Cashflow Duration and Immunization (Part II)

MATH 372 *Financial Mathematics I*

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Objectives

In this lesson we will

- explore the relationship between the market value of a fixed series of payments and the yield rate or term structure of interest rates used to value the payments,
- define and calculate the **duration** of a bond or general sequence of payments,
- determine a method for matching assets and liabilities (**immunization**).

Most of the formulas developed in this lesson will make liberal use of Taylor's Theorem.

Taylor's Theorem

Theorem

Suppose function $f \in C^n[a,b]$ and that $f^{(n+1)}$ exists on $[a,b]$ and let $x_0 \in [a,b]$. For every $x \in [a,b]$, there exists a number $z(x)$ between x_0 and x such that $f(x) = P_n(x) + R_n(x)$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

*is called the n**th Taylor polynomial** for f about x_0, and*

$$R_n(x) = \frac{f^{(n+1)}(z(x))}{(n+1)!}(x - x_0)^{n+1}$$

*is called the n**th Taylor remainder**.*

Taylor Polynomials

The 1st Taylor polynomial is a linear approximation to $f(x)$ at $x = x_0$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

The 2nd Taylor polynomial is a quadratic approximation to $f(x)$ at $x = x_0$.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$$

Modified Convexity

The **modified convexity** of the price $P(i)$ of an asset is defined as

$$C_{mod} = \frac{P''(i)}{P(i)}.$$

For a sequence of n equally spaced cash flows,

$$C_{mod} = \frac{\sum_{t=1}^{n} t(t+1)CF_t(1+i)^{-t-2}}{\sum_{t=1}^{n} CF_t(1+i)^{-t}}.$$

Incorporating the modified convexity improves the approximation of the price of an asset when the interest rate changes.

$$P(i) \approx P(i_0) - D_{mod}(i_0)P(i_0)(i-i_0) + \frac{1}{2}C_{mod}(i_0)P(i_0)(i-i_0)^2$$

This is called the **second-order modified approximation**.

Example

Consider an annuity immediate with 30 annual payments of \$250. If the annual effective interest rate is 11%, then

$$
\begin{aligned}
P(0.11) &= 250\, a_{\overline{30}|0.11} = \$2173.4481 \\
D_{mac}(0.11) &= \frac{(Ia)_{\overline{30}|0.11}}{a_{\overline{30}|0.11}} = \frac{75.8148}{8.6938} = 8.7206.
\end{aligned}
$$

- Use the first-order modified approximation to estimate the price of the annuity if $i = 0.10$.
- Use the second-order modified approximation to estimate the price of the annuity if $i = 0.10$.

Solution

$$
\begin{aligned}
P(0.11) &= 250 a_{\overline{30}|0.11} = \$2173.4481 \\
D_{mod}(0.11) &= 7.8564 \\
C_{mod}(0.11) &= 108.3647 \\
P(0.10) &\approx P(0.11) - D_{mod}(0.11) P(0.11)(0.10 - 0.11) \\
&= \$2,344.2021 \\
P(0.10) &\approx P(0.11) - D_{mod}(0.11) P(0.11)(0.10 - 0.11) \\
&\quad + \frac{1}{2} C_{mod}(0.11) P(0.11)(0.10 - 0.11)^2 \\
&= \$2,355.9784 \\
P(0.10) &= 250 a_{\overline{30}|0.10} = \$2,356.7286
\end{aligned}
$$

The second-order modified approximation is more accurate than the first-order modified approximation.

Example (1 of 3)

Consider an annuity-immediate with 30 annual payments of \$500. At an annual effective interest rate of 5%,

$$\begin{aligned}
\text{PV} &= 500\, a_{\overline{30|}0.05} = \$7,686.2255 \\
D_{mac} &= \frac{(Ia)_{\overline{30|}0.05}}{a_{\overline{30|}0.05}} = 11.9691 \\
D_{mod} &= \frac{D_{mac}}{(1+0.05)} = 11.3992 \\
C_{mod} &= \frac{\sum_{t=1}^{30} t(t+1)(1+0.05)^{-(t+2)}}{a_{\overline{30|}0.05}} = 202.038.
\end{aligned}$$

Example (2 of 3)

