n}{r\,a_{\overline{n}|j} + v_j^n}$$

Yield Rate	**Maturity**		
	3	**5**	**10**
0.05	2.7525	4.2535	7.2698
0.10	2.7355	4.1699	6.7590
0.15	2.7183	4.0830	6.2367

Surface Plot of Macaulay Duration

$$D_{mac} = \frac{r(Ia)_{\overline{n}|j} + n\,v_j^n}{r\,a_{\overline{n}|j} + v_j^n}$$



Insight and Interpretation

Using the definition of the derivative,

$$\begin{aligned} \frac{P(i+\Delta i) - P(i)}{\Delta i} &\approx \frac{dP}{di} \\ P(i+\Delta i) - P(i) &\approx (\Delta i)\,\frac{dP}{di} = -(\Delta i)(D_{mod})P(i) \end{aligned}$$

which approximates the change in the price of a bond for small changes in the interest rate.

- D_{mac} is the weighted average time when cash flows occur.
- D_{mod} is the proportionate change in price when interest rates change.

Approximating Change in Price

Recall the linear approximation formula:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

If $P(i)$ is the price of an investment depending on interest rate i, then

$$\begin{aligned} P(i) &\approx P(i_0) + P'(i_0)(i - i_0) \\ &= P(i_0) - D_{mod}(i_0)P(i_0)(i - i_0) \end{aligned}$$

which is called the **first-order modified approximation** for $P(i)$.

The change in price can be expressed as

$$\Delta P \approx -D_{mod}(i_0)P(i_0)\Delta i.$$

Example

Suppose the effective annual yield for all maturities of zero coupon bonds is 7.5%. Find the modified duration, Macaulay duration, the approximate change, and exact change in the price of zero coupon bonds with redemption amounts of 1000 with maturities 1, 3, and 5 years when the yield changes to 7.49%.

Solution: 1-Year Bond

$$
\begin{aligned}
P &= \frac{1000}{1+0.075} = \$930.2326 \\
D_{mac} &= 1 \text{ (zero-coupon bond)} \\
D_{mod} &= \frac{D_{mac}}{1+0.075} = 0.9302 \\
-D_{mod} P \Delta i &= -0.9302(930.2326)(0.0001) = 0.0865 \approx \Delta P \\
\Delta P &= \frac{1000}{1+0.0749} - \frac{1000}{1+0.075} = 0.0865
\end{aligned}
$$

Example

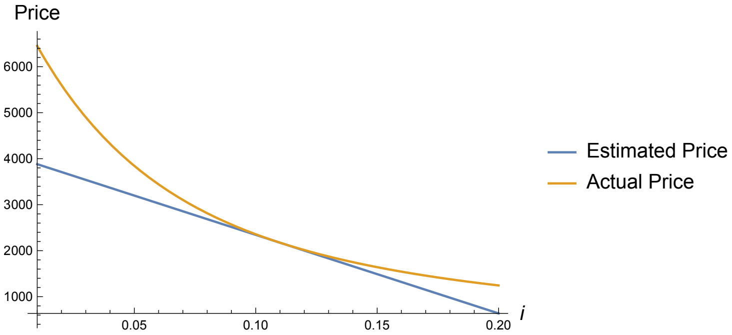
Consider an annuity immediate with 30 annual payments of \$250. If the annual effective interest rate is 11%, then

$$
\begin{aligned}
P(0.11) &= 250\, a_{\overline{30}|0.11} = \$2173.4481 \\
D_{mac}(0.11) &= \frac{(Ia)_{\overline{30}|0.11}}{a_{\overline{30}|0.11}} = \frac{75.8148}{8.6938} = 8.7206 \\
D_{mod}(0.11) &= \frac{D_{mac}}{1+0.11} = 7.8564.
\end{aligned}
$$

Thus the price of the annuity for $i \approx 0.11$ is approximated by the linear expression,

$$
\begin{aligned}
P(i) &\approx P(0.11) - D_{mod}(0.11)P(0.11)(i - 0.11) \\
&= 2173.4481 - 17075.3994(i - 0.11).
\end{aligned}
$$

Illustration



Since $P(i)$ is a concave up function, the linear approximation underestimates the actual price.

