ACTEX & a/s/m

# EXAM FM
## FORMULA CARD

## TIME VALUE OF MONEY

***Accumulation and Amount Function:*** $A(t) = Ka(t) \quad A(0) = K \quad a(0) = 1$

***Effective Interest Rate:*** $i_t = \frac{a(t)-a(t-1)}{a(t-1)} = \frac{A(t)-A(t-1)}{A(t-1)}$

***Simple Interest:*** $a(t) = 1 + it \qquad i_t = \frac{i}{1+i(t-1)}$

***Compound Interest:*** $a(t) = (1+i)^t \qquad i_t = i$

***Effective Discount Rate:*** $d_t = \frac{a(t)-a(t-1)}{a(t)} = \frac{A(t)-A(t-1)}{A(t)}$

***Discount Rate:*** $d = \frac{i}{1+i} = 1 - v = iv \qquad v = (1+i)^{-1} \qquad \frac{1}{d} - \frac{1}{i} = 1$

***Nominal Rates:*** $\left(1+\frac{i^{(m)}}{m}\right)^m = 1+i \qquad \left(1-\frac{d^{(m)}}{m}\right)^m = 1-d$

$i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right] \qquad d^{(m)} = m\left[1-(1-d)^{\frac{1}{m}}\right]$

***Force of Interest:*** $\delta_t = \frac{a'(t)}{a(t)} = \frac{A'(t)}{A(t)} \qquad a(t) = e^{\int_0^t \delta_r dr} \quad A(t) = A(0)e^{\int_0^t \delta_r dr}$

***Constant Force of Interest:*** $e^{\delta} = 1 + i \qquad \delta = \ln(1+i)$

***PV of $1 due in t years:***

$$PV = \frac{1}{a(t)} = (1+i)^{-t} = v^t = e^{-\delta t} = (1-d)^t = \left(1+\frac{i^{(m)}}{m}\right)^{-mt} = \left(1-\frac{d^{(m)}}{m}\right)^{mt}$$

***AV of $1 over t years:***

$$AV = a(t) = (1+i)^t = e^{\delta t} = (1-d)^{-t} = \left(1+\frac{i^{(m)}}{m}\right)^{mt} = \left(1-\frac{d^{(m)}}{m}\right)^{-mt}$$

***AV at time $t_2$ of $1 invested at time $t_1$:*** $AV = \frac{a(t_2)}{a(t_1)} = e^{\int_{t_1}^{t_2} \delta_t dt}$

***Method of Equated Time:*** $\bar{t} = \frac{A_1t_1 + A_2t_2 + \cdots + A_nt_n}{A_1 + A_2 + \cdots + A_n}$

## ANNUITIES

***Annuity-Immediate:***
(PV one period before first payment, AV at time of last payment)

$$PV = a_{\overline{n}|} = v + v^2 + \cdots + v^n = \frac{1-v^n}{i}$$

$$AV = s_{\overline{n}|} = 1 + (1+i) + \cdots + (1+i)^{n-1} = \frac{(1+i)^n - 1}{i} = a_{\overline{n}|}(1+i)^n$$

***Annuity-Due:***
(PV at time of first payment, AV one period after last payment)

$$PV = \ddot{a}_{\overline{n}|} = 1 + v + \cdots + v^{n-1} = \frac{1-v^n}{d} = \left(\frac{i}{d}\right)a_{\overline{n}|} = (1+i)a_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

$$AV = \ddot{s}_{\overline{n}|} = (1+i) + (1+i)^2 + \cdots + (1+i)^n = \frac{(1+i)^n - 1}{d} = \left(\frac{i}{d}\right)s_{\overline{n}|} = (1+i)s_{\overline{n}|} = s_{\overline{n+1}|} - 1$$

***Continuous Annuity:***

$$PV = \bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta} = \left(\frac{i}{\delta}\right)a_{\overline{n}|} = \int_0^n v^t\,dt = \int_0^n e^{-\delta t}\,dt$$

$$AV = \bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta} = \left(\frac{i}{\delta}\right)s_{\overline{n}|} = \int_0^n (1+i)^{n-t}\,dt = \int_0^n e^{\delta(n-t)}\,dt$$