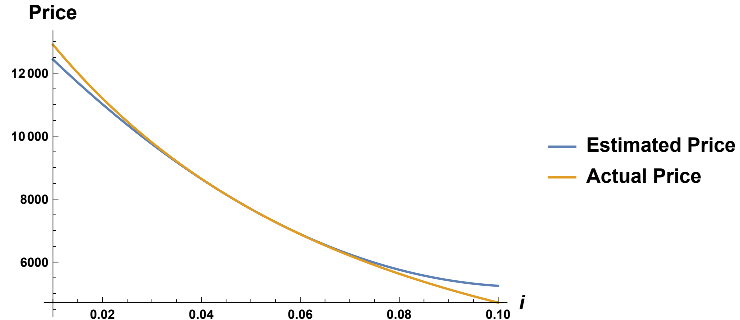
Three approximations to the price of the annuity as a function of i.

$$
\begin{aligned}
P(i) &\approx P(0.05) - D_{mod}(0.05)P(0.05)(i - 0.05) \\
&= 7686.2255 - (11.3992)(7686.2255)(i - 0.05) \\
P(i) &\approx P(0.05)\left(\frac{1+0.05}{1+i}\right)^{D_{mac}(0.05)} \\
&= 7686.2255\left(\frac{1+0.05}{1+i}\right)^{11.9691} \\
P(i) &\approx P(0.05) - D_{mod}(0.05)P(0.05)(i - 0.05) + \frac{1}{2}C_{mod}(0.05)(i - 0.05)^2 \\
&= 7686.2255 - (11.3992)(7686.2255)(i - 0.05) + \frac{1}{2}(202.038)(i - 0.05)^2
\end{aligned}
$$

Example (3 of 3)

i	Exact	C_{mod} Approx.	C_{mod} Error	D_{mac} Error	D_{mod} Error
0.01	$12,903.85$	$12,433.22$	470.63	668.75	1712.96
0.02	$11,198.23$	$11,013.54$	184.69	324.11	883.50
0.03	$9,800.22$	$9,749.14$	51.08	124.57	361.66
0.04	$8,646.02$	$8,640.04$	5.98	27.03	83.62
0.05	$7,686.23$	$7,686.23$	0.00	0.00	0.00
0.06	$6,882.42$	$6,887.70$	-5.29	20.56	72.36
0.07	$6,204.52$	$6,244.47$	-39.95	72.10	270.63
0.08	$5,628.89$	$5,756.54$	-127.64	142.62	571.17
0.09	$5,136.83$	$5,423.89$	-287.06	223.59	955.27
0.10	$4,713.46$	$5,246.53$	-533.07	308.96	$1,408.07$

Illustration



Macaulay Convexity

The **Macaulay convexity** for a sequence of n equally spaced cash flows is defined to be

$$C_{mac} = \frac{\sum_{t=1}^{n} t^2 CF_t (1+i)^{-t}}{\sum_{t=1}^{n} CF_t (1+i)^{-t}}.$$

The modified convexity, Macaulay duration, and Macaulay convexity are related through the formula,

$$C_{mod} = \frac{C_{mac} + D_{mac}}{(1+i)^2}.$$

Remark: when the term “convexity” is used alone it usually refers to modified convexity.

Example

Suppose a bond has a price of $965.35 and yields 6%. The bond's Macaulay duration $D_{mac} = 3.7177$ and the bond's Macaulay convexity $C_{mac} = 14.4095$.

- Find the first-order modified approximation for the bond's price with a yield of 7%.
- Find the first-order Macaulay approximation for the bond's price with a yield of 7%.
- Find the second-order modified approximation for the bond's price with a yield of 7%.

Solution

$$
\begin{aligned}
P(0.06) &= \$965.35 \\
D_{mod}(0.06) &= \frac{D_{mac}(0.06)}{1+0.06} = 3.5073 \\
C_{mod}(0.06) &= \frac{C_{mac}(0.06) + D_{mac}(0.06)}{(1+0.06)^2} = 16.1331 \\
P(0.07) &\approx P(0.06) - D_{mod}(0.06)P(0.06)(0.07-0.06) \\
&= \$931.4923 \\
P(0.07) &\approx P(0.06)\left(\frac{1+0.06}{1+0.07}\right)^{D_{mac}(0.06)} = \$932.2327 \\
P(0.07) &\approx 931.4923 + \frac{965.35}{2} C_{mod}(0.06)(0.07-0.06)^2 \\
&= \$932.2710
\end{aligned}
$$