Duration with Nominal Rates of Interest

If the interest rate is given as a nominal rate $i^{(m)}$ then,

$$
\begin{aligned}
D_{mod}(i^{(m)}) &= \frac{-P'(i^{(m)})}{P(i^{(m)})} \\
&= \frac{-1}{P(i^{(m)})} \frac{dP(i^{(m)})}{di^{(m)}} = \frac{-1}{P(i^{(m)})} \frac{P(i)}{di} \frac{di}{di^{(m)}} \\
&= \frac{-P'(i)}{P(i^{(m)})} m \left(1 + \frac{i^{(m)}}{m}\right)^{m-1} \frac{1}{m} \\
&= \frac{-P'(i)}{P(i)} \frac{\left(1 + \frac{i^{(m)}}{m}\right)^{m}}{1 + \frac{i^{(m)}}{m}} = \frac{-P'(i)}{P(i)} \frac{1+i}{1 + \frac{i^{(m)}}{m}} \\
D_{mod}(i^{(m)}) &= \frac{D_{mac}}{1 + \frac{i^{(m)}}{m}}.
\end{aligned}
$$

Change in Price of an Asset

At a nominal rate of interest $i^{(m)}$ the linear approximation to the price of an asset is

$$P(i^{(m)}) = P(i_0^{(m)}) - D_{mod}(i_0^{(m)})P(i_0^{(m)})(i^{(m)} - i_0^{(m)}).$$

Example

A 5-year bond with a face amount of $1,000 pays semiannual coupons at a 6% rate. Its nominal yield is 8% convertible semiannually. Using the first-order modified approximation, estimate the bond's price at a nominal yield of 8.2%.

Solution

- Let $i^{(2)} = 0.08$, then the price of the bond is

$$P(i^{(2)}) = 30\,a_{\overline{10}|\frac{i^{(2)}}{2}} + 1000\,v^{10}_{i^{(2)}/2} = \$918.89.$$

- Macaulay duration of the bond is

$$D_{mac} = \frac{1}{2}\left[\frac{0.03(Ia)_{\overline{10}|0.08/2} + 10\,v^{10}_{0.08/2}}{0.03\,a_{\overline{10}|0.08/2} + v^{10}_{0.08/2}}\right] = 4.3615 \text{ years.}$$

- Modified duration of the bond

$$D_{mod}(i^{(2)}) = \frac{D_{mac}}{1 + \frac{0.08}{2}} = \frac{4.3615}{1.04} = 4.1937.$$

- Estimated price of the bond

$$P(8.2\%) = 918.89 - (4.1937)(918.89)(0.082 - 0.08) = \$911.18$$

First-Order Macaulay Approximation

Comments:

- We will develop an approximation to the price of an asset which is more accurate than the first-order modified duration approximation.
- Let $P(i)$ be the present value of an asset (annuity, bond, *etc*.) and define $V_t(i) = P(i)(1+i)^t$ to be the time-t value of the asset.

Find a critical number for $V_t(i)$.

$$\begin{aligned} 0 &= \frac{d}{di}\left[V_t(i)\right] = P'(i)(1+i)^t + P(i)t\,(1+i)^{t-1} \\ P(i)t\,(1+i)^{t-1} &= -P'(i)(1+i)^t \\ t &= -(1+i)\frac{P'(i)}{P(i)} = D_{mac}(i) \end{aligned}$$

Macaulay Value Function

Define the function,

$$V(i) = P(i)(1+i)^{D_{mac}(i_0)}$$

where $V_t'(i_0) = 0$.