***Deferred Annuity:***

$${}_{m|}a_{\overline{n}|} = {}_{m+1|}\ddot{a}_{\overline{n}|} = v^m a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$$

***Perpetuity:***

$$a_{\overline{\infty}|} = \frac{1}{i} \qquad \ddot{a}_{\overline{\infty}|} = \frac{1}{d} \qquad \bar{a}_{\overline{\infty}|} = \frac{1}{\delta}$$

***Relationships:***

$$a_{\overline{2n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|} \qquad \frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n = \frac{\ddot{a}_{\overline{2n}|}}{\ddot{a}_{\overline{n}|}} \qquad \frac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n}$$

***Increasing Annuity - Payments in Arithmetic Progression:***

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \qquad (Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i} = \frac{s_{\overline{n+1}|} - (n+1)}{i}$$

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} \qquad (I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{s_{\overline{n+1}|} - (n+1)}{d}$$

***Decreasing Annuity - Payments in Arithmetic Progression:***

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i} \qquad (Ds)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{i}$$

$$(D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d} \qquad (D\ddot{s})_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{d}$$

***Increasing/Decreasing Perpetuity - Payments in Arithmetic Progression:***

$$(Ia)_{\overline{\infty}|} = \frac{1}{i} + \frac{1}{i^2} = \frac{1}{id} \qquad (I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$$

***m-thly Annuity:***

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}} \qquad (Ia)_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i^{(m)}} \qquad (I^{(m)}a)_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|}^{(m)} - nv^n}{i^{(m)}}$$

***Continuously Increasing Annuity:***

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} = \int_0^n tv^t\,dt = \int_0^n te^{-\delta t}\,dt$$

$$(\bar{I}\bar{s})_{\overline{n}|} = \frac{\bar{s}_{\overline{n}|} - n}{\delta} = \int_0^n t(1+i)^{n-t}\,dt = \int_0^n te^{\delta(n-t)}\,dt$$

***Annuity with Varying Annual Rate of Payment:***

$$PV = \int_0^n f(t)e^{-\delta t}\,dt = \int_0^n f(t)e^{-\int_0^t \delta_r dr}\,dt$$

$$AV = \int_0^n f(t)e^{\delta(n-t)}\,dt = \int_0^n f(t)e^{\int_t^n \delta_r dr}\,dt$$

***Geometric Annuity-Immediate:***
1st Payment is 1, Subsequent Payments Increasing by $k\%$

$$PV = \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i-k} \qquad Geometric\ Sum = (1st\ Term)\left(\frac{1 - Ratio^{Number\ of\ Terms}}{1 - Ratio}\right)$$

## LOANS

***Amortization Method:***
Level payment $R$, Outstanding balance $B_t$, Principal repayment $P_t$

$L = Ra_{\overline{n}|} \qquad R = P_t + I_t \qquad I_t = R - P_t = R(1 - v^{n-t+1})$

$P_t = R - I_t = Rv^{n-t+1} \qquad P_{t+s} = P_t(1+i)^s \qquad I_t = iB_{t-1}$

$\sum I_t = nR - L \qquad \sum P_t = L \qquad B_t = B_{t-1} - P_t$

$B_t = Ra_{\overline{n-t}|}$ (Prospective) $\qquad B_t = L(1+i)^t - Rs_{\overline{t}|}$ (Retrospective)

***Sinking Fund Method:***
Loan $L$, SF Deposit D, Interest on Loan $i$, Interest on SF $j$

$Annual\ Outlay = I + D \qquad I = iL \qquad L = Ds_{\overline{n}|j} \qquad D = \frac{L}{s_{\overline{n}|j}} \qquad \frac{1}{a_{\overline{n}|i}} = i + \frac{1}{s_{\overline{n}|i}}$

Sinking Fund Balance at time $t = SF_t = Ds_{\overline{t}|j}$

Net Amount of Interest paid in year $t = I_t = iL - j(SF_{t-1})$

Principal Repaid in year $t = P_t = SF_t - SF_{t-1} = D + j(SF_{t-1})$

## INTEREST RATE SWAPS

***Constant Notional Value:*** n-year swap rate $R = \frac{1-P_n}{\sum P_i}$

t-year deferred m-year swap rate: $R = \frac{P_t - P_{t+m}}{P_{t+1} + \cdots + P_{t+m}}$