Duration of a Portfolio of Cashflows

- Suppose there are m separate sequences of cashflows $\{c_t^{(k)}\}_{t=1}^{n}$ for $k = 1, 2, \ldots, m$.
- The present value of the kth sequence of cashflows is

$$X_k = \sum_{t=1}^{n} c_t^{(k)} (1+i)^{-t}$$

and the aggregate present value of all the cashflows is

$$X = \sum_{k=1}^{m} X_k.$$

- The Macaulay duration of the kth sequence of cashflows is

$$(D_{mac})_k = -\frac{(1+i)}{X_k} \frac{dX_k}{di} \iff (D_{mac})_k X_k = -(1+i) \frac{dX_k}{di}$$

and the Macaulay duration of the aggregate of all the cashflows is

$$D_{mac} = -\frac{(1+i)}{X} \frac{dX}{di} = \frac{1}{X} \sum_{k=1}^{m} -(1+i) \frac{dX_k}{di} = \sum_{k=1}^{m} (D_{mac})_k \frac{X_k}{X}.$$

- The duration of a portfolio of cashflows is the weighted average of the durations of the cashflows where the weights are the proportions the cashflows make up of the total.

Example

A portfolio consists of four bonds:

1. 3-year bond with face amount \$75,000 and 6% annual coupons.
2. 5-year bond with face amount \$90,000 and 5% annual coupons.
3. 10-year bond with face amount \$120,000 and 8% annual coupons.
4. 20-year bond with face amount \$100,000 and 10% annual coupons.

Find the Macaulay duration of the portfolio if the term structure is flat with effective annual interest rate of 8%.

Solution

$$D_{mac} = \frac{F\,r\,(Ia)_{\overline{n}|j} + n\,C\,v_j^n}{F\,r\,a_{\overline{n}|j} + C\,v_j^n}$$

$$D_{mod} = \frac{D_{mac}}{1+j}$$

Bond	**Price**	D_{mac}	**Weight**
3-year	71,134.36	2.8286	0.1824
5-year	79,219.68	4.5116	0.2031
10-year	120,000.00	7.2469	0.3077
20-year	119,636.29	10.1823	0.3068
Total	389,990.33		1.0000

$$\begin{aligned} D_{mac} &= (2.8286)(0.1824) + (4.5116)(0.2031) \\ &\quad + (7.2469)(0.3077) + (10.1823)(0.3068) \\ &= 6.7860 \end{aligned}$$

Modified Duration for a Portfolio

- The D_{mod} for a portfolio of bonds is the weighted average of the modified durations of the bonds in the portfolio.
- If the bonds have different yield rates, the D_{mod} for the portfolio will not be the D_{mac} for the portfolio divided by $1 + i$ since there is no single yield rate.

Immunization

Sometimes it is not possible to arrange an exact match between assets and liabilities for all $t > 0$. Thus a company may sometimes run a surplus when assets exceed liabilities and at other times a deficit when the company will have to borrow to pay liabilities.

Assume the term structure of interest rates is flat with a yield to all maturities of i_0. If the present values of assets and liabilities are the same then

$$\begin{aligned}
\mathrm{PV}_A(i_0) = \sum_{t=0}^{n} A_t v_{i_0}^t &= \sum_{t=0}^{n} L_t v_{i_0}^t = \mathrm{PV}_L(i_0) \\
\sum_{t=0}^{n} (A_t - L_t) v_{i_0}^t &= 0 \\
\sum_{t=0}^{n} (A_t - L_t)(1 + i_0)^{n-t} &= 0.
\end{aligned}$$

Remarks

$$\sum_{t=0}^{n}(A_t - L_t)(1 + i_0)^{n-t} = 0$$

- If the interest rate remains constant the balance of funds after the last liability is met will be zero.
- If interest rates change before the last liability is matched then there is a risk that $\mathrm{PV}_A(i) < \mathrm{PV}_L(i)$.
- The actuary F.M. Redington determined a method by which the asset/liability flow can be **immunized** against small changes in the interest rates ensuring that $\mathrm{PV}_A(i) \geq \mathrm{PV}_L(i)$.