The linear approximation to $V(i)$ can be expressed as

$$\begin{aligned} V(i) &\approx V(i_0) + V'(i_0)(i - i_0) = V(i_0) \\ P(i)(1+i)^{D_{mac}(i_0)} &\approx P(i_0)(1+i_0)^{D_{mac}(i_0)} \\ P(i) &\approx P(i_0)\left(\frac{1+i_0}{1+i}\right)^{D_{mac}(i_0)} . \end{aligned}$$

This approximation to $P(i)$ is called the **first-order Macaulay approximation.**

Example

Consider an annuity immediate with 30 annual payments of \$250. If the annual effective interest rate is 11%, then

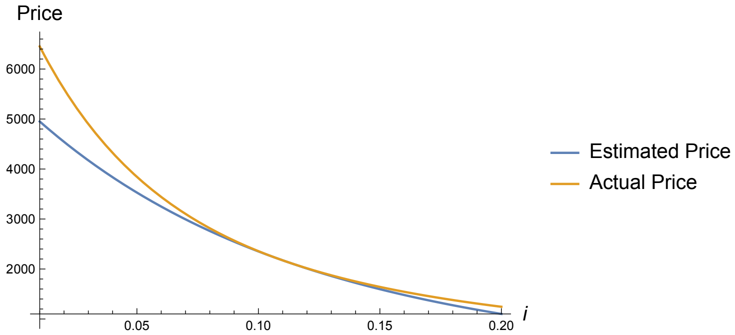
$$\begin{aligned} P(0.11) &= 250\, a_{\overline{30}|0.11} = \$2173.4481 \\ D_{mac}(0.11) &= \frac{(Ia)_{\overline{30}|0.11}}{a_{\overline{30}|0.11}} = \frac{75.8148}{8.6938} = 8.7206. \end{aligned}$$

Use the first-order Macaulay approximation to estimate the price of the annuity if $i = 0.06$, $i = 0.09$, $i = 0.13$.

Solution

$$\begin{aligned} P(i) &\approx P(0.11)\left(\frac{1+0.11}{1+i}\right)^{8.7206} \\ P(0.06) &\approx 3248.7015 \\ P(0.09) &\approx 2546.8951 \\ P(0.13) &\approx 1860.0185 \end{aligned}$$

Illustration



The first-order Macaulay approximation still underestimates the actual value but by a smaller amount than the first-order modified approximation.

Example

An investment pays \$3,000 in one year, \$2,000 in two years, and \$1,000 in three years. The investment is purchased to yield $i = 0.10$.

- Find the price of the investment.
- Find the price of the investment if $i = 0.105$.
- Use the first-order modified duration to estimate the price of the investment if $i = 0.105$.
- Use the first-order Macaulay duration to estimate the price of the investment if $i = 0.105$.

Solution

$$
\begin{aligned}
P(0.10) &= \frac{3000}{1+0.10} + \frac{2000}{(1+0.10)^2} + \frac{1000}{(1+0.10)^3} = \$5,131.4801 \\
P(0.105) &= \frac{3000}{1+0.105} + \frac{2000}{(1+0.105)^2} + \frac{1000}{(1+0.105)^3} = \$5,094.0623 \\
D_{mod}(0.10) &= \frac{3000(1.10)^{-2} + 4000(1.10)^{-3} + 3000(1.10)^{-4}}{P(0.10)} \\
&= \frac{7533.6384}{5131.4801} = 1.4681 \\
P(0.105) &\approx P(0.10) - D_{mod}(0.10)P(0.10)(0.105 - 0.10) = \$5,093.812 \\
D_{mac}(0.10) &= (1+0.10)D_{mod}(0.10) = 1.6149 \\
P(0.105) &\approx P(0.10)\left(\frac{1+0.10}{1+0.105}\right)^{D_{mac}(0.10)} = \$5,094.0351
\end{aligned}
$$

Homework

- Read Chapter 7
- Exercises: distributed on handout