***Variable Notional Values:*** $R = \frac{\sum Q_t f_{[t-1,t]} P_t}{\sum Q_t P_t}$

$P_t = (1+r_t)^{-t} \qquad f_{[t-1,t]} = \frac{(1+r_t)^t}{(1+r_{t-1})^{t-1}} - 1 = \frac{P_{t-1}}{P_t} - 1$

## BONDS

***Price Formulas:*** Number of coupon payments $n$, Coupon rate $r$

$P = Fra_{\overline{n}|} + Cv^n \qquad P = C + (Fr - Ci)a_{\overline{n}|}$

$P = K + \frac{g}{i}(C - K), \quad K = Cv^n, \quad g = \frac{Fr}{C}$

***Price between Coupon Dates:*** $B_{t+k} = B_t(1+i)^k, \quad 0 < k < 1$

***Quoted Price:*** $Q_t = B_{t+k} - kFr = B_t(1+i)^k - kFr$

***Callable Bonds*** - To calculate appropriate price:
If Bond is sold at a Premium, assume Early Redemption date
If Bond is sold at a Discount, assume Late Redemption date

If $g > i$, then $P > C$ and ***Premium*** $= P - C = (Fr - Ci)a_{\overline{n}|} = (Cg - Ci)a_{\overline{n}|}$

***Amortization of Premium Amount:*** $P_t = Fr - I_t = B_{t-1} - B_t = (Fr - Ci)v^{n-t+1}$

***Book Value:*** $B_t = B_{t-1} - P_t = Fra_{\overline{n-t}|} + Cv^{n-t}$

***Interest Earned:*** $I_t = iB_{t-1}$

If $g < i$, then $P < C$ and ***Discount*** $= C - P = (Ci - Fr)a_{\overline{n}|} = (Ci - Cg)a_{\overline{n}|}$

***Accumulation of Discount Amount:*** $P_t = I_t - Fr = B_t - B_{t-1} = (Ci - Fr)v^{n-t+1}$

***Book Value:*** $B_t = B_{t-1} + P_t = Fra_{\overline{n-t}|} + Cv^{n-t}$

***Interest Earned:*** $I_t = iB_{t-1}$

## GENERAL CASH FLOW

***Reinvestment Rates:*** Interest rate $i$, Reinvestment rate $i'$
A single deposit of \$1, $AV = 1 + is_{\overline{n}|i'}$
Deposits of \$1 at beginning of each year, $AV = n + i(Is)_{\overline{n}|i'}$

***Dollar-Weighted Rate of Return:***
$i_D = \frac{B - A - \sum C_t}{A + \sum C_t(1-t)}$ , Initial balance A, Ending balance B, Contribution $C_t$

***Time-Weighted Method:***
$1 + i_T = \frac{B_1}{B_0} \cdot \frac{B_2}{B_1 + C_1} \cdot \ldots \cdot \frac{B_n}{B_{n-1} + C_{n-1}}$, Balance $B_t$ before any contribution $C_t$

***Price of a Stock - Dividend Discount Model:***
$P = \frac{D}{i}$ , Level dividends
$P = \frac{D}{i-k}$ , Dividends increasing at rate $k$% with first $D$ at time 1, $k < i$

***Spot Rates:*** Effective annual yield rate on investment for $t$ years, $r_t$
Price of a $t$-year zero-coupon Bond: $P_t = (1 + r_t)^{-t}$

***Forward Rates:*** (Effective Annual)
$m$-year forward rate, deferred $t$ years:
$(1 + r_t)^t (1 + f_{[t,t+m]})^m = (1 + r_{t+m})^{t+m}$

One-year forward rate, deferred $t$ years:
$f_{[t,t+1]} = \frac{(1+r_{t+1})^{t+1}}{(1+r_t)^t} - 1 = \frac{P_t}{P_{t+1}} - 1$