Redington Immunization

If asset cashflows are A_t for $t = 0, 1, \ldots, n$ and liability cashflows are L_t for $t = 0, 1, \ldots, n$ then the liability cashflows are **Redington immunized** by the asset cashflows at valuation interest rate i_0 provided:

$$
\begin{aligned}
\mathrm{PV}_A(i_0) &= \mathrm{PV}_L(i_0) \\
\frac{d}{di}\left[\mathrm{PV}_A(i)\right]_{i=i_0} &= \frac{d}{di}\left[\mathrm{PV}_L(i)\right]_{i=i_0} \\
\frac{d^2}{di^2}\left[\mathrm{PV}_A(i)\right]_{i=i_0} &> \frac{d^2}{di^2}\left[\mathrm{PV}_L(i)\right]_{i=i_0}
\end{aligned}
$$

Justification

- Define $f(i) = \mathrm{PV}_A(i) - \mathrm{PV}_L(i)$.
- The first equation,

$$\mathrm{PV}_A(i_0) = \mathrm{PV}_L(i_0) \iff f(i_0) = 0,$$

  implying the cashflows are matched at interest rate i_0 in effect at $t = 0$.
- The second equation,

$$\frac{d}{di}\left[\mathrm{PV}_A(i)\right]_{i=i_0} = \frac{d}{di}\left[\mathrm{PV}_L(i)\right]_{i=i_0} \iff f'(i_0) = 0,$$

  implying function $f(i)$ has a critical number at $i = i_0$.
- The third equation,

$$\frac{d^2}{di^2}\left[\mathrm{PV}_A(i)\right]_{i=i_0} > \frac{d^2}{di^2}\left[\mathrm{PV}_L(i)\right]_{i=i_0} \iff f''(i_0) > 0,$$

  implying function $f(i)$ has a local minimum at $i = i_0$.

Remarks

- The first equation $\mathrm{PV}_A(i_0) = \mathrm{PV}_L(i_0)$ implies the asset and liability cashflows have the same present value.
- The second equation $\frac{d}{di}\left[\mathrm{PV}_A(i)\right]_{i=i_0} = \frac{d}{di}\left[\mathrm{PV}_L(i)\right]_{i=i_0}$ implies the asset and liability cashflows have the same modified duration.

$$D^A_{mod}(i_0) = D^L_{mod}(i_0)$$

- The inequality $\frac{d^2}{di^2}\left[\mathrm{PV}_A(i)\right]_{i=i_0} > \frac{d^2}{di^2}\left[\mathrm{PV}_L(i)\right]_{i=i_0}$ implies modified convexity for the asset cash flows is greater than the modified convexity for the liability cash flows.

$$C^A_{mod}(i_0) > C^L_{mod}(i_0)$$

Example

A company has a liability which requires a single payment of \$200,000 in 7 years. The interest rate for assets and liabilities is $i_0 = 0.06$. The company wishes to create an immunized portfolio by buying two zero-coupon bonds that will mature in 4 years and 10 years. Find the amounts of the two bonds which must be purchased.

Solution (1 of 2)

The first step is present value matching and duration matching. Let A_4 be the amount of the 4-year bond purchased and A_{10} be the amount of the 10-year bond purchased.

$$\begin{aligned}
\frac{A_4}{(1+0.06)^4} + \frac{A_{10}}{(1+0.06)^{10}} &= \frac{200,000}{(1+0.06)^7} \\
\frac{4A_4}{(1+0.06)^4} + \frac{10A_{10}}{(1+0.06)^{10}} &= \frac{(7)200,000}{(1+0.06)^7}
\end{aligned}$$

Solving this system of equations produces

$$\begin{aligned}
A_4 &= \$83,961.93 \\
A_{10} &= \$119,101.60.
\end{aligned}$$

Solution (2 of 2)

The portfolio is immunized since we can confirm that

$$\begin{aligned}
\frac{(4)^2 A_4}{(1+0.06)^4} + \frac{(10)^2 A_{10}}{(1+0.06)^{10}} &> \frac{(7)^2 200,000}{(1+0.06)^7} \\
\frac{(4)^2(83961.93)}{(1+0.06)^4} + \frac{(10)^2 119101.60}{(1+0.06)^{10}} &> \frac{(7)^2 200,000}{(1+0.06)^7} \\
7,714,662.52 &> 6,517,559.71.
\end{aligned}$$

Example

A company is terminating three employees of ages 53, 58, and 60. The company will provide a severance package that pays $20,000 per year to each employee until age 65 plus $100,000 at age 65. The company plans to match the liability cashflows with the proceeds from two zero coupon bonds due at time $t_1 = 5$ and $t_2 = 12$ (measured from the starting date of the severance package). Suppose the term structure of interest rates is flat at an effective annual rate of 10%.

1. Determine the amounts of each zero coupon bond that must be purchased.
2. Determine whether the overall asset/liability portfolio is in an immunized position.

Solution (1 of 3)

- Let A_5 be the amount of zero coupon bond purchased with maturity at $t_1 = 5$ and let A_{12} be the amount of zero coupon bond purchased with maturity at $t_2 = 12$.
- To match the assets and liabilities we need:

$$\begin{aligned} A_5 v_{0.10}^5 + A_{12} v_{0.10}^{12} &= 20,000 a_{\overline{12}|0.10} + 100,000 v_{0.10}^{12} \\ &\quad + 20,000 a_{\overline{7}|0.10} + 100,000 v_{0.10}^{7} \\ &\quad + 20,000 a_{\overline{5}|0.10} + 100,000 v_{0.10}^{5} \\ (0.6209) A_5 + (0.3186) A_{12} &= \$454,728.97 \end{aligned}$$

Solution (2 of 3)

- To match durations we need:

$$
\begin{aligned}
5A_5 v_{0.10}^5 + 12A_{12} v_{0.10}^{12} &= 20,000 \sum_{t=1}^{12} t\, v_{0.10}^t + 12(100,000) v_{0.10}^{12} \\
&\quad + 20,000 \sum_{t=1}^{7} t\, v_{0.10}^t + 7(100,000) v_{0.10}^{7} \\
&\quad + 20,000 \sum_{t=1}^{5} t\, v_{0.10}^t + 5(100,000) v_{0.10}^{5} \\
(3.1046)A_5 + (3.8236)A_{12} &= 20,000 \left[(Ia)_{\overline{12}|0.10} + (Ia)_{\overline{7}|0.10} + (Ia)_{\overline{5}|0.10} \right] \\
&\quad + 1,052,028.33 \\
&= 1,299,979.77 + 1,052,028.33 \\
&= 2,352,008.10
\end{aligned}
$$

Solution (3 of 3)

We must solve the system of equations:

$$\begin{aligned}(0.6209)A_5 + (0.3186)A_{12} &= 454,728.97 \\ (3.1046)A_5 + (3.8236)A_{12} &= 2,352,008.10\end{aligned}$$

In this case $A_5 = 714,360$ and $A_{12} = 35,099$.

The portfolio is immunized if

$$\begin{aligned}(5)^2 A_5 v_{0.10}^5 + (12)^2 A_{12} v_{0.10}^{12} & \\ &> 20,000\left[\sum_{t=1}^{12} t^2 v_{0.10}^t + \sum_{t=1}^{7} t^2 v_{0.10}^t + \sum_{t=1}^{5} t^2 v_{0.10}^t\right] \\ &\quad + 100,000\left[(12)^2 v_{0.10}^{12} + (7)^2 v_{0.10}^7 + (5)^2 v_{0.10}^5\right] \\ 12,699,500 &< 76,184,900\end{aligned}$$

so the portfolio is not immunized.

Full Immunization

Definition

A portfolio is **fully immunized** if

$$\sum_{t=1}^{n} A_t v^t \geq \sum_{t=1}^{n} L_t v^t$$

for any $i > 0$.

Example

Suppose we wish to fully immunize a liability payment of $L_{12} = \$150,000$ due in 12 years by purchasing zero coupon bonds A_2 and A_{15} due in 2 and 15 years respectively. If the term structure is flat with $i_0 = 0.10$, how much of each zero coupon bond should be purchased?

Solution (1 of 6)

The assets and liabilities are matched and have the same duration when

$$\begin{aligned} A_2 v_{0.10}^2 + A_{15} v_{0.10}^{15} &= L_{12} v_{0.10}^{12} \\ 2A_2 v_{0.10}^2 + 15A_{15} v_{0.10}^{15} &= 12L_{12} v_{0.10}^{12}. \end{aligned}$$

Multiply the first equation by 12 and subtract the second equation.

$$\begin{aligned} 10A_2 v_{0.10}^2 - 3A_{15} v_{0.10}^{15} &= 0 \\ 10A_2 v_{0.10}^2 &= 3A_{15} v_{0.10}^{15} \end{aligned}$$

We can use this relationship to eliminate A_{15} from the first equation.