***Inflation Rate:*** Real rate of interest $i'$, Inflation rate $r$
$1 + i = (1 + i')(1 + r), \qquad i = i' + r + i'r$

***Duration:***

$D_{mac}(i) = \frac{\sum tA_t v^t}{\sum A_t v^t} = -\frac{\frac{d}{d\delta}P}{P} \qquad D_{mod}(i) = \frac{\sum tA_t v^{t+1}}{\sum A_t v^t} = -\frac{\frac{d}{di}P}{P} = D_{mac}v$

Perpetuity: $D_{mac} = \frac{1+i}{i}$ — Mortgage or Level Annuity: $D_{mac} = \frac{(Ia)_{\overline{n}|}}{a_{\overline{n}|}}$

Bond: $D_{mac} = \frac{Fr(Ia)_{\overline{n}|} + nCv^n}{P}$ — Bond Sold at Par: $D_{mac} = \ddot{a}_{\overline{n}|}$

***Convexity:***

$C_{mac} = \frac{\sum t^2 A_t v^t}{\sum A_t v^t} = \frac{\frac{d^2}{d\delta^2}P}{P} \qquad C_{mod} = \frac{\sum t(t+1)A_t v^{t+2}}{\sum A_t v^t} = \frac{\frac{d^2}{di^2}P}{P} = \frac{C_{mac} + D_{mac}}{(1+i)^2}$

***Duration of a Portfolio:*** $D_t$ and $P_t$ are duration and price of components of Portfolio
$D(Portfolio) = \frac{D_1P_1 + D_2P_2 + \cdots + D_nP_n}{P_1 + P_2 + \cdots + P_n}$

***First-Order Modified Price Approximation:*** $P(i) \approx P(i_0) - P(i_0)(i - i_0)D_{mod}(i_0)$

***First-Order Macaulay Price Approximation:*** $P(i) \approx P(i_0)\left(\frac{1+i_0}{1+i}\right)^{D_{mac}(i_0)}$

## IMMUNIZATION

***Redington Immunization:***

| | | |
|---|---|---|
| (i) PV(Assets) = PV(Liabilities) | (i) $P_A = P_L$ | (i) $\sum A_t v^t = \sum L_t v^t$ |
| (ii) Duration(Assets) = Duration(Liabilities) | (ii) $P'_A = P'_L$ | (ii) $\sum tA_t v^t = \sum tL_t v^t$ |
| (iii) Convexity(Assets) > Convexity(Liabilities) | (iii) $P''_A > P''_L$ | (iii) $\sum t^2 A_t v^t > \sum t^2 L_t v^t$ |

***Full Immunization:*** (i) and (ii) as above, (iii) One Asset cash inflow before each Liability cash outflow and one after it.

***Exact Matching or Dedication:*** Match both the amount and the time of Assets and Liabilities

## DETERMINANTS OF INTEREST RATES

***U.S. Treasury Bills:***
$Price\ P = C\left(1 - \frac{n}{360}d\right) \qquad Quoted\ Rate\ d = \left(\frac{360}{n}\right)\left(\frac{C-P}{C}\right)$

***Government of Canada Treasury Bills:***
$Price\ P = \frac{C}{\left(1 + \frac{n}{365}i\right)} \qquad Quoted\ Rate\ i = \left(\frac{365}{n}\right)\left(\frac{C-P}{P}\right)$

***Default Risk:*** $x$ amount received with no default
$(1-q)y = x$, $y$ amount received with a default rate $q$
$(1-q)y + q(uy) = x$, with a partial recovery rate $u$

***Components of Interest Rate $R$:***
$r$ rate without default risk, $s$ rate for risk compensation, $i$ inflation rate

With Default Risk but No Inflation:
$R = (1+r)(1+s) - 1$ (effective) $\qquad R = r + s$ (continuous)

With Default Risk and Known Inflation:
$R = (1+r)(1+s)(1+i) - 1$ (effective) $\qquad R = r + s + i$ (continuous)

With Default Risk and Uncertain Inflation:
$R = r + s + i_e + i_u$, $i_e$ and $i_u$ continuous expected and unexpected inflation rates