Solution (2 of 6)

$$\begin{aligned}
L_{12} v_{0.10}^{12} &= A_2 v_{0.10}^{2} + A_{15} v_{0.10}^{15} \\
&= A_2 v_{0.10}^{2} + \frac{1}{3}\left(3 A_{15} v_{0.10}^{15}\right) \\
&= A_2 v_{0.10}^{2} + \frac{10}{3} A_2 v_{0.10}^{2} \\
L_{12} &= A_2 (1+i_0)^{10} + \frac{10}{3} A_2 (1+i_0)^{10} \\
&= A_2 (1+i_0)^{10} \left[1 + \frac{10}{3}\right]
\end{aligned}$$

We can use this to eliminate L_{12} in the matching equation.

Solution (3 of 6)

Define the function:

$$
\begin{aligned}
f(i) &= A_2 v_i^2 + A_{15} v_i^{15} - L_{12} v_i^{12} \\
&= v_i^{12} \left[A_2 (1+i)^{10} + A_{15} (1+i)^{-3} - L_{12} \right] \\
&= v_i^{12} (1+i_0)^{10} \left[A_2 \left(\frac{1+i}{1+i_0} \right)^{10} + A_{15} \frac{(1+i)^{-3}}{(1+i_0)^{10}} - \frac{1}{(1+i_0)^{10}} L_{12} \right]
\end{aligned}
$$

Recall that

$$
\begin{aligned}
A_{15} &= \frac{10}{3} A_2 (1+i_0)^{13} \\
L_{12} &= \frac{13}{3} A_2 (1+i_0)^{10}
\end{aligned}
$$

and substitute in the expression above.

Solution (4 of 6)

Using the expressions derived for

$$\begin{aligned} f(i) &= v_i^{12}(1+i_0)^{10}\left[A_2\left(\frac{1+i}{1+i_0}\right)^{10} + A_{15}\frac{(1+i)^{-3}}{(1+i_0)^{10}} - \frac{1}{(1+i_0)^{10}}L_{12}\right] \\ &= v_i^{12}(1+i_0)^{10}A_2\left[\left(\frac{1+i}{1+i_0}\right)^{10} + \frac{10}{3}\left(\frac{1+i}{1+i_0}\right)^{-3} - \frac{13}{3}\right] \\ &= v_i^{12}A_2(1+i_0)^{10}g(i) \end{aligned}$$

where

$$g(i) = \left(\frac{1+i}{1+i_0}\right)^{10} + \frac{10}{3}\left(\frac{1+i}{1+i_0}\right)^{-3} - \frac{13}{3}.$$

Note that $f(i_0) = g(i_0) = 0$.

Solution (5 of 6)

$$
\begin{aligned}
g(i) &= \left(\frac{1+i}{1+i_0}\right)^{10} + \frac{10}{3}\left(\frac{1+i}{1+i_0}\right)^{-3} - \frac{13}{3} \\
g'(i) &= \frac{10}{1+i_0}\left(\frac{1+i}{1+i_0}\right)^{9} - \frac{10}{1+i_0}\left(\frac{1+i}{1+i_0}\right)^{-4} \\
&= 10v_i\left[\left(\frac{1+i}{1+i_0}\right)^{10} - \left(\frac{1+i}{1+i_0}\right)^{-3}\right]
\end{aligned}
$$

Note:

- If $i < i_0$ then $g'(i) < 0$.
- If $i > i_0$ then $g'(i) > 0$.
- Thus $g(i)$ is decreasing for $i < i_0$ and increasing for $i > i_0$.
- Therefore $g(i)$ has an absolute minimum at $i = i_0$.
- Consequently $g(i) \geq g(i_0) = 0$ for all i.
- Since $f(i) = v_i^{12}(1+i_0)^{10} A_2 g(i)$ then $f(i) \geq 0$ for all i and the portfolio is fully immunized.

Solution (6 of 6)

- The assets and liabilities are matched if

$$\begin{aligned} A_2 v_{0.10}^2 + A_{15} v_{0.10}^{15} &= 150,000 v_{0.10}^{12} \\ (0.8264)A_2 + (0.2394)A_{15} &= 47,794.62. \end{aligned}$$

- The assets and liabilities have the same duration if

$$\begin{aligned} 2A_2 v_{0.10}^2 + 15A_{15} v_{0.10}^{15} &= 12(150,000) v_{0.10}^{12} \\ (1.6529)A_2 + (3.5909)A_{15} &= 573,535.47. \end{aligned}$$

- Solving this system of equations produces

$$\begin{aligned} A_2 &= \$13,345.73 \\ A_{15} &= \$153,576.92. \end{aligned}$$

Introduction to Stocks

- A share of a corporation's **stock** makes the shareholder a partial owner of the corporation.
- Shareholders are paid a portion of the corporation's profits in the form of **dividends**.
- The shareholder may also sell the stock for a higher price than at which it was purchased resulting in a **capital gain**.
- If the stock is sold for less than the price paid to purchase it the shareholder incurs a **capital loss**.

Dividend Discount Model

The **dividend discount model** is one way to determine the value of a share of stock. It assumes the value of the stock is the present value of all the dividends at a valuation interest rate of i.

Let Div_k be the amount of the dividend paid at $t = k$, then

$$\mathrm{PV} = \sum_k \frac{\mathrm{Div}_k}{(1+i)^k}.$$

Dividend Growth Model

Suppose the next dividend to be paid is Div and that each dividend to follow will increase by a factor of g. If the dividends are paid in perpetuity, then

$$\begin{aligned}
\mathrm{PV} &= \sum_{k=0}^{\infty} \frac{\mathrm{Div}}{1+i} \left(\frac{1+g}{1+i} \right)^{k} \\
&= \frac{\mathrm{Div}}{1+i} \frac{1}{1 - \frac{1+g}{1+i}} \\
&= \frac{\mathrm{Div}}{i-g}
\end{aligned}$$

Example

Find the value and duration of a stock that pays dividends at the end of each year in perpetuity. The dividend X is constant and the annual effective rate of interest is 6%.

Solution

- The present value of the stock is

$$P = X a_{\overline{\infty}|i} = \frac{X}{i}.$$

- The duration is

$$D_{mac} = -(1+i)\frac{1}{P}\frac{dP}{di} = -(1+i)\frac{i}{X}\left(-\frac{X}{i^2}\right) = \frac{1+i}{i}.$$

- If $i = 0.06$ then

$$D_{mac} = \frac{1+0.06}{0.06} = 17.6667.$$

Example

The stock of a company sells for \$100 per share assuming an effective annual interest rate of i. Dividends are paid at the end of each year in perpetuity. The first dividend is \$2 and each dividend is 2% larger than the previous one. Find the duration of the stock.

Solution

- Find i.

$$100 = \frac{2}{i - 0.02} \implies i = 0.04$$

- Find D_{mac}.

$$\begin{aligned}
P &= \frac{2}{i - 0.02} \implies \frac{dP}{di} = -\frac{2}{(i - 0.02)^2} \\
D_{mac} &= -(1 + i)\frac{1}{P}\frac{dP}{di} \\
D_{mac} &= \frac{(1 + 0.04)}{100}\frac{2}{(0.04 - 0.02)^2} = 52
\end{aligned}$$

Mutual Funds

- A **mutual fund** allows an investor to invest in a pool of stocks selected by professional investment managers.
- An investor buys shares of the mutual fund by paying the current price per share of the fund to the fund managers.
- An investor redeems shares by selling them back to the mutual fund and receiving the current price per share.
- Mutual funds give investors the advantage of **diversification** since the pool of stocks in the fund will be larger than most individual investors can afford to buy on their own.
- Some mutual funds may specialize in pooling stocks related to one industry, or one type of security (*e.g.* bonds or real estate), or one geographical region.

Certificates of Deposit

- **Certificates of deposit** (**CD**) are offered by banks, credit unions, and savings and loans.
- The interest rate earned by the CD is fixed until maturity.
- There is usually a minimum deposit required.
- If the CD is redeemed before maturity, the investor may have to forfeit a portion of the interest earned.

Money Market Funds

- A **money market fund** is a mutual fund specializing in short-term secure investments like Treasury bills.
- Money market funds are operated as savings accounts and pay interest.
- The advantage to the money market fund is the higher interest rate paid by the fund than the rate offered by banks.

Mortgage-Backed Securities

- Interest rates charged on mortgages are usually higher than those offered by bank savings accounts or CDs.
- Mortgage lenders will bundle large numbers of mortgages into a pool of loans called a **mortgage-backed security (MBS)**.
- The MBS can be sold to investors who receive returns that yield the mortgage interest rates without having to manage the mortgage loans themselves.
- Due to uncertainty about mortgage defaults and prepayment, these types of investments are risky.

Homework

- Read Chapter 7
- Exercises: distributed on handout